SUBSPACE-BASED DIRECTION FINDING IN ALPHA-STABLE NOISE ENVIRONMENTS

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ABSTRACT

There exist real world applications in which impulsive channels tend to produce large amplitude interferences more frequently than Gaussian channels. The stable law has been shown to successfully model noise over certain impulsive channels. In this paper, we propose subspace-based methods for the direction-of-arrival estimation problem in impulsive noise environments. We define the covariation matrix of the array sensor outputs and show that eigendecomposition-based methods, such as the MUSIC algorithm, can be applied to the sample covariation matrix to extract the bearing information from the measurements.

1. INTRODUCTION

The majority of subspace-based, high resolution methods for direction finding in array processing has been based on the assumption of Gaussian distributed signals and noise. In [1] we introduced optimal (in the ML sense) approaches to the direction-of-arrival (DOA) estimation problem in the presence of impulsive noise environments. The analysis was based on the assumption that the additive noise could be modeled as a complex *symmetric α-stable (SαS)* process. The optimal ML techniques employed in [1] are often regarded as exceedingly complex due to the high computational load of the multivariate nonlinear optimization problem involved with these techniques. Hence, sub-optimal methods need to be developed for the solution of the DOA estimation problem in the presence of impulsive noise, when reduced computational cost is a crucial design requirement.

In this paper, we present subspace source localization methods based on geometrical properties of the data model. Considerable research has been done in this area under the framework of Gaussian distributed signals and/or noise [2]. The better known of the so-called eigenvector-based methods are the MUSIC [3], Minimum Norm [4], and the ESPRIT method [5]. These methods estimate the bearings of the source signals by performing an eigendecomposition on the spatial covariance matrix of the array sensor outputs. Studies concerning the statistical efficiency of the most popular eigendecomposition-based method, namely the MUSIC algorithm, have been done in [6]-[7]. The relationship between the MUSIC and ML methods has also been studied in [6]. Since $S\alpha S$ processes do not possess finite pth order moments for $p \geq \alpha$, traditional subspace techniques employing second- and higher-order moments [8] cannot be applied in impulsive noise environments modeled under the stable law. Instead, properties of fractional lower-order moments (FLOM's) and covariations should be used.

The paper is organized as follows: In Section 2, we present some necessary preliminaries on α-stable processes. In Section 3, we discuss the development of subspace techniques in the presence of α-stable distributed signals and noise. The analysis is based on the formulation of the covariation matrix of the array sensor outputs. Finally, simulation experiments are presented in Section 4 and conclusions are drawn in Section 5.

2. MATHEMATICAL PRELIMINARIES

In this section we introduce the statistical model that will be used to describe the additive noise. The model is based on the class of *isotropic* $S\alpha S$ distributions, and is well-suited for describing impulsive noise processes.

Stable processes satisfy the stability property which states that linear combinations of jointly stable variables are indeed stable. They arise as limiting processes of sums of independent, identically-distributed random variables via the generalized central limit theorem. They are described by their characteristic exponent α, taking values $0 < \alpha \leq 2$. Gaussian processes are stable processes with $\alpha = 2$. Stable distributions have heavier tails than the normal distribution, possess finite pth order moments only for $p < \alpha$, and are appropriate for modeling noise with outliers. The main reason for the difficulty in developing signal processing methods based on stable processes is due to the fact that the linear space of a stable process is not a Hilbert space, as in the case of Gaussian processes, but either a Banach ($1 < \alpha < 2$) or a metric space ($0 < \alpha \leq 1$) both of which are more unyielding in their structure. An extensive review of the $S\alpha S$ family can be found in [9]. Here, we focus on

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the elements useful for our analysis later on the paper.

A complex random variable (r.v.) $X = X_1 + \jmath X_2$ is isotropic $S\alpha S$ if X_1 and X_2 are jointly $S\alpha S$ and have a symmetric distribution. The characteristic function of X is given by

$$\varphi(\omega) = \mathcal{E}\{\exp(\jmath\Re[\omega X^*])\} = \exp(-\gamma|\omega|^\alpha), \qquad (1)$$

where $\omega = \omega_1 + \jmath\omega_2$. The *characteristic exponent* α is restricted to the values $0 < \alpha \leq 2$ and it determines the shape of the distribution. The smaller the characteristic exponent α, the heavier the tails of the density. The *dispersion* γ ($\gamma > 0$) plays a role analogous to the role that the variance plays for second-order processes. Namely, it determines the spread of the probability density function around the origin.

Several complex r.v.'s are jointly $S\alpha S$ if their real and imaginary parts are jointly $S\alpha S$. When X and Y are jointly $S\alpha S$ with $1 < \alpha \leq 2$, the *covariation* of X and Y is defined by

$$[X,Y]_\alpha = \frac{\mathcal{E}\{XY^{<p-1>}\}}{\mathcal{E}\{|Y|^p\}}\gamma_Y, \quad 1 \leq p < \alpha, \qquad (2)$$

where $\gamma_Y = [Y,Y]_\alpha$ is the dispersion of the r.v. Y, and we use throughout the convention $Y^{<p>} = |Y|^{p-1}Y^*$. Also, the *covariation coefficient* of X and Y is defined by

$$\lambda_{X,Y} = \frac{[X,Y]_\alpha}{[Y,Y]_\alpha}, \qquad (3)$$

and by using (2), it can be expressed as

$$\lambda_{X,Y} = \frac{E\{XY^{<p-1>}\}}{E\{|Y|^p\}}, \qquad \text{for } 1 \leq p < \alpha. \qquad (4)$$

The covariation of complex jointly $S\alpha S$ r.v.'s is not generally symmetric and has the following properties :

P1 If X_1, X_2 and Y are jointly $S\alpha S$, then for any complex constants a and b,

$$[aX_1 + bX_2, Y]_\alpha = a[X_1,Y]_\alpha + b[X_2,Y]_\alpha;$$

P2 If Y_1 and Y_2 are independent and X_1, X_2 and Y are jointly $S\alpha S$, then for any complex constants a, b and c,

$$[aX_1, bY_1 + cY_2]_\alpha = ab^{<\alpha-1>}[X_1,Y_1]_\alpha + ac^{<\alpha-1>}[X_1,Y_2]_\alpha;$$

P3 If X and Y are independent $S\alpha S$, then $[X,Y]_\alpha = 0$.

3. SUBSPACE TECHNIQUES IN THE α-STABLE FRAMEWORK

Consider an array of r sensors that receive signals generated by q narrow-band sources with center frequency ω and locations $\theta_1, \theta_2, \ldots, \theta_q$. Since the signals are narrow-band, the propagation delay across the array is much smaller than the reciprocal of the signal bandwidth, and it follows that, by using a complex envelop representation, the array output can be expressed as

$$\mathbf{x}(t) = \mathbf{A}(\Theta)\mathbf{s}(t) + \mathbf{n}(t), \quad 1 \leq t \leq M, \qquad (5)$$

where $\mathbf{s}(t)$ is the vector of signals emitted by the sources as received at the reference sensor of the array, $\mathbf{n}(t)$ is the noise vector and $\mathbf{A}(\Theta)$ is the $r \times q$ matrix of the array steering vectors

$$\mathbf{A}(\Theta) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_q)]. \qquad (6)$$

We assume that the q signal waveforms are noncoherent, complex isotropic $S\alpha S$ random processes with diagonal covariation matrix $\mathbf{\Gamma}_S = \mathrm{diag}(\gamma_{s_1}, \ldots, \gamma_{s_q})$. Also, the noise vector $\mathbf{n}(t)$ is a complex $S\alpha S$ random process with the same characteristic exponent α as the signals. The noise is independent of the signals and has covariation matrix $\mathbf{\Gamma}_N = \gamma_n\mathbf{I}$.

We can write (5) as follows:

$$\mathbf{x}(t) = \mathbf{w}(t) + \mathbf{n}(t), \qquad (7)$$

where $\mathbf{w}(t) = \mathbf{A}(\Theta)\mathbf{s}(t)$. By the stability property, it follows that $\mathbf{w}(t)$ is also a complex $S\alpha S$ random vector, independent of $\mathbf{n}(t)$, with components:

$$w_i(t) = \mathbf{A}_i(\Theta)\mathbf{s}(t) = \sum_{k=1}^{q} a_i(\theta_k)s_k(t). \qquad (8)$$

Now, we define the *covariation matrix* of the observation vector $\mathbf{x}(t)$ as the matrix whose elements are the covariations $[x_i(t), x_j(t)]_\alpha$ of the components of $\mathbf{x}(t)$. We have that

$$\begin{aligned}[x_i(t), x_j(t)]_\alpha &= [w_i(t) + n_i(t), w_j(t) + n_j(t)]_\alpha \\ &= [w_i(t), w_j(t)]_\alpha + [w_i(t), n_j(t)]_\alpha + \\ &\quad [n_i(t), w_j(t)]_\alpha + [n_i(t), n_j(t)]_\alpha. \qquad (9)\end{aligned}$$

By the independence assumption of $\mathbf{w}(t)$ and $\mathbf{n}(t)$, and by property P3 we have that

$$[w_i(t), n_j(t)]_\alpha = 0, \quad [n_i(t), w_j(t)]_\alpha = 0. \qquad (10)$$

Also, by using (8) and properties P1 and P2 it follows that

$$\begin{aligned}[w_i(t), w_j(t)]_\alpha &= [\sum_{k=1}^{q} a_i(\theta_k)s_k(t), w_j(t)]_\alpha \\ &= \sum_{k=1}^{q} a_i(\theta_k)[s_k(t), w_j(t)]_\alpha \\ &= \sum_{k=1}^{q} a_i(\theta_k)[s_k(t), \sum_{l=1}^{q} a_j(\theta_l)s_l(t)]_\alpha \\ &= \sum_{k=1}^{q} a_i(\theta_k)a_j^{<\alpha-1>}(\theta_k)\gamma_{s_k}, \qquad (11)\end{aligned}$$

where $\gamma_{s_k} = [s_k, s_k]_\alpha$. Finally, due to the noise assumption made earlier, it holds that

$$[n_i(t), n_j(t)]_\alpha = \gamma_n\delta_{i,j}, \qquad (12)$$

where $\delta_{i,j}$ is the Kronecker delta function. Organizing (9)-(12) in matrix form we get the following expression for the covariation matrix of the observation vector:

$$[\mathbf{x}(t), \mathbf{x}(t)]_\alpha = \mathbf{A}(\Theta)\mathbf{\Gamma}_S\mathbf{A}^{<\alpha-1>}(\Theta) + \gamma_n\mathbf{I}, \qquad (13)$$

where the (i,j)th element of matrix $\mathbf{A}^{<\alpha-1>}(\Theta)$ results from the (j,i)th element of $\mathbf{A}(\Theta)$ according to the operation

$$[\mathbf{A}^{<\alpha-1>}(\Theta)]_{i,j} = [\mathbf{A}(\Theta)]_{j,i}^{<\alpha-1>} = |[\mathbf{A}(\Theta)]_{j,i}|^{\alpha-2}[\mathbf{A}(\Theta)]_{j,i}^{*}.$$

Clearly, when $\alpha = 2$, i.e., for Gaussian distributed signals and noise, the expression for the covariation matrix is reduced to the well-known form of the covariance matrix.

When the amplitude response of the sensors equals unity, i.e., for steering vectors of the form

$$\mathbf{a}(\theta_k) = [1, e^{-j\omega\tau_2(\theta_k)}, \ldots, e^{-j\omega\tau_r(\theta_k)}]^T,$$

it follows that

$$[\mathbf{x}(t), \mathbf{x}(t)]_\alpha = \mathbf{A}(\Theta)\mathbf{\Gamma}_S\mathbf{A}^H(\Theta) + \gamma_n\mathbf{I}, \qquad (14)$$

Hence, in this case, subspace techniques such as the MUSIC algorithm can be applied to the covariation matrix of the observation vector to extract the bearing information. We will refer to the new algorithm resulting from the eigendecomposition of the array covariation coefficient matrix as the **Robust Covariation-Based MUSIC** or **ROC-MUSIC**.

In practice, we have to estimate the covariation matrix from a finite number of array sensor measurements. A proposed estimator for the covariation coefficient $\lambda_{X,Y}$ is called the *fractional lower order (FLOM) estimator* and is given by [9]

$$\hat{\lambda}_{X,Y} = \frac{\sum_{i=1}^{n} X_i Y_i^{<p-1>}}{\sum_{i=1}^{n} |Y_i|^p} \qquad (15)$$

for some $1 \le p < \alpha$ and independent observations $(X_1, Y_1), \ldots, (X_n, Y_n)$. Unfortunately, this estimator, although it is unbiased, it has large variance. To circumvent this difficulty we introduce the *modified covariation coefficient function*

$$\lambda_{X,Y}(p) = \frac{E\{XY^{<p-1>}\}}{E\{|Y|^p\}}, \qquad (16)$$

for $1/2 < p < \alpha$ if X and Y are real $S\alpha S$ random variables, and $0 < p < \alpha$ if X and Y are complex isotropic $S\alpha S$ random variables. This modified covariation coefficient function is well defined (finite) for the aforementioned values of the parameter p, as shown in [10]. The simulation experiments in the following section give some significant insight on the performances of the MFLOM estimator and the proposed ROC-MUSIC algorithm.

4. SIMULATION RESULTS

We performed two simulation experiments to assess the performance of the MFLOM estimator and to compare the MUSIC and ROC-MUSIC algorithms.

4.1. Experiment #1

The purpose of this experiment is to study the influence of the parameter p to the performance of the MFLOM estimator of the covariation coefficient. Two real $S\alpha S$ $(1 < \alpha \le 2)$ random variables, X and Y, are defined as

$$X = a_{11}U_1 + a_{12}U_2,$$

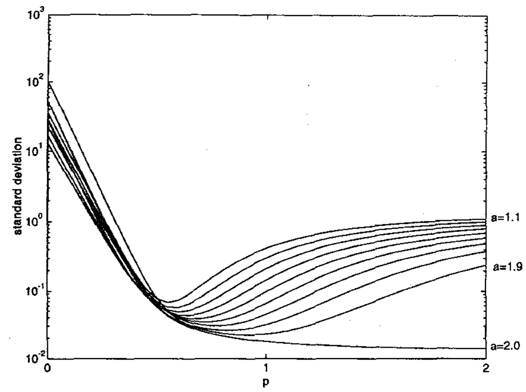

Figure 1: Standard deviation of the MFLOM estimates of the modified covariation coefficient as a function of the parameter p.

$$Y = a_{21}U_1 + a_{22}U_2,$$

where U_1, and U_2 are independent, $S\alpha S$ random variables. The model coefficients $\{a_{ij};\ i,j = 1,2\}$ are given by

$$[a_{ij}] = \begin{bmatrix} -0.75 & 0.25 \\ 0.18 & 0.78 \end{bmatrix}. \qquad (17)$$

It follows that the true covariation coefficient λ of X with Y is

$$\lambda = \frac{a_{11}a_{21}^{<\alpha-1>} + a_{12}a_{22}^{<\alpha-1>}}{|a_{21}|^\alpha + |a_{22}|^\alpha}. \qquad (18)$$

We generated $n = 5,000$ independent samples of U_1, U_2 and U_3 and we calculated the MFLOM estimator by means of the expression

$$\hat{\lambda}_{MFLOM}(p) = \frac{\sum_{i=1}^{n} X_i Y_i^{<p-1>}}{\sum_{i=1}^{n} |Y_i|^p} \qquad (19)$$

for different values of p in the range [0,2]. We run $K = 1,000$ Monte Carlo experiments. Figure 1 shows the standard deviation of the MFLOM estimator of the modified covariation coefficient as a function of the parameter p, and for different values of the characteristic exponent α. As we can see, for the case of non-Gaussian stable signals $(1 < \alpha < 2)$, values of p in the range $(1/2, \alpha/2)$ result into the smallest standard deviations. For Gaussian signals the optimal value of p is 2 and the resulting MFLOM estimator is simply the least-squares estimator, as expected.

4.2. Experiment #2

In this experiment, we compare the performance of the MUSIC and the proposed ROC-MUSIC algorithms in the presence of simulated $S\alpha S$ noise. The sample covariation coefficient matrix (SCCM), as estimated by (19), is not symmetric and hence it has complex eigenvalues in general. The more snapshots are available at the array sensors, the more

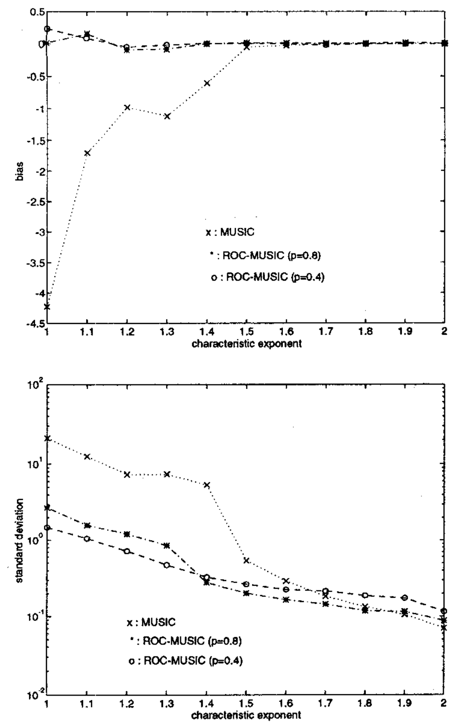

Figure 2: Bias and standard deviation of the estimators of DOA as a function of the characteristic exponent α.

nearly symmetric SCCM becomes. We come around this problem by performing the eigenvalue decomposition to the sum of the sample covariation coefficient matrix and its conjugate.

The array is linear with five sensors spaced a half wavelength apart. A single source is positioned at 20°. The noise is assumed to follow the complex isotropic $S\alpha S$ distribution with dispersion $\gamma = 1$. The so-called generalized signal-to-noise ratio $GSNR = 10\log(\frac{1}{\gamma M}\sum_{t=1}^{M}|s(t)|^2)$ is equal to 0dB. The number of snapshots available to the algorithms is $M = 50$. In every experiment we perform 100 Monte-Carlo runs and compute the bias and the standard deviation of the DOA estimates. We use two values of the parameter p in the estimation of the covariation (c.f. (19)): $p = 0.8$ and $p = 0.4$.

The results are depicted in Figure 2. All the curves are functions of the characteristic exponent α of the noise. Clearly, for very impulsive environments ($\alpha < 1.6$) MUSIC exhibits very large bias and standard deviation. On the other hand, for less impulsive noise ($\alpha > 1.8$), MUSIC and ROC-MUSIC have comparable performances. Comparing the ROC-MUSIC curves in Figure 2, we see that for $\alpha < 1.4$ the choice of $p = 0.4$ gives smaller standard deviation while for $\alpha > 1.4$ the choice of $p = 0.8$ gives better results. This is consistent with the results of the previous experiment and shows that values of p in the range $(1/2, \alpha/2)$ give estimators with smaller variances.

5. CONCLUDING REMARKS

We have formulated the covariation matrix of the array outputs for the case of $S\alpha S$ signals and noise. We showed that for the special case of array sensors with unit amplitude response, the covariation matrix has similar form to the covariance matrix of Gaussian distributed signals. Therefore, subspace-based bearing estimation techniques can be applied to the covariation matrix resulting to improved bearing estimates in the presence of impulsive noise environments.

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