CRAMÉR-RAO BOUNDS FOR TARGET ANGLE AND DOPPLER ESTIMATION FOR AIRBORNE RADAR IN CAUCHY INTERFERENCE

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ABSTRACT

We describe new methods on the modeling of the amplitude statistics of airborne radar clutter by means of alpha-stable distributions. We develop target angle and Doppler, maximum likelihood-based estimation techniques from radar measurements retrieved in the presence of impulsive noise modeled as a multivariate isotropic alpha-stable random process. We derive the Cramér-Rao bounds for the additive Cauchy interference scenario to assess the best-case estimation accuracy which can be achieved. The results are of great importance in the study of space-time adaptive processing (STAP) for airborne pulse Doppler radar arrays operating in impulsive interference environments.

1. INTRODUCTION

Future advanced airborne radar systems must be able to detect, identify, and estimate the parameters of a target in severe interference backgrounds. As a result, the problem of clutter and jamming suppression has been the focus of considerable research in the radar engineering community. It is recognized that effective clutter suppression can be achieved only on the basis of appropriate statistical modeling. Recently, experimental results have been reported where clutter returns are impulsive in nature. In addition, a statistical model of impulsive interference has been proposed, which is based on the theory of symmetric alpha-stable ($S\alpha S$) random processes [1]. The model is of a statistical-physical nature and has been shown to arise under very general assumptions and to describe a broad class of impulsive interference.

Until recently much of the work reported for radar systems has concentrated mostly on target detection [2]. In this paper, we address the target parameter estimation problem through the use of radar array sensor data retrieved in the presence of impulsive interference. In particular, we derive Cramér-Rao bounds on angle and Doppler estimator accuracy for the case of additive sub-Gaussian noise. Initially, we consider the case of additive multivariate Cauchy noise, assuming knowledge of the *underlying matrix* of the distribution. The results obtained here can be viewed as generalizations of the work done in [3] to the 2-D frequency estimation problem in impulsive interference backgrounds. In Section 2, we present some necessary preliminaries on α-stable processes. In Section 3, we define the space-time adaptive processing (STAP) problem for airborne radar and we form the maximum likelihood function. In Section 4, we present the Cramér-Rao analysis and derive bounds on the variances of the spatial and temporal frequency estimates. Finally, in Section 5, we give some examples on the joint target angle and Doppler estimation performance.

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2. SYMMETRIC ALPHA-STABLE DISTRIBUTIONS

In this section, we introduce the statistical model that will be used to describe the additive noise. The model is based on the class of *Complex Isotropic $S\alpha S$* distributions which are well suited for describing signals that are impulsive in nature.

The symmetric α-stable ($S\alpha S$) distribution is best defined by its characteristic function

$$\varphi(\omega) = \exp(j\delta\omega - \gamma|\omega|^{\alpha}) \tag{1}$$

where α is the *characteristic exponent* restricted to the values $0 < \alpha \leq 2, \delta(-\infty < \delta < \infty)$ is the *location parameter*, and γ is the *dispersion* of the distribution. The dispersion plays a role analogous to the role that the variance plays for second-order processes. The characteristic exponent α is the most important parameter of the $S\alpha S$ distribution and it determines the shape of the distribution: the smaller the characteristic exponent α is, the heavier the tails of the $S\alpha S$ density.

$S\alpha S$ densities obey two important properties which further justify their role in data modeling: the *stability property* and the *generalized central limit theorem.* Unfortunately, no closed form expressions exist for the general $S\alpha S$ probability density functions (pdf) except for the Cauchy and the Gaussian case. However, power series expansions can be derived for the general pdf's [1]. Here, we are interested in the family of complex isotropic $S\alpha S$ random

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variables. A complex $S\alpha S$ random variable $X = X_1 + jX_2$ is *isotropic* if and only if the bivariate distribution (X_1, X_2) has uniform spectral measure. In this case, the characteristic function of X can be written as

$$\varphi(\omega) = \exp(j\Re[\omega X^*]) = \exp(-\gamma|\omega|^\alpha). \tag{2}$$

An important difference between the Gaussian and the other distributions of the $S\alpha S$ family is that only moments of order less than α exist for the non-Gaussian family members. If X follows the isotropic stable distribution with dispersion γ, the so called *fractional lower order moments* (FLOM) are given by

$$E|X|^p = C_2(p,\alpha)\gamma^{\frac{p}{\alpha}} \quad \text{for } 0 < p < \alpha, \tag{3}$$

where

$$C_2(p,\alpha) = \frac{2^{p+1}\Gamma(\frac{p+2}{2})\Gamma(-\frac{p}{\alpha})}{\alpha\Gamma(-\frac{p}{2})}. \tag{4}$$

3. STAP PROBLEM FORMULATION AND MAXIMUM LIKELIHOOD FUNCTION

Space-time adaptive processing (STAP) refers to multidimensional adaptive algorithms that simultaneously combine the signals from the elements of an array antenna and the multiple pulses of a coherent radar waveform, to suppress interference and provide target detection [4, 2, 5].

Consider a uniformly spaced linear array radar antenna consisting of N elements, which transmits a coherent burst of M pulses at a constant pulse repetition frequency (PRF) f_r and over a certain range of directions of interest. The pulses repetition interval is T_r. A space-time snapshot refers to the $MN \times 1$ vector of samples corresponding to a single range gate. Given a single snapshot containing target at angle ϕ and Doppler frequency f, the space-time snapshot can be written as [4]

$$\mathbf{x} = \beta\mathbf{v}(\phi, f) + \mathbf{n} \tag{5}$$

where β is the target's complex amplitude given by

$$\beta = x + jy. \tag{6}$$

The vector $\mathbf{v}$ is an $NM \times 1$ vector called the *space-time steering vector*. It may be expressed as

$$\mathbf{v}(\phi, f) = \mathbf{b}(f) \otimes \mathbf{a}(\phi) \tag{7}$$

where $\mathbf{a}(\phi)$ is the $N \times 1$ *spatial steering vector* containing the interelement phase shifts for a target at ϕ, and b(f) is the $M \times 1$ *temporal steering vector* that contains the interpulse phase shifts for a target with Doppler f. It is assumed that the functional form of $\mathbf{v}(\phi, f)$ is known. In addition, we can write

$$v_i(\phi, f) = b_{f(i)}(f) \cdot a_{g(i)}(\phi) \tag{8}$$

where $v_i(\phi, f)$ is the i-th element of the space-time steering vector $\mathbf{v}(\phi, f)$, $1 \le f(i) \le M$, and $1 \le g(i) \le N$.

The snapshot also contains a noise component $\mathbf{n}$. Here, the noise includes clutter, jamming, thermal noise, and any other undesired signals. As a first approximation to the problem, we assume that the noise present at the array is statistically independent both along the array sensors and along time, and is modeled as a complex isotropic Cauchy process with marginal pdf given by

$$\chi_\gamma(r) = \frac{\gamma}{2\pi(r^2 + \gamma^2)^{3/2}}, \tag{9}$$

Under the independence assumption it follows from (5) and (7) that the joint density function for the case of a single snapshot is given by [3]

$$f(\mathbf{n}) = \prod_{i=1}^{MN} f(n_i) = \frac{\gamma^{MN}}{(2\pi)^{MN}\prod_{i=1}^{MN}(\gamma^2 + |x_i - \beta v_i|^2)^{3/2}}. \tag{10}$$

In the following, it will be convenient to work with the normalized spatial and temporal frequency variables:

$$\psi = \frac{2\pi d}{\lambda_0}\sin\phi \quad , \quad \omega = 2\pi f T_r. \tag{11}$$

The estimation problem involves four real valued parameters. We arrange them to form a 4×1 parameter vector

$$\mathbf{\Theta} = [\theta_1 \;\; \theta_2 \;\; \theta_3 \;\; \theta_4] = [\psi \;\; \omega \;\; x \;\; y]. \tag{12}$$

Then, given a single snapshot $\mathbf{x}$, the likelihood function $L(\mathbf{\Theta})$, ignoring the constant terms, is given by

$$L(\mathbf{\Theta}) = -\frac{3}{2}\sum_{i=1}^{NM}\log\left(\gamma^2 + |x_i - \beta v_i(\psi,\omega)|^2\right). \tag{13}$$

4. CRAMÉR-RAO BOUND ANALYSIS

The Cramér-Rao bound for the error variance of an unbiased estimator $\hat{\mathbf{\Theta}}$ satisfies

$$\mathbf{C}_{\hat{\mathbf{\Theta}}} - \mathbf{J}(\mathbf{\Theta}) \ge 0 \tag{14}$$

where $\mathbf{C}_{\hat{\mathbf{\Theta}}}$ is the covariance matrix of $\hat{\mathbf{\Theta}}$ and ≥ 0 is interpreted as meaning that the matrix is semidefinite positive. The matrix $\mathbf{J}(\mathbf{\Theta})$ is the Fisher information matrix given by

$$\mathbf{J}(\mathbf{\Theta}) = E\{[\partial L(\mathbf{\Theta})/\partial\mathbf{\Theta}][\partial L(\mathbf{\Theta})/\partial\mathbf{\Theta}]^T\}. \tag{15}$$

First, we calculate the derivatives of the log-likelihood function given in (13) with respect to the components of $\mathbf{\Theta}$. We have that

$$\frac{\partial L}{\partial \psi} = 3\sum_{i=1}^{MN}\frac{\Re\{\beta^* b^*_{f(i)}(d^a_{g(i)})^* n_i\}}{\gamma^2 + |n_i|^2} \tag{16}$$

where $d_i^a = \partial a_i/\partial\psi$, $i = 1, \cdots, N$. In addition

$$\frac{\partial L}{\partial \omega} = 3\sum_{i=1}^{MN}\frac{\Re\{\beta^* a^*_{g(i)}(d^b_{f(i)})^* n_i\}}{\gamma^2 + |n_i|^2} \tag{17}$$

where $d_i^b = \partial b_i/\partial\omega$, $i = 1, \cdots, M$. Additionally,

$$\frac{\partial L}{\partial x} = 3\sum_{i=1}^{MN}\frac{\Re\{a^*_{g(i)} b^*_{f(i)} n_i\}}{\gamma^2 + |n_i|^2} \tag{18}$$

and

$$\frac{\partial L}{\partial y} = -3 \sum_{i=1}^{MN} \frac{\Im\{a_{g(i)}^* b_{f(i)}^* n_i\}}{\gamma^2 + |n_i|^2}. \tag{19}$$

By performing the second derivatives and expectations in a similar way, the Fisher information matrix $\mathbf{J}(\boldsymbol{\Theta})$ is derived to be

$$\mathbf{J}(\boldsymbol{\Theta}) = \frac{3}{5\gamma^2} \cdot \begin{bmatrix} M|\beta|^2 \parallel \mathbf{d}_a \parallel^2 & |\beta|^2 \rho & yM\delta_a & xM\delta_a \\ |\beta|^2 \rho & N|\beta|^2 \parallel \mathbf{d}_b \parallel^2 & yN\delta_b & xN\delta_b \\ yM\delta_a & yN\delta_b & MN & 0 \\ xM\delta_a & xN\delta_b & 0 & MN \end{bmatrix},$$

where

$$\delta_a = \sum_{i=1}^{N} |d_i^a|, \quad \delta_b = \sum_{i=1}^{M} |d_i^b|, \quad \rho = \sum_{i=1}^{MN} |d_{g(i)}^a| |d_{f(i)}^b|,$$

and $\mathbf{d}_a = [d_1^a \ \cdots \ d_n^a]$, $\mathbf{d}_b = [d_1^b \ \cdots \ d_n^b]$. Since target angle and Doppler are the two parameters of primary interest, we shall focus on the upper left 2×2 block of the Fisher information matrix $\mathbf{J}_{2\times 2}$. The inverse of matrix $\mathbf{J}_{2\times 2}$ is obtained by applying the partitioned matrix inversion lemma. The result is

$$\mathbf{J}_{2\times 2}^{-1}(\boldsymbol{\Theta}) = \frac{1}{\xi} \cdot \frac{5\gamma^2}{3|\beta|^2} \cdot \begin{bmatrix} N(\parallel \mathbf{d}_b \parallel^2 - \frac{1}{M}\delta_b^2) & \delta_a\delta_b - \rho \\ \delta_a\delta_b - \rho & M(\parallel \mathbf{d}_a \parallel^2 - \frac{1}{N}\delta_a^2) \end{bmatrix}, \tag{20}$$

where $\xi = (M \parallel \mathbf{d}_a \parallel^2 - \frac{M}{N}\delta_a^2)(N \parallel \mathbf{d}_b \parallel^2 - \frac{N}{M}\delta_b^2) - (\delta_a\delta_b - \rho)^2$. The Cramér-Rao bounds of the resulting spatial and temporal frequency estimates are obtained from (20) as

$$CRB(\psi) = \frac{\gamma^2}{|\beta|^2} \cdot \frac{5N(\parallel \mathbf{d}_b \parallel^2 - \delta_b^2/M)}{3\xi} \tag{21}$$

and

$$CRB(\omega) = \frac{\gamma^2}{|\beta|^2} \cdot \frac{5M(\parallel \mathbf{d}_a \parallel^2 - \delta_a^2/N)}{3\xi}. \tag{22}$$

Finally, by using (11), we get

$$CRB(\phi) = CRB(\psi) \cdot \frac{\lambda_0^2}{(2\pi d)^2} \frac{1}{\cos^2(\phi)} \tag{23}$$

and

$$CRB(f) = CRB(\omega) \cdot \frac{1}{(2\pi T_r)^2} \tag{24}$$

A useful insight on the CRB can be gained if we consider the case of linear array whose sensors are spaced a half-wavelength apart, and a waveform with an uniform pulse repetition interval. The spatial and temporal steering vectors for such system are:

$$\mathbf{a}(\psi) = \begin{bmatrix} 1 \\ e^{-j\psi} \\ \vdots \\ e^{-j(N-1)\psi} \end{bmatrix}, \quad \mathbf{b}(\omega) = \begin{bmatrix} 1 \\ e^{-j\omega} \\ \vdots \\ e^{-j(M-1)\omega} \end{bmatrix}. \tag{25}$$

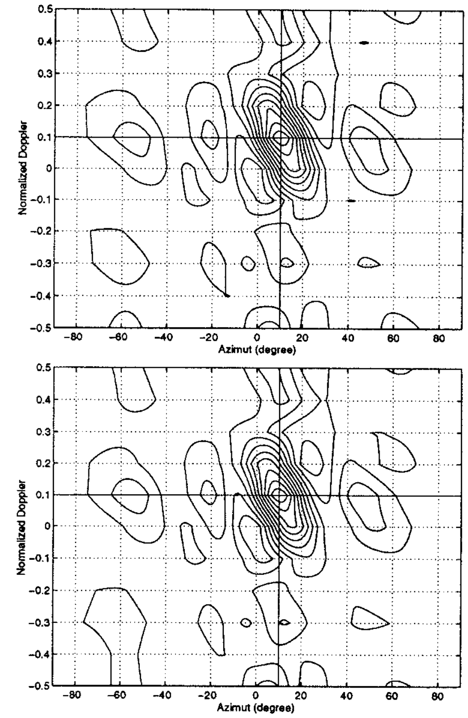

Figure 1: MLG (top) and MLC (bottom) angle-Doppler spectra ($N = 5$, $M = 4$, $\phi = -10°, fT_r = 0.1$). Additive Gaussian noise ($\alpha = 2$, $\gamma = 20, GSNR = 4$ dB).

In this case, it follows from (21) and (22) that

$$CRB(\psi) = \frac{\gamma^2}{|\beta|^2} \cdot \frac{20}{M^2 N^2 (N^2 - 1)} \tag{26}$$

and

$$CRB(\omega) = \frac{\gamma^2}{|\beta|^2} \cdot \frac{20}{M^2 N^2 (M^2 - 1)}. \tag{27}$$

5. SIMULATION RESULTS

In this simulation experiment, we test the robustness of the maximum likelihood estimator based on the Cauchy assumption (MLC). We assume a linear array with $N = 5$ elements that transmits a coherent burst of $M = 4$ pulses. We considered a single target located at $\phi = 10°$ and having Doppler such that $fT_r = \omega/2\pi = 0.1$. Since the alpha-stable family determines processes with infinite variance for $\alpha < 2$, we define an alternative signal-to-noise ratio (SNR). Namely, we define the *Generalized*-SNR (GSNR) to be the ratio of the signal power over the noise dispersion γ:

$$GSNR = 10 \log \left(\frac{|\beta|^2}{\gamma} \right) \tag{28}$$

In Figures 1 and 2 we plot isosurfaces of space-time spectral estimates (likelihood functions) for the maximum likelihood estimator based on the Gaussian assumption (MLG)

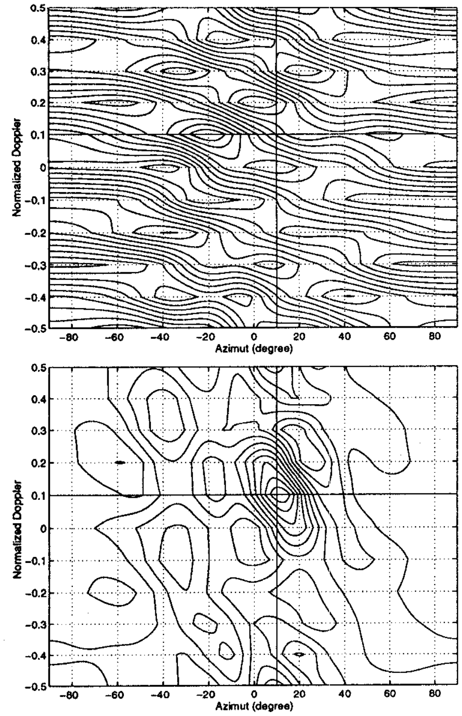

Figure 2: MLG (top) and MLC (bottom) angle-Doppler spectra ($N = 5$, $M = 4$, $\phi = -10°, fT_r = 0.1$). Additive stable noise ($\alpha = 1.5$, $\gamma = 20, GSNR = 4$ dB).

[2] and for the maximum likelihood estimator based on the Cauchy assumption (MLC).The likelihood functions are formed by using 50 space-time snapshots. In Figure 1, since the additive noise to the sensors is Gaussian ($\alpha = 2$), the MLG likelihood function is based on the correct assumption about the noise distribution. On the other hand, in Figure 2, the additive noise to the sensors is α-stable with $\alpha = 1.5$ and neither the MLG nor the MLC likelihood functions rely on the correct assumption about the noise distribution. As we can see from the figures, the MLC likelihood function, based on the Cauchy assumption, attains its maximum value very close to the true angle and Doppler values in both cases of additive stable noise. On the other hand, the MLG likelihood function, based on the Gaussian assumption, cannot localize the target accurately when the actual data distribution deviates from the Gaussian case.

The observed robustness of the MLC method is quantified in Figure 3 which shows the resulting mean-square error curve on the estimated Doppler as function of the characteristic exponent α of the additive noise. The results are based on 300 Monte Carlo runs. As we can clearly see, the Cauchy beamformer is practically insensitive to the changes of α. On the other hand, the MLG algorithm exhibits very large mean-square estimation error for non-Gaussian noise environments.

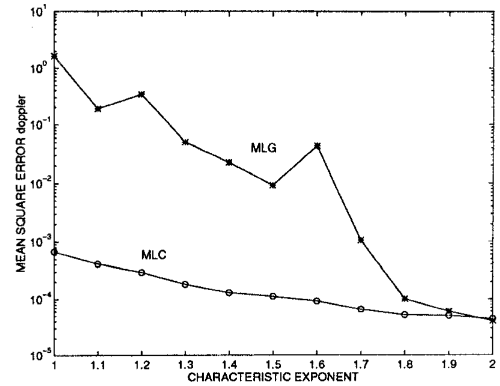

Figure 3: MSE of the estimated Doppler as a function of the characteristic exponent α.

6. CONCLUSIONS

We considered the problem of target angle and Doppler estimation with an airborne radar employing space-time adaptive processing. We derived Cramér-Rao bounds on angle and Doppler estimator accuracy for the case of additive multivariate Cauchy interference of known diagonal underlying matrix. The bounds are functions of a generalized SNR function, similarly to the Gaussian case where the bounds are functions of the SNR. As shown in (21) and (22), target angle accuracy is a function of Doppler frequency and vice-versa. In addition, we introduced a new joint spatial- and Doppler- frequency estimation technique based on the maximum likelihood Cauchy function (MLC) and we showed that the Cauchy estimator gives better results in a wide range of impulsive noise (clutter, jamming, thermal) environments.

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