OSCILLATIONS, SE(2)–SNAKES AND MOTION CONTROL: A STUDY OF THE ROLLER RACER

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Abstract

This paper is concerned with the problem of motion generation via cyclic variations in selected degrees of freedom (usually referred to as shape variables) in mechanical systems subject to nonholonomic constraints (here the classical one of a disk rolling without sliding on a flat surface). In earlier work, we identified an interesting class of such problems arising in the setting of Lie groups, and investigated these under a hypothesis on constraints, that naturally led to a purely kinematic approach. In the present work, the hypothesis on constraints does not hold, and as a consequence, it is necessary to take into account certain dynamical phenomena. Specifically we concern ourselves with the group $SE(2)$ of rigid motions in the plane and a concrete mechanical realization dubbed the 2–node, 1–module $SE(2)$–snake. In a restricted version, it is also known as the Roller Racer (a patented ride/toy).

Based on the work of Bloch, Krishnaprasad, Marsden and Murray, one recognizes in the example of this paper a balance law called the momentum equation, which is a direct consequence of the interaction of the $SE(2)$–symmetry of the problem with the constraints. The systematic use of this type of balance law results in certain structures in the example of this paper. We exploit these structures to demonstrate that the single shape freedom in this problem can be cyclically varied to produce a rich variety of motions of the $SE(2)$–snake.

In their study of the snakeboard, a patented modification of the skateboard that also admits the group $SE(2)$ as a symmetry group, Lewis, Ostrowski, Burdick and Murray exploited the same type of balance law as the one discussed here to generate motions. A key difference however is that, in the present paper, we have only one control variable and thus controllability considerations become somewhat more delicate.

In the present paper, we give a self–contained treatment of the geometry, mechanics and motion control of the Roller Racer.

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1 Introduction

The idea of using periodic driving signals to produce rectified movement appears in a number of settings in engineering. Some of the more inventive examples are associated with the design and operation of novel actuators exploiting vibratory transduction (Ueha and Tomikawa [30]; Venkataraman et al. [31]). In his paper [6], Brockett develops a mathematical basis for understanding such devices. Elsewhere, in the context of robotic machines with many degrees of freedom designed to mimic snake–like movements (Hirose [12]), periodic variations in the shape parameters are used in an essential way to generate global movements. In Krishnaprasad and Tsakiris [16], [17], we have developed a general mathematical formulation to study systems of this type. The study of periodic signal generators (also called central pattern generators), as sources of timing signals to compose movements has a long history in the neurophysiology of movement dating back to the early work of Sherrington, Brown and Bernstein.

Recent studies by neurophysiologists (Carling, Bowtell and Williams [7]) have attempted to bring together principles of motion control based on pattern generation in the spinal cord of the *lamprey*, its compliant body dynamics, and the fluid dynamics of its environment to achieve a comprehensive understanding of the swimming behavior of such anguilliform animals. These efforts have in part relied on continuum mechanical models of the body, and computational fluid dynamical (CFD) calculations. There appear to be some unifying themes that underlie this type of neural–mechanical approach to biological locomotion, and the work of the authors and others involving the study of land–based robotic machines subject to the constraint of 'no sliding'. As pointed out in (Krishnaprasad [15]), the connecting links between these two streams of research appear to be related to the manner in which systems of coupled oscillators are used to generate finite dimensional shape variations of the bodies of specialized robot designs, and the associated geometric–mechanical descriptions of the constraints to produce effective motion control strategies (see also the work of Collins and Stewart [9] for another dynamical systems perspective).

In the present paper, we report on a complete study of an interesting example, the (single module) $SE(2)$–snake, with a view towards deeper appreciation of the above–mentioned connections. In section 2, we present the basic geometry of the configuration space, and the applicable constraints. We also discuss a simplification that reduces the shape freedom to one variable, leading to the Roller Racer. The constraints of 'no sliding' are *insufficient* to determine the movement of the Roller Racer from shape variations alone. In sections 3 and 4, a model Lagrangian and the action of $SE(2)$, the rigid motion group in the plane as a symmetry group (of the Lagrangian and the constraints) are discussed. A balance law associated to the $SE(2)$–symmetry, the momentum equation, is derived, which is a consequence of the Lagrange–d'Alembert principle (The basic results behind momentum equations are to be found in Bloch, Krishnaprasad, Marsden and Murray [5]). This momentum equation is the key additional data that, together with the constraints, allows us to generate motion control laws. In section 5, we consider controllability and motion control issues.

G–snakes are kinematic chains with configurations taking values in products of several copies of a Lie group G, and subject to nonholonomic constraints (Krishnaprasad and Tsakiris [16]; Tsakiris [29]). The group G acts on the chain by diagonal action as a symmetry group. The shape space is the quotient by this action. Fig. 1 illustrates an $SE(2)$–snake composed of two modules and three nodes, where the configuration space Q is $SE(2)\times SE(2)\times SE(2)$.

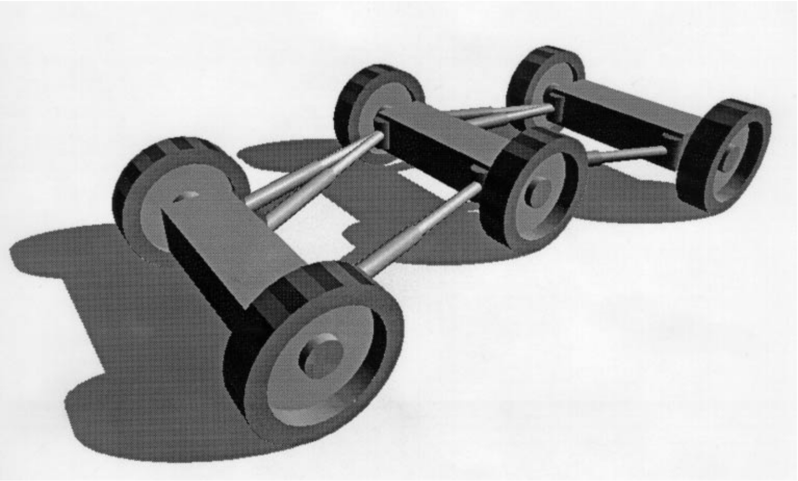


Figure 1: The 2–module $SE(2)$–Snake

The machine in fig. 1 is composed of three axles and linearly actuated linkages connecting each adjacent pair of axles, resulting in an assembly of two identical modules. Altering the lengths of the connecting linkages leads to changes in the shapes of component modules. The wheels mounted on each axle are independent and are *not* actuated but subject to the constraint of 'no sliding'. In this case there are three constraints, the shape space $\mathcal{S}$ is $SE(2)\times SE(2)$, the constraints define a principal connection on the bundle $(Q, SE(2), \mathcal{S})$, away from a set of nonholonomic singularities, and it is possible to generate global movement of the assembly by periodic variations in the module shapes. The entire situation can be understood at a kinematic level as long as the shapes are control variables (Krishnaprasad and Tsakiris [16], [17]; Tsakiris [29]).

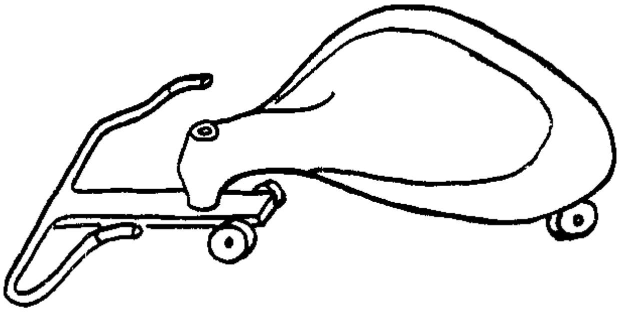


Figure 2: The Roller Racer

When one of the modules is removed from the machine in fig. 1, leaving us with two axles connected by linkages and two nonholonomic constraints, the resulting problem is kinematically under–constrained. It is no longer possible to define a connection without using additional information. It is this type of 1–module $SE(2)$–snake that is of interest here. Matters can be simplified by limiting the extent of shape freedom. In 1972, W.E. Hendricks was awarded U.S. patent no. $3,663,038$ for a toy illustrated in fig. 2 and dubbed the Roller Racer, that serves as one such simplification. The rider, on the seat shown, has to merely oscillate the handle–bars from side–to–side to generate forward propulsion, a behavior for which Hendricks did not claim to have an explanation.

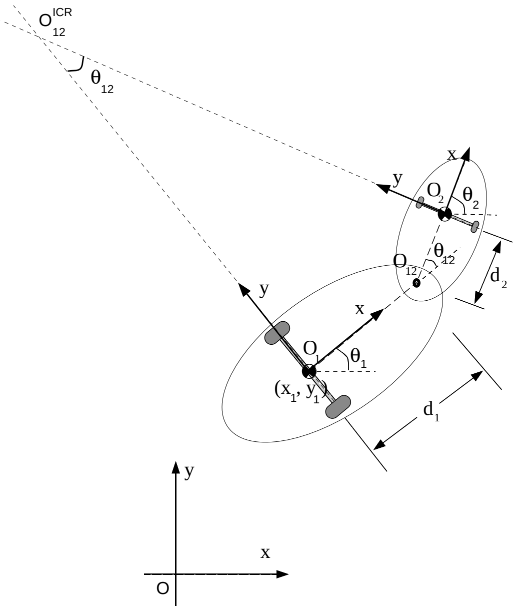


Figure 3: The Roller Racer Model

The model of fig. 3 will be used in our analysis. Two planar platforms with centers of mass (c.o.m.) located at points O_1 and O_2 are connected with a rotary joint at $O_{1,2}$. A pair of idler wheels is attached on each of the platforms, with the axis of the wheels perpendicular to the line connecting the c.o.m. with the joint. A coordinate frame centered at the c.o.m. and with its x–axis along the line $O_iO_{1,2}$ for $i = 1, 2$ connecting the c.o.m. with the joint, will be used to describe the configuration of each platform with respect to a global coordinate system at some reference point O. For simplicity, it will be assumed that the axis of the wheels passes through the c.o.m. of each platform.

The effects of the rider's body motion will be ignored at first approximation. Experiments with the Roller Racer and our analysis below, show that, even though these body motions may amplify the resulting motion of the system, the fundamental means of its propulsion is the pivoting of the steering arm around the joint axis and the nonholonomic constraints coming from the wheels' rolling–without–slipping on the plane supporting the vehicle. In this respect, this system is very different from the Snakeboard, a variation of the skateboard, where the motion of the rider is essential for the propulsion of the system (Lewis et al. [19]).

Riderless prototypes of the Roller Racer built at the Intelligent Servosystems Laboratory verified this. The propulsion and steering mechanism in these vehicles comes from a rotary motor at the joint $O_{1,2}$, whose torque can be considered as the control of our system. As discussed in Krishnaprasad and Tsakiris [17], the purely kinematic analysis of such a system does not allow us to determine the global motion of the system by just the shape variations (the joint velocity in this case), since (unlike the 2-module case) it does not possess a sufficient number of nonholonomic constraints for this to happen. Our goal here is to complement this kinematic analysis with the dynamics of the system, which will provide the necessary information. Thus, certain fundamental behaviors of the system ("straight–line" motion, "turning" motion) can be achieved by proper oscillatory relative motions of the two platforms. In both numerical simulations and experiments with prototypes, we observed such behaviors, as described in section 6.

In order to study the dynamics of this system, an alternative to the usual approach of solving the full Lagrange–d'Alembert equations of motion of the system is considered here. In Bloch, Krishnaprasad, Marsden and Murray [5], the notion of the momentum map is examined for systems with nonholonomic constraints and symmetries and its evolution law, the momentum equation, is derived from the Lagrange–d'Alembert equations. By applying this method to the problem at hand, a useful decomposition of the equations of motion is obtained: Given a shape–space trajectory (which corresponds to the controls of our system), first we compute the nonholonomic momentum from the momentum equation. This only involves the solution of a linear ordinary differential equation. Subsequently, we use the momentum to reconstruct the group trajectory, which corresponds to the global motion of the system. The corresponding velocities depend linearly on the momentum. This process is very useful for the derivation of motion control laws for this system and can be extended to 1–module $SE(2)$–snakes with more general shape–changing mechanisms.

2 Kinematics of the Roller Racer

Consider a left–invariant dynamical system on a matrix Lie group G with an n–dimensional Lie algebra $\mathcal{G}$ and a curve $g(\cdot) \subset G$. Then, there exists a curve $\xi(\cdot) \subset \mathcal{G}$ such that

$$\dot{g} = g\,\xi\ . \tag{1}$$

Let $\{\mathcal{A}_i,\ i = 1, \ldots, n\}$ be a basis of $\mathcal{G}$ and let $[\ ,\ ]$ be the usual Lie bracket on $\mathcal{G}$ defined by $[\mathcal{A}_i, \mathcal{A}_j] = \mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i$. Then, there exist constants $\Gamma^k_{i,j}$, called *structure constants,* such that

$$[\mathcal{A}_i, \mathcal{A}_j] = \sum_{k=1}^{n} \Gamma^k_{i,j}\mathcal{A}_k\ , \quad i, j = 1, \ldots, n. \tag{2}$$

Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$, i.e. the space of linear functions from $\mathcal{G}$ to $I\!R$. Let $\{\mathcal{A}^\flat_i,\ i = 1, \ldots, n\}$ be the basis of $\mathcal{G}^*$ such that

$$\mathcal{A}^\flat_i(\mathcal{A}_j) = \delta^j_i, \quad i, j = 1, \ldots, n, \tag{3}$$

where δ^j_i is the Kronecker symbol. Then the curve $\xi(\cdot) \subset \mathcal{G}$ can be represented as

$$\xi = \sum_{i=1}^{n} \underset{i}{\xi}\mathcal{A}_i = \sum_{i=1}^{n} \mathcal{A}^\flat_i(\xi)\mathcal{A}_i\ , \tag{4}$$

for $\underset{i}{\xi} \overset{\text{def}}{=} \mathcal{A}^\flat_i(\xi) \in I\!R,\ i = 1, \ldots, n.$

Let now the Lie group G be $SE(2)$, the Special Euclidean group of rigid motions on the plane, and $\mathcal{G}$ be $se(2)$, the corresponding Lie algebra with the basis

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\ . \tag{5}$$

Observe that

$$[\mathcal{A}_1\ ,\ \mathcal{A}_2] = \mathcal{A}_3\ ,\ [\mathcal{A}_1\ ,\ \mathcal{A}_3] = -\mathcal{A}_2\ ,\ [\mathcal{A}_2\ ,\ \mathcal{A}_3] = 0\ . \tag{6}$$

From equation 4, an element $\xi \in \mathcal{G}$ is represented as

$$\xi = \sum_{i=1}^{3} \underset{i}{\xi}\mathcal{A}_i = \begin{pmatrix} 0 & -\underset{1}{\xi} & \underset{2}{\xi} \\ \underset{1}{\xi} & 0 & \underset{3}{\xi} \\ 0 & 0 & 0 \end{pmatrix}\ . \tag{7}$$

The homogeneous matrix representation of an element $g \in SE(2)$ with coordinates $(x,\ y,\ \theta)$ is

$$g = \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix}\ . \tag{8}$$

From equations 1, 7 and 8, we get

$$\xi_1 = \dot{\phi}\ ,\ \ \xi_2 = \dot{x}\cos\phi + \dot{y}\sin\phi\ ,\ \ \xi_3 = -\dot{x}\sin\phi + \dot{y}\cos\phi\ . \tag{9}$$

Let $g_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & x_i \\ \sin\theta_i & \cos\theta_i & y_i \\ 0 & 0 & 1 \end{pmatrix} \in SE(2)$, for $i = 1, 2$, be the configuration of platform i with respect to the global coordinate frame at O, where x_i, y_i and θ_i are indicated in fig. 3. Let $g_{1,2} = \begin{pmatrix} \cos\theta_{1,2} & -\sin\theta_{1,2} & x_{1,2} \\ \sin\theta_{1,2} & \cos\theta_{1,2} & y_{1,2} \\ 0 & 0 & 1 \end{pmatrix} \in SE(2)$ be the configuration of platform 2 with respect to the coordinate frame of platform 1 at O_1. Because of the special structure of the joint, we have

$$x_{1,2} = d_1 + d_2 \cos\theta_{1,2} \ , \quad y_{1,2} = d_2 \sin\theta_{1,2} \ , \tag{10}$$

where $\theta_{1,2}$ is the relative angle of the two platforms and d_i is the distance of O_i from the joint $O_{1,2}$, as indicated in fig. 3. We consider non–negative d_1 and d_2. In fact, we assume $d_1 > 0$. However, we allow for the case $d_2 = 0$ and we examine it in detail.

Since the platforms form a kinematic chain, we have

$$g_2 = g_1 g_{1,2} \ , \tag{11}$$

thus

$$\begin{aligned} \theta_2 &= \theta_1 + \theta_{1,2} \ , \\ x_2 &= x_1 + x_{1,2}\cos\theta_1 - y_{1,2}\sin\theta_1 = x_1 + d_1\cos\theta_1 + d_2\cos\theta_2 \ , \\ y_2 &= y_1 + x_{1,2}\sin\theta_1 + y_{1,2}\cos\theta_1 = y_1 + d_1\sin\theta_1 + d_2\sin\theta_2 \ . \end{aligned} \tag{12}$$

The system kinematics are a special case of the n–module $SE(2)$–snake (n–VGT) assembly ([17], [29]), i.e. for $\xi_i = \begin{pmatrix} 0 & -\xi^i_1 & \xi^i_2 \\ \xi^i_1 & 0 & \xi^i_3 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{G} = se(2)$ we have:

$$\dot{g}_i = g_i \xi_i \ , \ i = 1, 2 \tag{13}$$

and

$$\dot{g}_{1,2} = g_{1,2} \xi_{1,2} \ , \tag{14}$$

where $\xi_{1,2} = \begin{pmatrix} 0 & -\xi^{1,2}_1 & \xi^{1,2}_2 \\ \xi^{1,2}_1 & 0 & \xi^{1,2}_3 \\ 0 & 0 & 0 \end{pmatrix} = \dot{\theta}_{1,2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & d_2 \\ 0 & 0 & 0 \end{pmatrix}$. By differentiating 12, we get

$$\begin{aligned} \dot{\theta}_2 &= \dot{\theta}_1 + \dot{\theta}_{1,2} \ , \\ \dot{x}_2 &= \dot{x}_1 - \dot{\theta}_1[d_1\sin\theta_1 + d_2\sin(\theta_1 + \theta_{1,2})] - \dot{\theta}_{1,2} d_2 \sin(\theta_1 + \theta_{1,2}) \ , \\ \dot{y}_2 &= \dot{y}_1 + \dot{\theta}_1[d_1\cos\theta_1 + d_2\cos(\theta_1 + \theta_{1,2})] + \dot{\theta}_{1,2} d_2 \cos(\theta_1 + \theta_{1,2}) \ . \end{aligned} \tag{15}$$

From 12 we see that the configuration space for the Roller Racer system is $Q = SE(2) \times S^1$. Its shape space is $\mathcal{S} = S^1$. Then, $Q = G \times \mathcal{S}$.

Consider the bases $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ for $\mathcal{G} = se(2)$ (given by 5) and $\{\mathcal{A}_1^\flat, \mathcal{A}_2^\flat, \mathcal{A}_3^\flat\}$ for its dual space $\mathcal{G}^*$. The *nonholonomic constraints* on the wheels of the two platforms can be expressed as:

$$\xi^1_3 = \mathcal{A}_3^\flat(\xi_1) = -\dot{x}_1 \sin\theta_1 + \dot{y}_1 \cos\theta_1 = 0 \;, \tag{16}$$

$$\xi^2_3 = \mathcal{A}_3^\flat(\xi_2) = -\dot{x}_2 \sin\theta_2 + \dot{y}_2 \cos\theta_2 = 0 \;, \tag{17}$$

From 15 and 17, we get

$$\xi^2_3 = \mathcal{A}_3^\flat(\xi_2) = -\dot{x}_1 \sin(\theta_1 + \theta_{1,2}) + \dot{y}_1 \cos(\theta_1 + \theta_{1,2}) + \dot{\theta}_1(d_1 \cos\theta_{1,2} + d_2) + \dot{\theta}_{1,2} d_2 = 0 \;. \tag{18}$$

Observe that for $d_2 = 0$, neither one of the constraints 16 and 18 involves $\dot{\theta}_{1,2}$. From 16 and 18, we get:

$$\xi^2_3 = \mathcal{A}_3^\flat(\xi_2) = -(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1)\sin\theta_{1,2} + (d_1 \cos\theta_{1,2} + d_2)\dot{\theta}_1 + d_2\dot{\theta}_{1,2} = 0 \;. \tag{19}$$

It can be easily seen that the nonholonomic constraints 16 and 18 are linearly independent for all $q \in Q$. The *constraint one–forms* can be defined as

$$\begin{aligned} \omega_q^1 &= -\sin\theta_1 dx_1 + \cos\theta_1 dy_1 \;, \\ \omega_q^2 &= -\sin(\theta_1 + \theta_{1,2})dx_1 + \cos(\theta_1 + \theta_{1,2})dy_1 + (d_1 \cos\theta_{1,2} + d_2)d\theta_1 + d_2 d\theta_{1,2} \;. \end{aligned} \tag{20}$$

The *constraint distribution* $\mathcal{D}_q$ is the subspace of T_qQ which is the intersection of the kernels of the constraint one–forms, i.e.

$$\mathcal{D}_q = \mathrm{Ker}\ \omega_q^1 \cap \mathrm{Ker}\ \omega_q^2 \;. \tag{21}$$

Since the constraints are linearly independent, we know that $\mathcal{D}_q$ is always 2–dimensional. A basis for $\mathcal{D}_q$ is given by

$$\mathcal{D}_q = \mathrm{sp}\{\xi^1_Q, \xi^2_Q\} \;, \tag{22}$$

where in the case $d_2 \neq 0$:

$$\begin{aligned} \xi_Q^1 &= d_2(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_{1,2}} \;, \\ \xi_Q^2 &= d_2 \frac{\partial}{\partial\theta_1} - (d_1 \cos\theta_{1,2} + d_2)\frac{\partial}{\partial\theta_{1,2}} \;, \end{aligned} \tag{23}$$

while in the case $d_2 = 0$:

$$\begin{aligned} \xi_Q^1 &= d_1 \cos\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1} \;, \\ \xi_Q^2 &= \frac{\partial}{\partial\theta_{1,2}} \;. \end{aligned} \tag{24}$$

(Reader: Do not confuse ξ^i_Q here with infinitesimal generators of group actions).

In order to model friction in the bearings of the Roller Racer wheels, the relationship between the wheel angular velocities and the configuration velocities is needed. For each platform i, $i = 1, 2$, let $\phi_{l,i}$ and $\phi_{r,i}$ be the angle of its left and right wheel with respect to some reference position of the wheel. The relationship of the angular velocities $\dot{\phi}_{l,i}, \dot{\phi}_{r,i}$, $i = 1, 2$, of the left and right wheel of platform i with the configuration velocities $\dot{q} = (\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2})^\top$ of the system, is as follows:

$$
\begin{aligned}
\dot{\phi}_{l,1} &= \frac{1}{R_1}[-\frac{L_1}{2}\dot{\theta}_1 + \dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1] \,, \\
\dot{\phi}_{r,1} &= \frac{1}{R_1}[\frac{L_1}{2}\dot{\theta}_1 + \dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1] \,, \\
\dot{\phi}_{l,2} &= \frac{1}{R_2}[-\frac{L_2}{2}(\dot{\theta}_1 + \dot{\theta}_{1,2}) + (\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1)\cos\theta_{1,2} + \dot{\theta}_1 d_1 \sin\theta_{1,2}] \,, \\
\dot{\phi}_{r,2} &= \frac{1}{R_2}[\frac{L_2}{2}(\dot{\theta}_1 + \dot{\theta}_{1,2}) + (\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1)\cos\theta_{1,2} + \dot{\theta}_1 d_1 \sin\theta_{1,2}] \,,
\end{aligned}
\tag{25}
$$

where R_i and L_i are, respectively, the wheel radius and the length of the wheel axis of platform i.

3 Symmetry of the Roller Racer

Consider now the effect of symmetries on this system. We first present some material on actions of Lie groups, which is based on Marsden and Ratiu [21].

Let Q be a smooth manifold. A (left) *action* of a Lie group G on Q is a smooth mapping $\Phi : G \times Q \longrightarrow Q$, such that, for all $q \in Q$, $\Phi(e, x) = x$ and, for every $g, h \in G$, $\Phi(g, \Phi(h, q)) = \Phi(gh, q)$. For every $g \in G$, define $\Phi_g : Q \longrightarrow Q : q \longrightarrow \Phi(g, q)$. This can be shown to be a diffeomorphism (i.e. one–to–one, onto and both Φ_g and $(\Phi_g)^{-1}$ are smooth).

For $q \in Q$, the *orbit* (or Φ–orbit) of q is $\mathrm{Orb}(q) \overset{\mathrm{def}}{=} \{\Phi_g(q) \mid g \in G\}$. An action Φ is *free* if, for each $q \in Q$, $g \mapsto \Phi_g(q)$ is one–to–one, i.e. the identity e is the only element of G with a fixed point. An action Φ is *proper* if and only if the map $\tilde{\Phi} : G \times Q \longrightarrow Q \times Q : (g, q) \mapsto \tilde{\Phi}(g, q) = (q, \Phi(g, q))$ is proper, i.e. if the set $K \subset Q \times Q$ is compact, then its inverse image $\tilde{\Phi}^{-1}(K)$ is also compact.

Let $\Phi : G \times Q \longrightarrow Q$ be a smooth action and let $\mathcal{G}$ be the Lie algebra of G. If $\xi \in \mathcal{G}$, then $\Phi^{\xi} : \mathbb{R} \times Q \longrightarrow Q : (t, q) \mapsto \Phi(\exp t\xi, q)$ is an $\mathbb{R}$–action on Q, i.e. is a flow on Q. The corresponding vector field on Q is called the *infinitesimal generator* of Φ corresponding to ξ, is denoted by $\xi_Q(q)$ and is given by

$$\xi_Q(q) = \frac{d}{dt}\Phi(\exp t\xi, q)\Big|_{t=0} . \tag{26}$$

The tangent space to the orbit $\mathrm{Orb}(q)$ of q is then

$$T_q\mathrm{Orb}(q) = \{\xi_Q(q) \mid \xi \in \mathcal{G}\} . \tag{27}$$

Consider the action Φ of the group $G = SE(2)$ on the configuration space $Q = SE(2) \times S^1$ of the Roller Racer, defined by

$$\begin{aligned} \Phi : \quad & G \times Q \to Q \\ & (g, (g_1, \theta_{1,2})) \mapsto (gg_1, \theta_{1,2}) \\ & ((x, y, \theta), (x_1, y_1, \theta_1, \theta_{1,2})) \mapsto \\ & \qquad (x_1 \cos\theta - y_1 \sin\theta + x, x_1 \sin\theta + y_1 \cos\theta + y, \theta_1 + \theta, \theta_{1,2}) , \end{aligned} \tag{28}$$

where $g = g(x, y, \theta) \in G$. The tangent space at $q \in Q$ to the orbit of Φ is given by

$$T_q\mathrm{Orb}(q) = \mathrm{sp}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial \theta_1}\} . \tag{29}$$

Notice that the sum of the subspaces $\mathcal{D}_q$ and $T_q\mathrm{Orb}(q)$ gives the entire T_qQ :

$$\mathcal{D}_q + T_q\mathrm{Orb}(q) = T_qQ . \tag{30}$$

In Bloch, Krishnaprasad, Marsden and Murray [5], this is referred to as the *principal* case. Our goal is to show that the nonholonomic constraints, together with a momentum equation, can specify a connection on the principal fiber bundle $Q \longrightarrow Q/G$.

An important observation, that we prove below, is that the intersection $\mathcal{S}_q$ of $\mathcal{D}_q$ and $T_q\mathrm{Orb}(q)$ is *non–trivial.* Contrast this with the $(n-1)$–module G–snake with $\dim G = n$, where $T_qQ = \mathcal{D}_q \oplus T_q\mathrm{Orb}(q)$, thus the intersection of $\mathcal{D}_q$ and $T_q\mathrm{Orb}(q)$ is trivial ([16], [29]); this is referred to as the *purely kinematic* case. We specify a basis for $\mathcal{S}_q$ as follows:

Proposition 3.1 *Consider the intersection*

$$\mathcal{S}_q \overset{\mathrm{def}}{=} \mathcal{D}_q \cap T_q\mathrm{Orb}(q) \ . \tag{31}$$

In the case $d_1 \neq d_2$, the distribution $\mathcal{S}_q$ is 1–dimensional and is given by

$$\mathcal{S}_q = \mathrm{sp}\{\xi^q_Q\} \ , \tag{32}$$

where

$$\xi^q_Q = (d_1\cos\theta_{1,2} + d_2)(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1} \ . \tag{33}$$

Proof

Consider $X_q \in \mathcal{S}_q$. Since $X_q \in \mathcal{D}_q$, we have $X_q = u_1\xi^1_Q + u_2\xi^2_Q$, for some $u_i \in I\!R$. Since $X_q \in T_q\mathrm{Orb}(q)$, we have $X_q = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial\theta_1}$, for some $v_i \in I\!R$. In order for X_q to lie in the intersection of the two spaces, we should have:

$$u_1\xi^1_Q + u_2\xi^2_Q = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial\theta_1} \ . \tag{34}$$

i) In the case $d_2 \neq 0$, we have from 23:

$$\begin{aligned}[d_2(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_{1,2}}]u_1 + [d_2\frac{\partial}{\partial\theta_1} - (d_1\cos\theta_{1,2} + d_2)\frac{\partial}{\partial\theta_{1,2}}]u_2 \\ = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial\theta_1} \ .\end{aligned} \tag{35}$$

This corresponds to a system of four equations:

$$\begin{pmatrix} d_2\cos\theta_1 & 0 & -1 & 0 & 0 \\ d_2\sin\theta_1 & 0 & 0 & -1 & 0 \\ 0 & d_2 & 0 & 0 & -1 \\ \sin\theta_{1,2} & -(d_1\cos\theta_{1,2} + d_2) & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ .$$

When $d_1 \neq d_2$, the 4×5 matrix above is always of maximal rank, thus $\dim\mathcal{S}_q = 5-4 = 1$, for all $q \in Q$. Pick $u_1 = (d_1\cos\theta_{1,2} + d_2)u_5$ and $u_2 = \sin\theta_{1,2}u_5$. Then, $v_1 = d_2\cos\theta_1(d_1\cos\theta_{1,2} + d_2)u_5$, $v_2 = d_2\sin\theta_1(d_1\cos\theta_{1,2} + d_2)u_5$ and $v_3 = d_2\sin\theta_{1,2}u_5$. Thus

$$X_q = [(d_1\cos\theta_{1,2} + d_2)(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1}]d_2u_5 \ ,$$

for arbitrary u_5. Observe that when $d_1 \neq d_2$, the vector field X_q is nontrivial for all $q \in Q$.

ii) In the case $d_2 = 0$, we have from 24:

$$\left[d_1 \cos\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1}\right]u_1 + u_2 \frac{\partial}{\partial\theta_{1,2}} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial y_1} + v_3 \frac{\partial}{\partial\theta_1} \ .$$

From this, we get:

$$u_2 = 0 \ , \quad v_1 = d_1 \cos\theta_{1,2}\cos\theta_1 u_1 \ , \quad v_2 = d_1 \cos\theta_{1,2}\sin\theta_1 u_1 \ , \quad v_3 = \sin\theta_{1,2} u_1 \ . \tag{36}$$

Therefore,

$$X_q = [d_1 \cos\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1}]u_1 \ ,$$

for arbitrary u_1. Thus, $\mathcal{S}_q$ is again a 1–dimensional distribution.
The two cases can be unified in the expression 33.

∎

The infinitesimal generators for the action Φ of $SE(2)$ on Q defined in 28, corresponding to the basis elements of $\mathcal{G} = se(2)$ defined in 5, at the point $q \in Q$, are

$$\mathcal{A}_{1}{}_Q^q = -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} + \frac{\partial}{\partial\theta_1} \ , \quad \mathcal{A}_{2}{}_Q^q = \frac{\partial}{\partial x_1} \ , \quad \mathcal{A}_{3}{}_Q^q = \frac{\partial}{\partial y_1} \ . \tag{37}$$

The infinitesimal generator corresponding to $\xi^q = \xi_1\mathcal{A}_1 + \xi_2\mathcal{A}_2 + \xi_3\mathcal{A}_3 \in \mathcal{G}$ is

$$\xi_Q^q = (\xi_2 - y_1\xi_1)\frac{\partial}{\partial x_1} + (\xi_3 + x_1\xi_1)\frac{\partial}{\partial y_1} + \xi_1 \frac{\partial}{\partial\theta_1} \ . \tag{38}$$

A given vector field $\xi_Q^q = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial y_1} + v_3 \frac{\partial}{\partial\theta_1}$ can be considered as the infinitesimal generator of an element $\xi^q \in \mathcal{G} = se(2)$, under the action Φ. This element ξ^q is

$$\xi^q = v_3\mathcal{A}_1 + (v_1 + y_1 v_3)\mathcal{A}_2 + (v_2 - x_1 v_3)\mathcal{A}_3 \ . \tag{39}$$

The vector field ξ_Q^q in 33 corresponds then to the following element ξ^q of $se(2)$:

$$\begin{aligned} \xi^q \quad = \quad & \sin\theta_{1,2}\mathcal{A}_1 + [(d_1\cos\theta_{1,2} + d_2)\cos\theta_1 + y_1\sin\theta_{1,2}]\mathcal{A}_2 \\ & \qquad + [(d_1\cos\theta_{1,2} + d_2)\sin\theta_1 - x_1\sin\theta_{1,2}]\mathcal{A}_3 \ . \end{aligned} \tag{40}$$

By differentiating 40, we get

$$\begin{aligned} \frac{d\xi^q}{dt} &= \cos\theta_{1,2}\dot\theta_{1,2}\mathcal{A}_1 \\ &+ [-d_1\sin\theta_{1,2}\cos\theta_1\dot\theta_{1,2} - (d_1\cos\theta_{1,2} + d_2)\sin\theta_1\dot\theta_1 + \dot y_1\sin\theta_{1,2} + y_1\cos\theta_{1,2}\dot\theta_{1,2}]\mathcal{A}_2 \\ &+ [-d_1\sin\theta_{1,2}\sin\theta_1\dot\theta_{1,2} + (d_1\cos\theta_{1,2} + d_2)\cos\theta_1\dot\theta_1 - \dot x_1\sin\theta_{1,2} - x_1\cos\theta_{1,2}\dot\theta_{1,2}]\mathcal{A}_3 \ . \end{aligned} \tag{41}$$

The corresponding infinitesimal generator is given by 38 as

$$\begin{aligned}\left[\frac{d\xi^q}{dt}\right]_Q \;&=\; [\,-d_1\sin\theta_{1,2}\cos\theta_1\dot{\theta}_{1,2}-(d_1\cos\theta_{1,2}+d_2)\sin\theta_1\dot{\theta}_1+\dot{y}_1\sin\theta_{1,2}]\frac{\partial}{\partial x_1}\\ &\qquad +[\,-d_1\sin\theta_{1,2}\sin\theta_1\dot{\theta}_{1,2}+(d_1\cos\theta_{1,2}+d_2)\cos\theta_1\dot{\theta}_1-\dot{x}_1\sin\theta_{1,2}]\frac{\partial}{\partial y_1}\\ &\qquad\qquad +\cos\theta_{1,2}\dot{\theta}_{1,2}\frac{\partial}{\partial\theta_1}\;.\end{aligned} \tag{42}$$

Finally, by differentiating ξ^q_Q in 33, we get

$$\begin{aligned}\frac{d\xi^q_Q}{dt} \;&=\; [\,-d_1\sin\theta_{1,2}\cos\theta_1\dot{\theta}_{1,2}-(d_1\cos\theta_{1,2}+d_2)\sin\theta_1\dot{\theta}_1]\frac{\partial}{\partial x_1}\\ &\qquad +[\,-d_1\sin\theta_{1,2}\sin\theta_1\dot{\theta}_{1,2}+(d_1\cos\theta_{1,2}+d_2)\cos\theta_1\dot{\theta}_1]\frac{\partial}{\partial y_1}\\ &\qquad\qquad +\cos\theta_{1,2}\dot{\theta}_{1,2}\frac{\partial}{\partial\theta_1}\;.\end{aligned} \tag{43}$$

4 Dynamics of the Roller Racer

4.1 The Lagrange–d'Alembert Equations of Motion

The Lagrangian dynamics of the Roller Racer are set up under the assumption that the mass and linear momentum of platform 2 are much smaller than those of platform 1 and can be ignored. However, the inertia of platform 2 is not ignored. Thus, we consider the following Lagrangian:

$$L(q,\dot{q}) = \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}I_{z_1}\dot{\theta}_1^2 + \frac{1}{2}I_{z_2}(\dot{\theta}_1+\dot{\theta}_{1,2})^2 \ , \tag{44}$$

for $q=(x_1,y_1,\theta_1,\theta_{1,2}) \in Q$ and $\dot{q}=(\dot{x}_1,\dot{y}_1,\dot{\theta}_1,\dot{\theta}_{1,2}) \in T_qQ$, where m_i and I_{z_i} is respectively the mass and moment of inertia of platform i. From 44, we get by differentiation

$$\frac{\partial L}{\partial \dot{q}} = \begin{pmatrix} m_1\dot{x}_1 \\ m_1\dot{y}_1 \\ (I_{z_1}+I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2} \\ I_{z_2}\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2} \end{pmatrix} . \tag{45}$$

The equations of motion of the Roller Racer are derived using the *Lagrange–d'Alembert principle* for a system with nonholonomic constraints (Vershik and Faddeev [32], Yang [34]).

Proposition 4.1 *(Lagrange–d'Alembert Principle)*

In the case of linear constraints on the velocities, the Lagrange–d'Alembert principle for the Roller Racer with the Lagrangian $L(q,v)$ given by 44, with $q=(x_1,y_1,\theta_1,\theta_{1,2}) \in Q$ and $v=(v_1,v_2,v_3,v_4)=\dot{q} \in T_qQ$, takes the form:

$$\left(\frac{d}{dt}(\frac{\partial L}{\partial v}) - \frac{\partial L}{\partial q}\right)\cdot u = \alpha_e \cdot u \ , \tag{46}$$

where (q,v) satisfy the nonholonomic constraints:

$$\begin{aligned} \omega_q^1(v) &= -\sin\theta_1 v_1 + \cos\theta_1 v_2 = 0 \ , \\ \omega_q^2(v) &= -\sin(\theta_1+\theta_{1,2})v_1 + \cos(\theta_1+\theta_{1,2})v_2 + (d_1\cos\theta_{1,2}+d_2)v_3 + d_2v_4 = 0 \ , \end{aligned} \tag{47}$$

and the test vector $u=(u_1,u_2,u_3,u_4) \in T_qQ$ satisfies:

$$\begin{aligned} \frac{\partial}{\partial v}\omega_q^1(v)\cdot u &= -\sin\theta_1 u_1 + \cos\theta_1 u_2 = 0 \ , \\ \frac{\partial}{\partial v}\omega_q^2(v)\cdot u &= -\sin(\theta_1+\theta_{1,2})u_1 + \cos(\theta_1+\theta_{1,2})u_2 + (d_1\cos\theta_{1,2}+d_2)u_3 + d_2u_4 = 0 \ , \end{aligned} \tag{48}$$

while α_e is the 1–form describing the external forcing to the system.

Using Lagrange multipliers, the Lagrange–d'Alembert principle for the case of a system with two linear (in the velocity) nonholonomic constraints, takes the form:

$$\frac{d}{dt}(\frac{\partial L}{\partial v}) - \frac{\partial L}{\partial q} = \alpha_e + \lambda_1\frac{\partial \omega_q^1}{\partial v} + \lambda_2\frac{\partial \omega_q^2}{\partial v} \ , \tag{49}$$

for functions λ_1 and λ_2 on TQ and for (q, v) such that the nonholonomic constraints 47 are satisfied.

Consider external forcing to the system described by the 1–form

$$\alpha_e = (F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}}) \ , \tag{50}$$

where $F_{\theta_{1,2}}$ may be the torque applied by the motor that actuates the rotary joint $O_{1,2}$ and $F_{x_1}, F_{y_1}, F_{\theta_1}$ may be the result of friction in the bearings of the wheels. The equations of motion of the Roller Racer are given below.

Proposition 4.2 *(Lagrange–d'Alembert Equations of Motion)*
i) In the case $d_2 \neq 0$, the equations of motion for the Roller Racer are:

$$\begin{aligned}
&(I_{z_2}\sin^2\theta_{1,2} + m_1 d_2^2)\dot{\nu}_1 - I_{z_2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}\dot{\nu}_2\\
&\qquad + I_{z_2}\sin^2\theta_{1,2}\cos\theta_{1,2}\nu_1^2\\
&\qquad - I_{z_2}\sin\theta_{1,2}[d_1(\cos^2\theta_{1,2} - \sin^2\theta_{1,2}) + d_2\cos\theta_{1,2}]\nu_1\nu_2\\
&\qquad - I_{z_2} d_1 \sin^2\theta_{1,2}(d_1\cos\theta_{1,2} + d_2)\nu_2^2\\
&\qquad\qquad = d_2(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \sin\theta_{1,2}F_{\theta_{1,2}} \ ,\\
&-I_{z_2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}\dot{\nu}_1 + (I_{z_1} d_2^2 + I_{z_2} d_1^2\cos^2\theta_{1,2})\dot{\nu}_2\\
&\qquad - I_{z_2} d_1 \sin\theta_{1,2}\cos^2\theta_{1,2}\nu_1^2\\
&\qquad + I_{z_2} d_1 \cos\theta_{1,2}[d_1(\cos^2\theta_{1,2} - \sin^2\theta_{1,2}) + d_2\cos\theta_{1,2}]\nu_1\nu_2\\
&\qquad + I_{z_2} d_1^2(d_1\cos\theta_{1,2} + d_2)\sin\theta_{1,2}\cos\theta_{1,2}\nu_2^2\\
&\qquad\qquad = d_2 F_{\theta_1} - r(\theta_{1,2})F_{\theta_{1,2}} \ ,
\end{aligned} \tag{51}$$

where $\nu_1 \overset{\text{def}}{=} \frac{1}{d_2}(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)$, $\nu_2 \overset{\text{def}}{=} \frac{1}{d_2}\dot{\theta}_1$ and $r(\theta_{1,2}) \overset{\text{def}}{=} d_1\cos\theta_{1,2} + d_2$.
ii) In the case $d_2 = 0$, the equations of motion are

$$\begin{aligned}
&[(I_{z_1} + I_{z_2})\sin^2\theta_{1,2} + m_1 d_1^2\cos^2\theta_{1,2}]\dot{\nu}_1 + I_{z_2}\sin\theta_{1,2}\dot{\nu}_2\\
&\qquad + (I_{z_1} + I_{z_2} - m_1 d_1^2)\sin\theta_{1,2}\cos\theta_{1,2}\nu_1\nu_2\\
&\qquad\qquad = d_1\cos\theta_{1,2}(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \sin\theta_{1,2}F_{\theta_1} \ ,\\
&I_{z_2}\sin\theta_{1,2}\dot{\nu}_1 + I_{z_2}\dot{\nu}_2 + I_{z_2}\cos\theta_{1,2}\nu_1\nu_2 = F_{1,2} \ ,
\end{aligned} \tag{52}$$

where $\nu_1 \overset{\text{def}}{=} \frac{1}{d_1\cos\theta_{1,2}}(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)$, in the case $\cos\theta_{1,2} \neq 0$, or $\nu_1 \overset{\text{def}}{=} \frac{1}{\sin\theta_{1,2}}\dot{\theta}_1$ otherwise and $\nu_2 \overset{\text{def}}{=} \dot{\theta}_{1,2}$.

Proof

The Lagrange–d'Alembert principle 46 for the Lagrangian given by 44, for $u = (u_1, u_2, u_3, u_4), v = (v_1, v_2, v_3, v_4) \in \mathcal{D}_q$ and for $\alpha_e = (F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}})$, takes the form:

$$\begin{aligned}
&m_1\dot{v}_1 u_1 + m_1\dot{v}_2 u_2 + [(I_{z_1} + I_{z_2})\dot{v}_3 + I_{z_2}\dot{v}_4]u_3 + I_{z_2}(\dot{v}_3 + \dot{v}_4)u_4\\
&\qquad = F_{x_1}u_1 + F_{y_1}u_2 + F_{\theta_1}u_3 + F_{\theta_{1,2}}u_4 \ .
\end{aligned} \tag{53}$$

i) Let $d_2 \neq 0$:

Any $u \in \mathcal{D}_q$ can be represented as $u = \alpha_1 \xi_Q^1 + \alpha_2 \xi_Q^2$, for ξ_Q^1 and ξ_Q^2 given by 23 and for some $\alpha_1, \alpha_2 \in I\!R$. Then its components are

$$u_1 = \alpha_1 d_2 \cos\theta_1 \ , \quad u_2 = \alpha_1 d_2 \sin\theta_1 \ , \quad u_3 = \alpha_2 d_2 \ , \quad u_4 = \alpha_1 \sin\theta_{1,2} - \alpha_2 r(\theta_{1,2}) \ . \qquad (54)$$

Similarly, any $v \in \mathcal{D}_q$ can be represented as $v = \nu_1 \xi_Q^1 + \nu_2 \xi_Q^2$, for some $\nu_1, \nu_2 \in I\!R$. Its components are

$$v_1 = \nu_1 d_2 \cos\theta_1 \ , \quad v_2 = \nu_1 d_2 \sin\theta_1 \ , \quad v_3 = \nu_2 d_2 \ , \quad v_4 = \nu_1 \sin\theta_{1,2} - \nu_2 r(\theta_{1,2}) \ . \qquad (55)$$

These relationships can be used to derive ν_1 and ν_2 as follows:

$$\nu_1 = \frac{1}{d_2}(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) \ , \quad \nu_2 = \frac{1}{d_2}\dot{\theta}_1 \ . \qquad (56)$$

By differentiating 55 we get

$$\begin{aligned} \dot{v}_1 &= \dot{\nu}_1 d_2 \cos\theta_1 - \nu_1 d_2 \sin\theta_1 \dot{\theta}_1 \ , \\ \dot{v}_2 &= \dot{\nu}_1 d_2 \sin\theta_1 + \nu_1 d_2 \cos\theta_1 \dot{\theta}_1 \ , \\ \dot{v}_3 &= \dot{\nu}_2 d_2 \ , \\ \dot{v}_4 &= \dot{\nu}_1 \sin\theta_{1,2} - \dot{\nu}_2 r(\theta_{1,2}) + \nu_1 \cos\theta_{1,2}\dot{\theta}_{1,2} + \nu_2 d_1 \sin\theta_{1,2}\dot{\theta}_{1,2} \ . \end{aligned} \qquad (57)$$

Introducing 54 and 57 in 53, we get

$$\begin{aligned} & m_1 \dot{\nu}_1 d_2^2 \alpha_1 \\ & \quad + [(I_{z_1} + I_{z_2})\dot{\nu}_2 d_2 + I_{z_2}(\dot{\nu}_1 \sin\theta_{1,2} - \dot{\nu}_2 r(\theta_{1,2}) + \nu_1 \cos\theta_{1,2}\dot{\theta}_{1,2} + \nu_2 d_1 \sin\theta_{1,2}\dot{\theta}_{1,2})] d_2 \alpha_2 \\ & \quad + I_{z_2}[\dot{\nu}_2 d_2 + \dot{\nu}_1 \sin\theta_{1,2} - \dot{\nu}_2 r(\theta_{1,2}) + \nu_1 \cos\theta_{1,2}\dot{\theta}_{1,2} + \nu_2 d_1 \sin\theta_{1,2}\dot{\theta}_{1,2}] \cdot \\ & \qquad\qquad\qquad\qquad\qquad\qquad \cdot [\sin\theta_{1,2}\alpha_1 - r(\theta_{1,2})\alpha_2] \\ & \qquad = F_{x_1} d_2 \cos\theta_1 \alpha_1 + F_{y_1} d_2 \sin\theta_1 \alpha_1 + F_{\theta_1} d_2 \alpha_2 + F_{\theta_{1,2}}[\sin\theta_{1,2}\alpha_1 - r(\theta_{1,2})\alpha_2] \ , \end{aligned} \qquad (58)$$

for arbitrary $\alpha_1, \alpha_2 \in I\!R$. From 55, we have

$$\dot{\theta}_{1,2} \equiv v_4 = \nu_1 \sin\theta_{1,2} - \nu_2 r(\theta_{1,2}) \ . \qquad (59)$$

Since α_1, α_2 are arbitrary, equation 58 splits (after using 59) in the following two equations:

$$\begin{aligned} & (I_{z_2} \sin^2\theta_{1,2} + m_1 d_2^2)\dot{\nu}_1 - I_{z_2} d_1 \sin\theta_{1,2} \cos\theta_{1,2} \dot{\nu}_2 \\ & \qquad + I_{z_2} \sin\theta_{1,2}(\cos\theta_{1,2}\nu_1 + d_1 \sin\theta_{1,2}\nu_2)(\nu_1 \sin\theta_{1,2} - \nu_2 r(\theta_{1,2})) \\ & \qquad\qquad\qquad = (F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) d_2 + F_{\theta_{1,2}} \sin\theta_{1,2} \ , \\ & -I_{z_2} d_1 \sin\theta_{1,2} \cos\theta_{1,2} \dot{\nu}_1 + (I_{z_1} d_2^2 + I_{z_2} d_1^2 \cos^2\theta_{1,2})\dot{\nu}_2 \\ & \qquad - I_{z_2} d_1 \cos\theta_{1,2}(\nu_1 \sin\theta_{1,2} - \nu_2 r(\theta_{1,2})) \\ & \qquad\qquad\qquad = F_{\theta_1} d_2 - F_{\theta_{1,2}} r(\theta_{1,2}) \ . \end{aligned} \qquad (60)$$

By rearranging terms, we obtain 51. It can be easily seen that the first of the equations 51 is equation 46 with test vector $u = \xi_Q^1$, while the second is 46 with $u = \xi_Q^2$.

ii) Let $d_2 = 0$:

Any $u \in \mathcal{D}_q$ can be represented as $u = \alpha_1 \xi_Q^1 + \alpha_2 \xi_Q^2$, for ξ_Q^1 and ξ_Q^2 given by 24 and for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Then its components are

$$u_1 = \alpha_1 d_1 \cos\theta_{1,2} \cos\theta_1 \ , \quad u_2 = \alpha_1 d_1 \cos\theta_{1,2} \sin\theta_1 \ , \quad u_3 = \alpha_1 \sin\theta_{1,2} \ , \quad u_4 = \alpha_2 \ . \tag{61}$$

Similarly, any $v \in \mathcal{D}_q$ can be represented as $v = \nu_1 \xi_Q^1 + \nu_2 \xi_Q^2$, for some $\nu_1, \nu_2 \in \mathbb{R}$. Its components are

$$v_1 = \nu_1 d_1 \cos\theta_{1,2} \cos\theta_1 \ , \quad v_2 = \nu_1 d_1 \cos\theta_{1,2} \sin\theta_1 \ , \quad v_3 = \nu_1 \sin\theta_{1,2} \ , \quad v_4 = \nu_2 \ . \tag{62}$$

These relationships can be used to derive ν_1 and ν_2 as follows:

$$\begin{aligned} \nu_1 &= \frac{1}{d_1 \cos\theta_{1,2}} (\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) \ , \quad \text{in the case } \cos\theta_{1,2} \neq 0 \ , \\ &= \frac{1}{\sin\theta_{1,2}} \dot{\theta}_1 \ , \quad \text{otherwise} \ , \\ \nu_2 &= \dot{\theta}_{1,2} \ . \end{aligned} \tag{63}$$

By differentiating 62 we get

$$\begin{aligned} \dot{v}_1 &= \dot{\nu}_1 d_1 \cos\theta_{1,2} \cos\theta_1 - \nu_1 d_1 (\cos\theta_{1,2} \sin\theta_1 \dot{\theta}_1 + \sin\theta_{1,2} \cos\theta_1 \dot{\theta}_{1,2}) \ , \\ \dot{v}_2 &= \dot{\nu}_1 d_1 \cos\theta_{1,2} \sin\theta_1 + \nu_1 d_1 (\cos\theta_{1,2} \cos\theta_1 \dot{\theta}_1 - \sin\theta_{1,2} \sin\theta_1 \dot{\theta}_{1,2}) \ , \\ \dot{v}_3 &= \dot{\nu}_1 \sin\theta_{1,2} + \nu_1 \cos\theta_{1,2} \dot{\theta}_{1,2} \ , \\ \dot{v}_4 &= \dot{\nu}_2 \ . \end{aligned} \tag{64}$$

Introducing 61 and 64 in 53, we get

$$\begin{aligned} & m_1 (\dot{\nu}_1 d_1 \cos\theta_{1,2} \cos\theta_1 - \nu_1 d_1 \sin\theta_{1,2} \cos\theta_1 \dot{\theta}_{1,2}) d_1 \cos\theta_{1,2} \cos\theta_1 \alpha_1 \\ & \quad + m_1 (\dot{\nu}_1 d_1 \cos\theta_{1,2} \sin\theta_1 - \nu_1 d_1 \sin\theta_{1,2} \sin\theta_1 \dot{\theta}_{1,2}) d_1 \cos\theta_{1,2} \sin\theta_1 \alpha_1 \\ & \quad + [(I_{z_1} + I_{z_2})(\dot{\nu}_1 \sin\theta_{1,2} + \nu_1 \cos\theta_{1,2} \dot{\theta}_{1,2}) + I_{z_2} \dot{\nu}_2] \sin\theta_{1,2} \alpha_1 \\ & \quad + I_{z_2} (\dot{\nu}_1 \sin\theta_{1,2} + \nu_1 \cos\theta_{1,2} \dot{\theta}_{1,2} + \dot{\nu}_2) \alpha_2 \\ & \qquad = F_{x_1} d_1 \cos\theta_{1,2} \cos\theta_1 \alpha_1 + F_{y_1} d_1 \cos\theta_{1,2} \sin\theta_1 \alpha_1 + F_{\theta_1} \sin\theta_{1,2} \alpha_1 + F_{\theta_{1,2}} \alpha_2 \ , \end{aligned} \tag{65}$$

for arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$. Since α_1, α_2 are arbitrary, 65 splits in the following two equations:

$$\begin{aligned} & [(I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1 d_1^2 \cos^2\theta_{1,2}] \dot{\nu}_1 + I_{z_2} \sin\theta_{1,2} \dot{\nu}_2 \\ & \qquad + (I_{z_1} + I_{z_2} - m_1 d_1^2) \sin\theta_{1,2} \cos\theta_{1,2} \nu_1 \dot{\theta}_{1,2} \\ & \qquad\qquad = d_1 \cos\theta_{1,2} (F_{x_1} \cos\theta_1 + F_{y_1} \sin\theta_1) + \sin\theta_{1,2} F_{\theta_1} \ , \\ & I_{z_2} \sin\theta_{1,2} \dot{\nu}_1 + I_{z_2} \dot{\nu}_2 + I_{z_2} \cos\theta_{1,2} \nu_1 \dot{\theta}_{1,2} = F_{1,2} \ . \end{aligned} \tag{66}$$

By using $\dot{\theta}_{1,2} \equiv v_4 = \nu_2$ from 62 and by rearranging terms, we obtain 52. Again, the first of the equations 52 is equation 46 with $u = \xi_Q^1$, while the second is 46 with $u = \xi_Q^2$.

■

Remark 4.1 The quantities $\dot{\nu}_1$ and $\dot{\nu}_2$ above can be interpreted as accelerations in the constrained directions.

Suppose *friction* is present in the joints of the Roller Racer wheels with their axes. We consider a simple viscous friction model, where the frictional forces are introduced in the Lagrange–d'Alembert equations through the following Rayleigh dissipation function that involves the wheel angular velocities $\dot{\phi}_{l,i}, \dot{\phi}_{r,i},\ i = 1, 2$ defined in 25:

$$\begin{aligned}
\mathcal{R} &= \frac{1}{2}k_1\dot{\phi}_{l,1}^2 + \frac{1}{2}k_1\dot{\phi}_{r,1}^2 + \frac{1}{2}k_2\dot{\phi}_{l,2}^2 + \frac{1}{2}k_2\dot{\phi}_{r,2}^2 \\
&= \frac{k_1}{R_1^2}\left[\frac{L_1^2}{4}\dot{\theta}_1^2 + (\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)^2\right] \\
&\qquad + \frac{k_2}{R_2^2}\left[\frac{L_2^2}{4}(\dot{\theta}_1 + \dot{\theta}_{1,2})^2 + ((\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\cos\theta_{1,2} + \dot{\theta}_1 d_1\sin\theta_{1,2})^2\right] ,
\end{aligned} \tag{67}$$

where $k_1 > 0$ and $k_2 > 0$ are friction coefficients, $q = (x_1, y_1, \theta_1, \theta_{1,2})^\top \in Q$ and $\dot{q} = (\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2})^\top$. The external force 1–form α_e is $\alpha_e = \mathcal{T}_{1,2} - \frac{\partial\mathcal{R}}{\partial\dot{q}}$, where $\mathcal{T}_{1,2} \overset{25}{=} (0,\ 0,\ 0,\ \tau_{1,2})^\top$, with $\tau_{1,2}$ being the torque applied by the motor at the joint $O_{1,2}$. The corresponding components of α_e are

$$\begin{aligned}
F_{x_1} &= -2\Big[\Big(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\Big)(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) \\
&\qquad\qquad + \frac{k_2}{R_2^2}d_1\sin\theta_{1,2}\cos\theta_{1,2}\dot{\theta}_1\Big]\cos\theta_1 , \\
F_{y_1} &= -2\Big[\Big(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\Big)(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) \\
&\qquad\qquad + \frac{k_2}{R_2^2}d_1\sin\theta_{1,2}\cos\theta_{1,2}\dot{\theta}_1\Big]\sin\theta_1 , \\
F_{\theta_1} &= -2\Big[\frac{k_2}{R_2^2}d_1\sin\theta_{1,2}\cos\theta_{1,2}(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) \\
&\qquad\qquad + \Big(\frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}(\frac{L_2^2}{4} + d_1^2\sin^2\theta_{1,2})\Big)\dot{\theta}_1\Big] , \\
F_{\theta_{1,2}} &= \tau_{1,2} - 2\frac{k_2}{R_2^2}\frac{L_2^2}{4}\dot{\theta}_{1,2} .
\end{aligned} \tag{68}$$

4.2 Nonholonomic Momentum and the Momentum Equation

The symmetries of an unconstrained system or of a system with holonomic constraints, described by the invariance of its Lagrangian with respect to a Lie group action, imply the existence of conserved quantities, called momenta (Noether Theorem). For systems with nonholonomic constraints, whose Lagrangian and constraints are invariant with respect to an appropriate Lie group action, it is still possible to define momentum–like quantities. These, however, are not necessarily conserved; instead, they evolve according to a law called the momentum equation [5].

It is easy to verify that, for the Roller Racer, the nonholonomic constraints 16 and 18 and the Lagrangian 44 are invariant under the action Φ given by 28 [18].

Momentum–like quantities can be defined for a constrained system by

$$p = \frac{\partial L}{\partial v} \cdot u \ ,$$

where $v \in T_q Q$ and $u \in \mathcal{D}_q$, the constraint distribution. In the present case, it is particularly advantageous (see Proposition 4.4 below) to restrict u to $\mathcal{S}_q$, the intersection of the subspaces $\mathcal{D}_q$ and $T_q Orb(q)$.

We define, then, the *nonholonomic momentum* as

$$p \overset{\text{def}}{=} \sum_i \frac{\partial L}{\partial \dot{q}^i} (\xi_Q^q)^i \ , \tag{69}$$

where $\xi_Q^q \in \mathcal{S}_q$. From 45 and 33, we get the nonholonomic momentum for the Roller Racer:

$$p = m_1(d_1 \cos\theta_{1,2} + d_2)(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) + [(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}] \sin\theta_{1,2} \ . \tag{70}$$

The nonholonomic momentum p given by equation 70 is (up to a scale factor) the angular momentum about the point of intersection O_{12}^{ICR} of the two wheel axles (cf. fig. 3). It can be easily seen that $O_1 O_{12}^{\mathrm{ICR}} = \frac{d_1 \cos\theta_{1,2} + d_2}{\sin\theta_{1,2}}$, when $\sin\theta_{1,2} \neq 0$.

Let

$$\Delta(\theta_{1,2}) \overset{\text{def}}{=} (I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1(d_1 \cos\theta_{1,2} + d_2)^2 \ . \tag{71}$$

For $d_1 \neq d_2$, we have $\Delta > 0$, for all $q \in Q$. Also, let

$$\delta(\theta_{1,2}) \overset{\text{def}}{=} I_{z_2} \sin^2\theta_{1,2} + m_1 d_2 (d_1 \cos\theta_{1,2} + d_2) \ . \tag{72}$$

Proposition 4.3 *The angular velocity $\dot{\theta}_1$ is an affine function of the nonholonomic momentum*

$$\dot{\theta}_1 = \frac{1}{\Delta(\theta_{1,2})} [\sin\theta_{1,2} \ \ p - \delta(\theta_{1,2})\dot{\theta}_{1,2}] \ . \tag{73}$$

Proof

Multiplying both sides of 70 by $\sin\theta_{1,2}$ and using 18, we get

$$\begin{aligned} \sin\theta_{1,2} \ \ p \ &= \ m_1(d_1 \cos\theta_{1,2} + d_2)[(d_1 \cos\theta_{1,2} + d_2)\dot{\theta}_1 + d_2\dot{\theta}_{1,2}] \\ &\qquad + [(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}] \sin^2\theta_{1,2} \ . \end{aligned} \tag{74}$$

Solving for $\dot{\theta}_1$, the result follows.

∎

The momentum equation presented in the next result is derived in (Bloch, Krishnaprasad, Marsden and Murray [5]) from the Lagrange–d'Alembert principle by considering only variations that satisfy the constraints and that depend on the symmetry, as it is expressed by a free group action. The equation does not depend on internal torques and depends only on the shape variables and not on the group variables. It is given below for the case where external torques are not present.

Proposition 4.4 *Consider a Lagrangian L which is invariant under the action Φ of a group G on a configuration space Q. Let $\mathcal{D}_q$ be a constraint distribution on T_qQ and consider the intersection $\mathcal{S}_q$ of $\mathcal{D}_q$ with the tangent space to the orbit of Φ at q. Let $\xi^q_Q \in \mathcal{S}_q$ and let ξ^q be the corresponding element of the Lie algebra $\mathcal{G}$. The evolution of the nonholonomic momentum p, defined as in equation 69, satisfies the equation*

$$\frac{dp}{dt} = \sum_i \frac{\partial L}{\partial \dot{q}^i} \left[\frac{d\xi^q}{dt} \right]^i_Q . \tag{75}$$

This result generalizes the classical Noether Theorem, which specifies conserved quantities for solutions of the Euler–Lagrange equations (Arnold [2], Abraham and Marsden [1], Marsden and Ratiu [21]). In the present paper, the subspace $\mathcal{S}_q$ is one–dimensional and one has a scalar p. In general, one could have a vector nonholonomic momentum.

Proposition 4.5 *(Momentum Equation without External Forces)*
The momentum equation for the Roller Racer is

$$\frac{dp}{dt} = A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 , \tag{76}$$

where

$$A_1^4(\theta_{1,2}) \overset{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})}\beta(\theta_{1,2}) \sin\theta_{1,2} \tag{77}$$

and

$$A_2^4(\theta_{1,2}) \overset{\text{def}}{=} \frac{m_1}{\Delta(\theta_{1,2})}\lambda(\theta_{1,2})\gamma(\theta_{1,2}) , \tag{78}$$

with

$$\begin{aligned} \beta(\theta_{1,2}) &\overset{\text{def}}{=} (I_{z_1} + I_{z_2})\cos\theta_{1,2} - m_1 d_1 (d_1 \cos\theta_{1,2} + d_2) , \\ \gamma(\theta_{1,2}) &\overset{\text{def}}{=} -I_{z_1} d_2 + I_{z_2} d_1 \cos\theta_{1,2} , \\ r(\theta_{1,2}) &\overset{\text{def}}{=} d_1 \cos\theta_{1,2} + d_2 , \quad \lambda(\theta_{1,2}) \overset{\text{def}}{=} d_1 + d_2\cos\theta_{1,2} , \quad I \overset{\text{def}}{=} I_{z_1} + I_{z_2} . \end{aligned} \tag{79}$$

Proof

From 75, 45 and 42 we get

$$\begin{aligned}\frac{dp}{dt} &= -m_1 d_1(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1)\sin\theta_{1,2}\dot{\theta}_{1,2} \\ &\qquad + m_1(d_1\cos\theta_{1,2} + d_2)(-\dot{x}_1\sin\theta_1 + \dot{y}_1\cos\theta_1)\dot{\theta}_1 \\ &\qquad\qquad + [(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}]\cos\theta_{1,2}\dot{\theta}_{1,2} \ . \end{aligned} \tag{80}$$

Introducing the nonholonomic constraints 16 and 18, in 80 above, we get

$$\frac{dp}{dt} = [(I_{z_1} + I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2} + d_2)]\dot{\theta}_1\dot{\theta}_{1,2} + [I_{z_2}\cos\theta_{1,2} - m_1 d_1 d_2]\dot{\theta}^2_{1,2} \ . \tag{81}$$

By substituting $\dot{\theta}_1$ from 73 in the above expression, the result follows.

∎

Proposition 4.6 *The solution of the momentum equation 76 is*

$$p(t) = \Phi(t,t_0)p(t_0) + \int_{t_0}^{t} \Phi(t,\tau)A^4_2(\theta_{1,2}(\tau))\dot{\theta}^2_{1,2}(\tau)d\tau \ , \tag{82}$$

where

$$\Phi(t,t_0) \overset{\text{def}}{=} \exp\Big[\int_{t_0}^{t} A^4_1(\theta_{1,2}(\tau))\dot{\theta}_{1,2}(\tau)d\tau\Big] = \sqrt{\frac{\Delta(\theta_{1,2}(t))}{\Delta(\theta_{1,2}(t_0))}} \tag{83}$$

is the state transition matrix of the time–varying linear ordinary differential equation 76.

Proof

Equation 76 is a first–order linear time–varying ODE with state transition matrix $\Phi(t,t_0)$. Thus, 82 is obvious. To compute the state transition matrix $\Phi(t,t_0)$, observe that we get from 71

$$\frac{d\Delta}{d\theta_{1,2}} = 2\beta(\theta_{1,2})\sin\theta_{1,2} \ . \tag{84}$$

From this and from the definition of A^4_1 in 76 we get

$$A^4_1(\theta_{1,2}) = \frac{\beta(\theta_{1,2})}{\Delta(\theta_{1,2})}\sin\theta_{1,2} = \frac{1}{2\Delta}\frac{d\Delta}{d\theta_{1,2}} \ . \tag{85}$$

Thus

$$\begin{aligned}\Phi(t,t_0) &= \exp\Big[\int_{t_0}^{t} A^4_1(\theta_{1,2}(\tau))\dot{\theta}_{1,2}(\tau)d\tau\Big] = \exp\Big[\int_{\theta_{1,2}(t_0)}^{\theta_{1,2}(t)} A^4_1(\theta_{1,2})d\theta_{1,2}\Big] \\ &= \exp\Big[\int_{\Delta(\theta_{1,2}(t_0))}^{\Delta(\theta_{1,2}(t))} \frac{1}{2}\frac{d\Delta}{\Delta}\Big] = \exp\Big[\ln\big(\sqrt{\frac{\Delta(\theta_{1,2}(t))}{\Delta(\theta_{1,2}(t_0))}}\big)\Big] = \sqrt{\frac{\Delta(\theta_{1,2}(t))}{\Delta(\theta_{1,2}(t_0))}} \ . \end{aligned} \tag{86}$$

$\blacksquare$

Equation 82 can be used to derive qualitative information about the momentum, which can be useful in motion control.

Proposition 4.7 *(Sign of the Nonholonomic Momentum)*

Assume $d_1 > d_2$.

a) Let $I_{z_1} d_2 > I_{z_2} d_1$.

Assume further that the initial momentum of the system is non–positive. Then, the momentum p *is negative at all subsequent times.*

b1) Let $I_{z_1} d_2 < I_{z_2} d_1$.

Suppose, in addition, that the angle $\theta_{1,2}$ *remains in an* $\tilde{\epsilon}$*–neighborhood of* $\theta_{1,2} = 0$*, with* $\tilde{\epsilon} \leq \cos^{-1}(\frac{I_{z_1} d_2}{I_{z_2} d_1})$*. Assume further that the initial momentum of the system is non–negative. Then, the momentum* p *is positive at all subsequent times.*

b2) Let $I_{z_1} d_2 < I_{z_2} d_1$.

Suppose, in addition, that the angle $\theta_{1,2}$ *remains outside an* $\tilde{\epsilon}$*–neighborhood of* $\theta_{1,2} = 0$*, with* $\tilde{\epsilon} \leq \cos^{-1}(\frac{I_{z_1} d_2}{I_{z_2} d_1})$*. Assume further that the initial momentum of the system is non–positive. Then, the momentum* p *is negative at all subsequent times.*

Proof

Since $d_1 > d_2$, we know that $\Delta > 0$ and $\lambda = d_1 + d_2 \cos\theta_{1,2} > 0$, for all $\theta_{1,2}$.

a) In the case $I_{z_1} d_2 > I_{z_2} d_1$ we have $\gamma = -I_{z_1} d_2 + I_{z_2} d_1 \cos\theta_{1,2} < 0$, for all $\theta_{1,2}$. Thus $A^4_2 = \frac{m_1}{\Delta}\lambda\gamma < 0$, for all $\theta_{1,2}$ and, thus, the second term of 82 is negative. If $p(t_0) \leq 0$, then $p(t) < 0$, $\forall t > t_0$.

b1) In the case $I_{z_1} d_2 < I_{z_2} d_1$, by our choice of the $\tilde{\epsilon}$–neighborhood we have $\gamma > 0$, for all $\theta_{1,2}$ in this neighborhood. Then, $A^4_2 = \frac{m_1}{\Delta}\lambda\gamma > 0$ and the second term of 82 is positive. If $p(t_0) \geq 0$, then $p(t) > 0$, $\forall t > t_0$.

b2) In the case $I_{z_1} d_2 < I_{z_2} d_1$, by our choice of the $\tilde{\epsilon}$–neighborhood we have $\gamma < 0$, for all $\theta_{1,2}$ outside this neighborhood. Then, $A^4_2 = \frac{m_1}{\Delta}\lambda\gamma < 0$ and the second term of 82 is negative. If $p(t_0) \leq 0$, then $p(t) < 0$, $\forall t > t_0$.

$\blacksquare$

As shown in [5], the momentum equation 76 can be derived directly from the Lagrange–d'Alembert Principle 46, by considering a test vector u in the space $\mathcal{S}_q \subset \mathcal{D}_q \subset T_qQ$. This approach is used below to derive the momentum equation for the case when external forces, of the type considered in equation 50, are acting on the Roller Racer.

Proposition 4.8 *(Momentum Equation with External Forces)*

Consider external forcing to the system described by the 1–form $\alpha_e = (F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}})$*. The nonholonomic momentum evolves according to the equation below:*

$$\frac{dp}{dt} = A^4_1(\theta_{1,2})\dot{\theta}_{1,2}p + A^4_2(\theta_{1,2})\dot{\theta}^2_{1,2} + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \sin\theta_{1,2}F_{\theta_1} \ , \qquad (87)$$

Proof

Consider the Lagrange–d'Alembert principle 53 with u restricted to $\mathcal{S}_q \subset \mathcal{D}_q$, instead of belonging to the whole $\mathcal{D}_q$.

i) For $d_2 \neq 0$, the vectors $v \in \mathcal{D}_q$ and $\dot{v}$ in this equation are given by 55 and 57, while $u \in \mathcal{S}_q \subset \mathcal{D}_q$ is given by 54, where $\alpha_1 = \frac{r(\theta_{1,2})}{d_2}$ and $\alpha_2 = \frac{\sin\theta_{1,2}}{d_2}$ (cf. equation 33). Thus, 53 takes the form 58, with α_1 and α_2 as specified above, which gives:

$$\begin{aligned}(I_{z_2}\sin^2\theta_{1,2} + m_1 d_2 r(\theta_{1,2}))\dot{\nu}_1 + (I_{z_1} d_2 - I_{z_2} d_1 \cos\theta_{1,2})\sin\theta_{1,2}\dot{\nu}_2 \\ + I_{z_2}(\nu_1\cos\theta_{1,2} + \nu_2 d_1 \sin\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2} \\ = r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2}\ ,\end{aligned} \tag{88}$$

from which we get

$$\begin{aligned}\delta(\theta_{1,2})\dot{\nu}_1 - \gamma(\theta_{1,2})\sin\theta_{1,2}\ \dot{\nu}_2 \ &= \ -I_{z_2}(\nu_1\cos\theta_{1,2} + \nu_2 d_1\sin\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2} \\ &\quad + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2}\ .\end{aligned} \tag{89}$$

Consider the nonholonomic momentum defined in 70, i.e.

$$p = m_1 r(\theta_{1,2})(v_1\cos\theta_1 + v_2\sin\theta_1) + [(I_{z_1} + I_{z_2})v_3 + I_{z_2}v_4]\sin\theta_{1,2}\ , \tag{90}$$

with $v = (v_1,\ v_2,\ v_3,\ v_4) \in T_qQ$. By restricting v to $\mathcal{D}_q$, we get for p (using, for $d_2 \neq 0$, the expression 55):

$$\begin{aligned} p \ &= \ m_1 r(\theta_{1,2}) d_2 \nu_1 + (I_{z_1} + I_{z_2}) d_2 \sin\theta_{1,2}\nu_2 + I_{z_2}\sin\theta_{1,2}(\sin\theta_{1,2}\nu_1 - r(\theta_{1,2})\nu_2) \\ &= \ \delta(\theta_{1,2})\nu_1 - \gamma(\theta_{1,2})\sin\theta_{1,2}\nu_2\ .\end{aligned} \tag{91}$$

The last of the equations 55 (the one for $v_4 \equiv \dot{\theta}_{1,2}$) and equation 91 are linear in ν_1 and ν_2. By solving them, we get

$$\nu_1 \ = \ \frac{1}{d_2\Delta(\theta_{1,2})}[r(\theta_{1,2})p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}]\ ,\ \ \nu_2 = \frac{1}{d_2\Delta(\theta_{1,2})}[\sin\theta_{1,2}p - \delta(\theta_{1,2})\dot{\theta}_{1,2}]\ . \tag{92}$$

By differentiating 91, we get

$$\begin{aligned}\frac{dp}{dt} \ &= \ \delta(\theta_{1,2})\dot{\nu}_1 - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\nu}_2 \\ &\quad + \Big(\frac{\partial\delta}{\partial\theta_{1,2}}\nu_1 - \frac{\partial\gamma}{\partial\theta_{1,2}}\sin\theta_{1,2}\nu_2 - \gamma(\theta_{1,2})\cos\theta_{1,2}\nu_2\Big)\dot{\theta}_{1,2}\ .\end{aligned} \tag{93}$$

Replacing the first two terms of the RHS above with their expression from 89, using 92 and using the definitions 77 and 78, we get 87.

ii) For $d_2 = 0$, the vectors $v \in \mathcal{D}_q$ and $\dot{v}$ in equation 53 are given by 62 and 64, while $u \in \mathcal{S}_q \subset \mathcal{D}_q$ is given by 61, where $\alpha_1 = 1$ and $\alpha_2 = 0$ (cf. equation 33). From 53 we get

$$\begin{aligned}\Delta(\theta_{1,2})\dot{\nu}_1 + I_{z_2}\sin\theta_{1,2}\dot{\nu}_2 &= -\beta(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\ \nu_1 \\ &\quad + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \sin\theta_{1,2}F_{\theta_1}\ .\end{aligned} \tag{94}$$

From the definition of the nonholonomic momentum (equation 70) and by restricting the corresponding v to $\mathcal{D}_q$, we get

$$p = \Delta(\theta_{1,2})\ \nu_1 + I_{z_2}\sin\theta_{1,2}\nu_2\ . \tag{95}$$

From this and 62, we can get ν_1 and ν_2 as functions of p and $\dot{\theta}_{1,2}$

$$\nu_1 = \frac{1}{\Delta(\theta_{1,2})}(p - I_{z_2}\sin\theta_{1,2}\dot{\theta}_{1,2})\ ,\ \ \nu_2 = \dot{\theta}_{1,2}\ . \tag{96}$$

By differentiating p, we get

$$\frac{dp}{dt} = \Delta(\theta_{1,2})\dot{\nu}_1 + I_{z_2}\sin\theta_{1,2}\dot{\nu}_2 + 2\beta(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\ \nu_1 + I_{z_2}\cos\theta_{1,2}\dot{\theta}_{1,2}\ \nu_2 \tag{97}$$

Replacing the first two terms of the RHS above with their expression from 94, using 96 and using the definitions 77 and 78 for $d_2 = 0$, we get 87.

∎

In the case when viscous friction, of the type considered in section 4.1, is present, the momentum equation takes the form below.

Proposition 4.9 *(Momentum Equation with Friction)*
In the presence of friction, the nonholonomic momentum evolves according to the equation

$$\frac{dp}{dt} = [A_1^4(\theta_{1,2})\dot{\theta}_{1,2} - A_1^5(\theta_{1,2})]\ p + [A_2^4(\theta_{1,2})\dot{\theta}_{1,2} + A_2^5(\theta_{1,2})]\ \dot{\theta}_{1,2}\ , \tag{98}$$

where

$$\begin{aligned}A_1^5(\theta_{1,2}) &\overset{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})}[\eta_1(\theta_{1,2})\sin\theta_{1,2} + \eta_2(\theta_{1,2})r(\theta_{1,2})]\ , \\ A_2^5(\theta_{1,2}) &\overset{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})}[\eta_1(\theta_{1,2})\delta(\theta_{1,2}) + \eta_2(\theta_{1,2})\gamma(\theta_{1,2})\sin\theta_{1,2}]\ , \\ \eta_1(\theta_{1,2}) &\overset{\text{def}}{=} 2\left[\frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}\frac{L_2^2}{4} + \frac{k_2}{R_2^2}d_1\lambda(\theta_{1,2})\right]\sin\theta_{1,2}\ , \\ \eta_2(\theta_{1,2}) &\overset{\text{def}}{=} 2\frac{k_1}{R_1^2}r(\theta_{1,2}) + 2\frac{k_2}{R_2^2}\lambda(\theta_{1,2})\cos\theta_{1,2}\ .\end{aligned} \tag{99}$$

If $d_1 > d_2$, *then* $A_1^5(\theta_{1,2}) > 0$, *for all* $\theta_{1,2}$.

Proof

We consider the momentum equation 87 with an external force 1–form α_e which is due to friction and to the torque $\tau_{1,2}$ at the joint $O_{1,2}$. Thus, α_e has the form 68. The force-related terms from the RHS of 87 take, then, the form

$$r(\theta_{1,2})(F_{x_1}\cos\theta_1+F_{y_1}\sin\theta_1)+F_{\theta_1}\sin\theta_{1,2} = -\eta_1(\theta_{1,2})\dot\theta_1-\eta_2(\theta_{1,2})(\dot x_1\cos\theta_1+\dot y_1\sin\theta_1)\ , \tag{100}$$

where η_1 and η_2 are defined in 99.

i) Let $d_2 \neq 0$. For $v = (\dot x_1, \dot y_1, \dot\theta_1, \dot\theta_{1,2}) \in \mathcal{D}_q$, we have from 55

$$\dot x_1\cos\theta_1+\dot y_1\sin\theta_1 = \nu_1\ d_2 \quad \text{and} \quad \dot\theta_1 = \nu_2\ d_2\ ,$$

for $\nu_1, \nu_2 \in I\!R$. Thus,

$$r(\theta_{1,2})(F_{x_1}\cos\theta_1+F_{y_1}\sin\theta_1)+F_{\theta_1}\sin\theta_{1,2} = -\eta_1\nu_2 d_2 - \eta_2\nu_1 d_2\ .$$

Using 92, we get

$$\begin{aligned} r(\theta_{1,2})(F_{x_1}\cos\theta_1+F_{y_1}\sin\theta_1)&+F_{\theta_1}\sin\theta_{1,2} \\ &= -\frac{1}{\Delta(\theta_{1,2})}[\eta_1(\theta_{1,2})\sin\theta_{1,2}+\eta_2(\theta_{1,2})r(\theta_{1,2})]p \\ &\qquad + \frac{1}{\Delta(\theta_{1,2})}[\eta_1(\theta_{1,2})\delta(\theta_{1,2})+\eta_2(\theta_{1,2})\gamma(\theta_{1,2})\sin\theta_{1,2}]\dot\theta_{1,2}\ . \end{aligned} \tag{101}$$

From this, the result follows.

ii) Let $d_2 = 0$. For $v \in \mathcal{D}_q$, we have from 62

$$\dot x_1\cos\theta_1+\dot y_1\sin\theta_1 = \nu_1\ d_1\cos\theta_{1,2} \quad \text{and} \quad \dot\theta_1 = \nu_1\ \sin\theta_{1,2}\ ,$$

for $\nu_1, \nu_2 \in I\!R$. From 96 and 100, the result follows.

It is easy to check from the definitions 99 that

$$A_1^5(\theta_{1,2}) \ \geq\ \frac{2}{\Delta(\theta_{1,2})}\frac{k_2}{R_2^2}\lambda^2(\theta_{1,2})\ . \tag{102}$$

If $d_1 > d_2$, then $\Delta(\theta_{1,2}) > 0$ and also $\lambda(\theta_{1,2}) > 0$ for all $\theta_{1,2}$. Thus, $A_1^5(\theta_{1,2}) > 0$.

∎

Comparing the momentum equations 98 and 76, it is possible to identify the extra terms that are due to friction.

When the shape of the Roller Racer is held constant ($\dot\theta_{1,2} = 0$), the momentum equation for the model without friction 76 takes the form $\dot p = 0$, i.e. the momentum is conserved. However, in the same case, the momentum equation for the model with friction 98 takes the form

$$\frac{dp}{dt} = -A_1^5(\theta_{1,2}(0))p\ , \tag{103}$$

thus

$$p(t) = e^{-A_1^5(\theta_{1,2}(0))t} p(0) \ , \tag{104}$$

for a constant $A_1^5(\theta_{1,2}(0)) > 0$, which is the rate at which the momentum decreases exponentially and at which the system will come to rest. This provides a *braking mechanism* for the Roller Racer, which has been noticed in experiments with the ISL's Roller Racer prototypes.

4.3 Reconstruction of Group Motion

Assume that a shape–space trajectory $\theta_{1,2}(\cdot) \subset \mathcal{S}$ has been specified. The corresponding nonholonomic momentum can be determined from the solution of the momentum equation 76 in the case of the Roller Racer model without external forces, or from the solution of the momentum equation 98 in the case of the Roller Racer model with friction. From the definition of the nonholonomic momentum (equation 70) and from the nonholonomic constraints (equations 16 and 18), we can reconstruct the group trajectory $g_1(\cdot) = g_1(x_1(\cdot), y_1(\cdot), \theta_1(\cdot)) \subset SE(2)$. This can be done by first specifying $(\dot{x}_1, \dot{y}_1, \dot{\theta}_1)$ and then integrating to find (x_1, y_1, θ_1).

Proposition 4.10 *(Reconstruction of Group Trajectory)*
For $\underset{1}{g} = \underset{1}{g}(x_1, y_1, \theta_1) \in SE(2)$, the corresponding curve in the Lie algebra $\underset{1}{\xi} = \underset{1}{g}^{-1}\underset{1}{\dot{g}}$ is given by

$$\underset{1}{\xi} = \underset{1}{\xi}^1(\theta_{1,2}, \dot{\theta}_{1,2})\mathcal{A}_1 + \underset{2}{\xi}^1(\theta_{1,2}, \dot{\theta}_{1,2})\mathcal{A}_2 \ , \tag{105}$$

where for $d_1 \neq d_2$, the components of $\underset{1}{\xi}$ are

$$\begin{aligned} \underset{1}{\xi}^1(\theta_{1,2}, \dot{\theta}_{1,2}) &= \dot{\theta}_1 = \frac{1}{\Delta(\theta_{1,2})}[\sin\theta_{1,2}\, p - \delta(\theta_{1,2})\dot{\theta}_{1,2}] \ , && (106) \\ \underset{2}{\xi}^1(\theta_{1,2}, \dot{\theta}_{1,2}) &= \dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1 = \frac{1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}] \ . && (107) \end{aligned}$$

The group trajectory is given by solving

$$\begin{aligned} \dot{\theta}_1 &= \underset{1}{\xi}^1 = \frac{1}{\Delta(\theta_{1,2})}[\sin\theta_{1,2}\, p - \delta(\theta_{1,2})\dot{\theta}_{1,2}] \ , \\ \dot{x}_1 &= \cos\theta_1\ \underset{2}{\xi}^1 = \frac{\cos\theta_1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}] \ , \\ \dot{y}_1 &= \sin\theta_1\ \underset{2}{\xi}^1 = \frac{\sin\theta_1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}] \ . \end{aligned} \tag{108}$$

The solution can be obtained by quadratures.

Proof

Equation 106 is immediate from 73.

When $d_1 \neq d_2$, either $\sin\theta_{1,2} \neq 0$ or $r(\theta_{1,2}) = d_1 \cos\theta_{1,2} + d_2 \neq 0$.

First consider $\sin\theta_{1,2} \neq 0$. From equations 19 and 73 we get

$$(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1)\sin\theta_{1,2} = \frac{r(\theta_{1,2})}{\Delta}\sin\theta_{1,2}\; p - \frac{\gamma(\theta_{1,2})}{\Delta}\sin^2\theta_{1,2}\;\dot{\theta}_{1,2}\ . \tag{109}$$

Since $\sin\theta_{1,2} \neq 0$, equation 107 follows.
Now let $r(\theta_{1,2}) \neq 0$. From equation 70 we get

$$m_1 r(\theta_{1,2})(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) = p - (I_{z_1} + I_{z_2})\sin\theta_{1,2}\dot{\theta}_1 - I_{z_2}\sin\theta_{1,2}\dot{\theta}_{1,2}\ . \tag{110}$$

From equation 73 we get

$$m_1 r(\theta_{1,2})(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) = p - \frac{I_{z_1} + I_{z_2}}{\Delta}\sin^2\theta_{1,2}\; p + \frac{(I_{z_1} + I_{z_2})\delta - I_{z_2}\Delta}{\Delta}\sin\theta_{1,2}\dot{\theta}_{1,2}\ . \tag{111}$$

Observe that $(I_{z_1} + I_{z_2})\delta(\theta_{1,2}) - I_{z_2}\Delta(\theta_{1,2}) = -m_1 r(\theta_{1,2})\gamma(\theta_{1,2})$. Then

$$m_1 r(\theta_{1,2})(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) = \frac{m_1 r(\theta_{1,2})}{\Delta}[r(\theta_{1,2})\; p - \gamma(\theta_{1,2})\sin\theta_{1,2}\;\dot{\theta}_{1,2}]\ . \tag{112}$$

Since $r(\theta_{1,2}) \neq 0$, equation 107 follows.

Finally, equations 108 are immediate from 107 and 16.

∎

Now observe that from 106 and 107 we get

$$\begin{pmatrix} \xi_1^1 \\ \xi_1^2 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} \sin\theta_{1,2} & -\delta(\theta_{1,2}) \\ r(\theta_{1,2}) & -\gamma(\theta_{1,2})\sin\theta_{1,2} \end{pmatrix}\begin{pmatrix} p \\ \dot{\theta}_{1,2} \end{pmatrix} \overset{\text{def}}{=} B(\theta_{1,2})\begin{pmatrix} p \\ \dot{\theta}_{1,2} \end{pmatrix} \tag{113}$$

and notice that

$$r(\theta_{1,2})\delta(\theta_{1,2}) - \gamma(\theta_{1,2})\sin^2\theta_{1,2} = d_2\Delta(\theta_{1,2})\ , \tag{114}$$

therefore

$$\det B(\theta_{1,2}) = \frac{d_2}{\Delta(\theta_{1,2})}\ . \tag{115}$$

Thus, in the case $d_2 = 0$, given a group trajectory $\xi_1 \subset \mathcal{G}$, we cannot always solve 113 for p and $\dot{\theta}_{1,2}$.

When $d_1 \neq d_2$ and $d_2 \neq 0$, from 113 and 115, we get

$$\begin{pmatrix} p \\ \dot{\theta}_{1,2} \end{pmatrix} = B^{-1}(\theta_{1,2})\begin{pmatrix} \xi_1^1 \\ \xi_1^2 \end{pmatrix} = \frac{1}{d_2}\begin{pmatrix} -\gamma(\theta_{1,2})\sin\theta_{1,2} & \delta(\theta_{1,2}) \\ -r(\theta_{1,2}) & \sin\theta_{1,2} \end{pmatrix}\begin{pmatrix} \xi_1^1 \\ \xi_1^2 \end{pmatrix}\ . \tag{116}$$

4.4 Principal Fiber Bundles and Connections

The following material on principal fiber bundles and connections is based on Bleecker [4] and Nomizu [23]. These references consider principal fiber bundles where the group action is a right action. Here, we consider left actions and modify appropriately the definition of a principal fiber bundle, as for instance done in (Yang [34]).

Let $\mathcal{S}$ be a differentiable manifold and G a Lie group. A differentiable manifold Q is called a (differentiable) *principal fiber bundle*, if the following conditions are satisfied:
1) G acts on Q to the left, freely and differentiably:

$$\Phi : \ G \times Q \to Q : \ (g, q) \mapsto g \cdot q \overset{\text{def}}{=} \Phi_g \cdot q \, . \tag{117}$$

2) $\mathcal{S}$ is the quotient space of Q by the equivalence relation induced by G, i.e. $\mathcal{S} = Q/G$ and the canonical projection $\pi : Q \to \mathcal{S}$ is differentiable.
3) Q is locally trivial, i.e. every point $s \in \mathcal{S}$ has a neighborhood U such that $\pi^{-1}(U) \subset Q$ is isomorphic with $U \times G$, in the sense that $q \in \pi^{-1}(U) \mapsto (\pi(q), \phi(q)) \in U \times G$ is a diffeomorphism such that $\phi : \pi^{-1}(U) \to G$ satisfies $\phi(g \cdot q) = g\phi(q), \forall g \in G$.
For $s \in \mathcal{S}$, the *fiber over* s is a closed submanifold of Q which is differentiably isomorphic with G. For any $q \in Q$, the *fiber through* q is the fiber over $s = \pi(q)$. When $Q = \mathcal{S} \times G$, then Q is said to be a *trivial* principal fiber bundle (fig. 4).

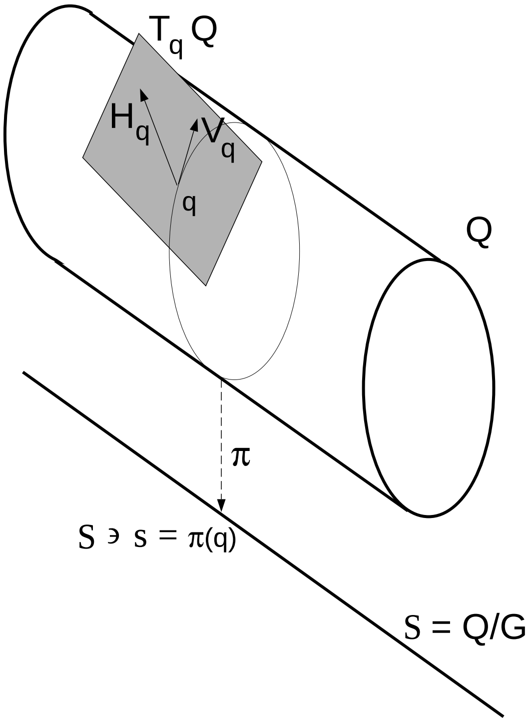


Figure 4: Connection on a Principal Fiber Bundle

Definition 4.1 *Let* $(Q, \mathcal{S}, \pi, G)$ *be a principal fiber bundle. The kernel of* $T_q\pi$*, denoted* $V_q \overset{\text{def}}{=} \{v \in T_qQ \mid T_q\pi(v) = 0\}$*, is the subspace of* T_qQ *tangent to the fiber through* q *and*

is called the vertical subspace. A connection on the principal fiber bundle is a choice of a tangent subspace $H_q \subset T_qQ$ at each point $q \in Q$, called the horizontal subspace, such that
1) $T_qQ = H_q \oplus V_q$.
2) For every $g \in G$ and $q \in Q$, $T_q\Phi_g \cdot H_q = H_{g\cdot q}$.
3) H_q depends differentiably on q.

4.5 The Nonholonomic Connection

In this section, we explicitly realize a connection on the bundle $(Q, \mathcal{S}, \pi, G)$, the nonholonomic connection of [5] for the Roller Racer.

Consider the *kinetic energy inner product* $\ll \, , \, \gg$ specified by the Lagrangian 44:

$$\ll v, \tilde{v} \gg \overset{\text{def}}{=} v^\top \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & I_{z_1} + I_{z_2} & I_{z_2} \\ 0 & 0 & I_{z_2} & I_{z_2} \end{pmatrix} \tilde{v} \, , \tag{118}$$

for $v, \tilde{v} \in T_qQ$.

Proposition 4.11 *The orthogonal complement H_q of the subspace $\mathcal{S}_q$ with respect to the constraint subspace $\mathcal{D}_q$, i.e.*

$$\mathcal{S}_q \oplus H_q = \mathcal{D}_q \, , \tag{119}$$

where orthogonality is defined with respect to the kinetic energy inner product $\ll, \gg$, is given by

$$H_q = \mathrm{sp}\{\xi_Q^H\} \, , \tag{120}$$

where

$$\xi_Q^H = \gamma(\theta_{1,2}) \sin\theta_{1,2} (\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \delta(\theta_{1,2}) \frac{\partial}{\partial \theta_1} - \Delta(\theta_{1,2}) \frac{\partial}{\partial \theta_{1,2}} \, . \tag{121}$$

Proof

Since dim $\mathcal{S}_q = 1$ and dim $\mathcal{D}_q = 2$, we should have dim $H_q = 1$. Consider an element $\xi_Q^H \in H_q$. As ξ_Q^H also belongs to $\mathcal{D}_q$, it can be written, as a function of the basis elements of $\mathcal{D}_q$, as

$$\xi_Q^H = \alpha_1 \xi_Q^1 + \alpha_2 \xi_Q^2 \, , \tag{122}$$

for some $\alpha_1, \alpha_2 \in I\!R$.

i) When $d_2 \neq 0$, we have from 23

$$\xi_Q^H \;=\; \alpha_1 d_2 (\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \alpha_2 d_2 \frac{\partial}{\partial \theta_1} + [\alpha_1 \sin\theta_{1,2} - \alpha_2 r(\theta_{1,2})] \frac{\partial}{\partial \theta_{1,2}} \, . \tag{123}$$

The vector ξ_Q^H should be orthogonal to every $\xi_Q^q \in \mathcal{S}_q$, i.e. $\ll \xi_Q^q, \xi_Q^H \gg = 0$. This gives $\delta(\theta_{1,2})\alpha_1 - \gamma(\theta_{1,2}) \sin\theta_{1,2} \alpha_2 = 0$. Choose $\alpha_1 = \gamma(\theta_{1,2}) \sin\theta_{1,2}$ and $\alpha_2 = \delta(\theta_{1,2})$. Using 114 and dividing by d_2, we get 121.

ii) In the case $d_2 = 0$, we have from 24:

$$\xi_Q^H = \alpha_1 d_1 \cos\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \alpha_1 \sin\theta_{1,2} \frac{\partial}{\partial \theta_1} + \alpha_2 \frac{\partial}{\partial \theta_{1,2}} \ . \tag{124}$$

From orthogonality to ξ_Q^q, we obtain $\alpha_1 = I_{z_2} \sin\theta_{1,2}$ and $\alpha_2 = [(I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1 d_1^2 \cos^2\theta_{1,2}]$, thus

$$\begin{aligned} \xi_Q^H = {} & I_{z_2} d_1 \sin\theta_{1,2} \cos\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + I_{z_2} \sin^2\theta_{1,2} \frac{\partial}{\partial \theta_1} \\ & - [(I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1 d_1^2 \cos^2\theta_{1,2}] \frac{\partial}{\partial \theta_{1,2}} \ . \end{aligned} \tag{125}$$

Note that, when $d_2 = 0$, this expression is the same as 121.

■

Proposition 4.12 *The orthogonal complement $\mathcal{U}_q$ of the subspace $\mathcal{S}_q$ with respect to the subspace $T_qOrb(q)$, i.e.*

$$\mathcal{S}_q \oplus \mathcal{U}_q = T_qOrb(q) \ , \tag{126}$$

where orthogonality is defined with respect to the kinetic energy inner product $\ll, \gg$, is given by

$$\mathcal{U}_q = \mathrm{sp}\{\xi_Q^{\mathcal{U}_1}, \ \xi_Q^{\mathcal{U}_2}\} \ , \tag{127}$$

where

$$\begin{aligned} \xi_Q^{\mathcal{U}_1} &= -\sin\theta_1 \frac{\partial}{\partial x_1} + \cos\theta_1 \frac{\partial}{\partial y_1} \ , \\ \xi_Q^{\mathcal{U}_2} &= (I_{z_1} + I_{z_2}) \sin\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) - m_1(d_1 \cos\theta_{1,2} + d_2) \frac{\partial}{\partial \theta_1} \ . \end{aligned} \tag{128}$$

Proof

Since $\dim \mathcal{S}_q = 1$ and $\dim T_qOrb(q) = 3$, we have $\dim \mathcal{U}_q = 2$. Let $\xi_Q^{\mathcal{U}_1}$ and $\xi_Q^{\mathcal{U}_2}$ be two basis elements of $\mathcal{U}_q$. Since the $\xi_Q^{\mathcal{U}_i}$, $i = 1, 2$ also belong to $T_qOrb(q)$, they can be expressed as a function of its basis elements as $\xi_Q^{\mathcal{U}_i} = u_1^i \frac{\partial}{\partial x_1} + u_2^i \frac{\partial}{\partial y_1} + u_3^i \frac{\partial}{\partial \theta_1}$, for some $u_j^i \in I\!R$. The $\xi_Q^{\mathcal{U}_i}$ need to be mutually linearly independent and orthogonal to $\xi_Q^q \in \mathcal{S}_q$. This last requirement gives

$$m_1 r(\theta_{1,2})(u_1^i \cos\theta_1 + u_2^i \sin\theta_1) + (I_{z_1} + I_{z_2}) \sin\theta_{1,2} u_3^i = 0, \ i = 1, 2 \ .$$

Two linearly independent vectors that fulfill this condition are the ones given in 128.

■

The configuration space for the Roller Racer is $Q = SE(2) \times S^1$. From left invariance of the system's kinematics, the tangent space T_qQ to the configuration space is

$$T_qQ = \{(\dot{g}_1, \dot{\theta}_{1,2}) \mid g_1 \in SE(2), \theta_{1,2} \in S^1\} = \{(g_1\xi_1, \dot{\theta}_{1,2}) \mid \xi_1 \in se(2), \dot{\theta}_{1,2} \in I\!R\} \; . \tag{129}$$

Consider, then, the configuration space $Q = SE(2) \times S^1$, the group $G = SE(2)$, the shape space $\mathcal{S} = S^1$ of the Roller Racer, which is the quotient space of Q by G, and the canonical projection

$$\pi : \; Q \longrightarrow \mathcal{S} : \; (g_1, \theta_{1,2}) \mapsto \theta_{1,2} \; . \tag{130}$$

The projection π is differentiable and its differential at $q = (g_1, \; \theta_{1,2}) \in Q$ is

$$T_q\pi : \; T_qQ \longrightarrow T_{\pi(q)}\mathcal{S} : \; (g_1\xi_1, \dot{\theta}_{1,2}) \mapsto \dot{\theta}_{1,2} \; . \tag{131}$$

The quadruple $(Q, \mathcal{S}, \pi, G)$, together with the left action Φ of G on Q defined by equation 28, is a (trivial) *principal fiber bundle.* This bundle expresses the ultimate dependence of all configuration variables on the shape $\theta_{1,2}$.

By considering the Lagrangian dynamics in addition to the kinematic constraints, we can synthesize a principal connection for this system, which reflects the dependence of all configuration velocities on the shape variation $\dot{\theta}_{1,2}$.

Proposition 4.13 *(Nonholonomic Connection)*

The nonholonomic kinematic constraints and the system dynamics determine a connection on the principal fiber bundle $(Q, \mathcal{S}, \pi, G)$. The horizontal subspace of the connection is the subspace H_q defined in 120 and 121, i.e. the orthogonal complement of the subspace $\mathcal{S}_q$ with respect to the constraint distribution $\mathcal{D}_q$, with orthogonality defined with respect to the kinetic energy inner product. When $d_1 \neq d_2$, the horizontal subspace is

$$\begin{aligned} H_q \; &\overset{\text{def}}{=} \; \{v \in T_qQ \mid v \in \text{sp}\{\xi^H_Q\}\} \\ &= \; \{v = (\dot{x}_1, \; \dot{y}_1, \; \dot{\theta}_1, \; \dot{\theta}_{1,2}) \in T_qQ \mid \\ &\qquad \dot{x}_1 = -\frac{\gamma(\theta_{1,2}) \sin\theta_{1,2}}{\Delta(\theta_{1,2})} \cos\theta_1 \; \dot{\theta}_{1,2}, \\ &\qquad \dot{y}_1 = -\frac{\gamma(\theta_{1,2}) \sin\theta_{1,2}}{\Delta(\theta_{1,2})} \sin\theta_1 \; \dot{\theta}_{1,2}, \\ &\qquad \dot{\theta}_1 = -\frac{\delta(\theta_{1,2})}{\Delta(\theta_{1,2})} \; \dot{\theta}_{1,2} \; \} \; . \end{aligned} \tag{132}$$

The vertical subspace of the connection is

$$V_q \overset{\text{def}}{=} \{v \in T_qQ \mid T_q\pi = 0\} = \{v = (\dot{x}_1, \; \dot{y}_1, \; \dot{\theta}_1, \; \dot{\theta}_{1,2}) \in T_qQ \mid \; \dot{\theta}_{1,2} = 0\} \; . \tag{133}$$

Proof

It is easy to see that the horizontal subspace H_q defined in 132 is

$$H_q = \{(g_{_1}\xi_{_1},\ \dot{\theta}_{1,2}) \mid g_{_1} \in SE(2)\ ,\ \xi_{_1} = -A_{loc}(\theta_{1,2})\,\dot{\theta}_{1,2}\}\ , \tag{134}$$

where the local form A_{loc} of the connection [5] is

$$A_{loc}(\theta_{1,2}) \overset{\text{def}}{=} \frac{\delta(\theta_{1,2})}{\Delta(\theta_{1,2})}\mathcal{A}_1 + \frac{\gamma(\theta_{1,2})\sin\theta_{1,2}}{\Delta(\theta_{1,2})}\mathcal{A}_2\ \in se(2)\ . \tag{135}$$

To show property (1) of Definition 4.1, consider a non–zero vector $v \in V_q \cap H_q$. Since v is non–zero and belongs to H_q, we have from 134 that $\dot{\theta}_{1,2} \neq 0$. But then, because of 133, v cannot belong also to V_q, as we supposed. Thus, $V_q \cap H_q = \{0\}$. Moreover, dim V_q+dim H_q = $3 + 1 = 4 = \dim T_qQ$. Thus, $V_q \oplus H_q = T_qQ$.

To show property (2) of Definition 4.1, consider a $g \in G$. From left–invariance

$$\begin{aligned} T_q\Phi_g \cdot H_q &= g \cdot H_q = g \cdot \{(g_{_1}\xi_{_1}, \dot{\theta}_{1,2}) \mid \xi_{_1} = -A_{loc}(\theta_{1,2})\dot{\theta}_{1,2}\} \\ &\overset{\text{def}}{=} \{(gg_{_1}\xi_{_1}, \dot{\theta}_{1,2}) \mid \xi_{_1} = -A_{loc}(\theta_{1,2})\dot{\theta}_{1,2}\} \end{aligned} \tag{136}$$

and

$$H_{g\cdot q} = \{v \in T_{g\cdot q}Q \mid \xi_{_1} = -A_{loc}(\theta_{1,2})\dot{\theta}_{1,2}\} = \{(gg_{_1}\xi_{_1}, \dot{\theta}_{1,2}) \mid \xi_{_1} = -A_{loc}(\theta_{1,2})\dot{\theta}_{1,2}\}\ . \tag{137}$$

Then, obviously, $T_q\Phi_g \cdot H_q = H_{g\cdot q}$.

The differentiability of H_q with respect to $q \in Q$ (property (3) of Definition 4.1) follows from the smooth dependence of A_{loc} on the shape $\theta_{1,2}$ and from the left–invariance of our system.

■

Physically, V_q is the set of all possible rigid motions of the system on the plane that keep shape constant; these "frozen–shape" motions do not need to satisfy the nonholonomic constraints. On the other hand, H_q is the set of all possible motions of the system on the plane that comply with the nonholonomic constraints. Observe that all such motions are due to shape variations.

Let the set of Lie algebra elements, whose infinitesimal generators belong to $\mathcal{S}_q$, be denoted as $\mathcal{G}^q$. From 40: $\mathcal{G}^q = \mathrm{sp}\{\xi^q\}$. The *locked inertia tensor* $\mathbb{I}(q)$ relative to $\mathcal{G}^q$ is defined in [21], [5] as

$$\mathbb{I}(q) :\ \mathcal{G}^q \longrightarrow (\mathcal{G}^q)^\star :\ \xi^q \longmapsto < \mathbb{I}(q)\,\xi^q,\ \cdot\ >\ , \tag{138}$$

where, for $\eta^q \in \mathcal{G}^q$, with corresponding infinitesimal generator $\eta^q_Q \in \mathcal{S}_q$, we define

$$< \mathbb{I}(q)\,\xi^q,\ \eta^q > \overset{\text{def}}{=} \ll \xi^q_Q,\ \eta^q_Q \gg\ , \tag{139}$$

and where $\ll\ ,\ \gg$ is the kinetic energy inner product defined in 118.

It is easy to verify from 33 and 71, that for the Roller Racer

$$< \mathbb{I}(q)\,\xi^q,\ \xi^q > = \ll \xi^q_Q,\ \xi^q_Q \gg = \Delta(\theta_{1,2})\ . \tag{140}$$

Since $\eta^q_Q = \beta\, \xi^q_Q$, for some $\beta \in I\!R$, we have

$$< I\!I(q)\, \xi^q,\ \eta^q > = \ll \xi^q_Q,\ \eta^q_Q \gg = \beta\ \ll \xi^q_Q,\ \xi^q_Q \gg = \beta\, \Delta(\theta_{1,2})\ . \tag{141}$$

With the above definition of A_{loc} in equation 135, the reconstructed group trajectory equations 105, 106 and 107 take the form:

$$\xi_{_1} = g_{_1}^{-1} \dot{g}_{_1} = -A_{loc}(\theta_{1,2})\, \dot{\theta}_{1,2} + I\!I^{-1}_{loc}(\theta_{1,2})\, p\ , \tag{142}$$

where

$$I\!I^{-1}_{loc}(\theta_{1,2}) = \frac{\sin\theta_{1,2}}{\Delta(\theta_{1,2})}\mathcal{A}_1 + \frac{r(\theta_{1,2})}{\Delta(\theta_{1,2})}\mathcal{A}_2 \tag{143}$$

is the local form of the inverse of the locked inertia tensor of the Roller Racer.

4.6 The Reduced Dynamics

A Lagrangian reduction procedure for systems with nonholonomic constraints is developed in (Bloch, Krishnaprasad, Marsden and Murray [5]). Its goal is to lower the dimension of the system's dynamics by passing to an appropriate quotient space. The reduced equations are composed of a set of Euler–Lagrange equations on the shape space of the system, where some of the forcing terms are due to the curvature of the nonholonomic connection, and a set of momentum equations. These reduced equations, together with the reconstruction ones and the nonholonomic constraints, give the full set of equations of motion of the system.

In the case of the Roller Racer, the reduced dynamics are composed of an Euler–Lagrange equation on the one–dimensional shape space $\mathcal{S}$ and of the momentum equation derived in section 4.2. The reduced Euler–Lagrange equation is a second–order equation on the shape variable $\theta_{1,2}$ and involves the reduced Lagrangian

$$l_c(\theta_{1,2},\ \dot{\theta}_{1,2},\ p) \overset{\text{def}}{=} L\Big(q, \dot{q}(q,\ \dot{\theta}_{1,2},\ p)\Big) \overset{44,108}{=} \frac{1}{2}\frac{1}{\Delta(\theta_{1,2})}\, p^2 + \frac{1}{2}\frac{\Delta_1(\theta_{1,2})}{\Delta(\theta_{1,2})}\, \dot{\theta}^2_{1,2}\ , \tag{144}$$

as described in [5]. The curvature of the nonholonomic connection of section 4.5 can be easily seen to be zero, since the shape space is one–dimensional, thus the corresponding forcing terms are zero.

In our derivation of the reduced dynamics of the Roller Racer, we take a shortcut around [5], by employing directly the Lagrange–d'Alembert principle with a test vector horizontal with respect to the nonholonomic connection, but still in the constraint distribution, thus belonging to the orthogonal complement H_q of $\mathcal{S}_q$ with respect to $\mathcal{D}_q$.

Let

$$\Delta_1(\theta_{1,2}) \overset{\text{def}}{=} I_{z_1} I_{z_2} \sin^2\theta_{1,2} + m_1(I_{z_1} d_2^2 + I_{z_2} d_1^2 \cos^2\theta_{1,2})\ . \tag{145}$$

Observe that $\Delta_1(\theta_{1,2}) > 0,\ \forall q \in Q$.

Proposition 4.14 *(Reduced Dynamics with External Forces)*

Consider external forcing to the system described by the 1–form $\alpha_e = (F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}})$. *The reduced dynamics of the Roller Racer take the form*

$$\begin{aligned}
\ddot{\theta}_{1,2} &= B_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + B_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \\
&\qquad\qquad + B_3^4(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + B_4^4(\theta_{1,2})F_{\theta_1} + B_5^4(\theta_{1,2})F_{\theta_{1,2}} \ , \\
\frac{dp}{dt} &= A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \sin\theta_{1,2}F_{\theta_1} \ ,
\end{aligned} \tag{146}$$

where

$$\begin{aligned}
B_1^4(\theta_{1,2}) &\overset{\text{def}}{=} -\frac{A_2^4(\theta_{1,2})}{\Delta_1(\theta_{1,2})} \ , \\
B_2^4(\theta_{1,2}) &\overset{\text{def}}{=} -\frac{1}{2}\frac{\Delta}{\Delta_1}\frac{\partial}{\partial\theta_{1,2}}\left(\frac{\Delta_1}{\Delta}\right) = \frac{m_1\gamma(\theta_{1,2})\sin\theta_{1,2}}{\Delta(\theta_{1,2})\Delta_1(\theta_{1,2})}[\gamma(\theta_{1,2})\cos\theta_{1,2} + d_1\delta(\theta_{1,2})] \ , \\
B_3^4(\theta_{1,2}) &\overset{\text{def}}{=} -\frac{\gamma(\theta_{1,2})\sin\theta_{1,2}}{\Delta_1(\theta_{1,2})} \ , \quad B_4^4(\theta_{1,2}) \overset{\text{def}}{=} -\frac{\delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})} \ , \quad B_5^4(\theta_{1,2}) \overset{\text{def}}{=} \frac{\Delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})} \ .
\end{aligned} \tag{147}$$

For all $q \in Q$, *we have* $B_5^4(\theta_{1,2}) > 0$.

Proof

Consider the Lagrange–d'Alembert principle 53 with a test vector $u \in \mathcal{D}_q$, which we restrict to the orthogonal complement H_q of $\mathcal{S}_q$ with respect to $\mathcal{D}_q$. Orthogonality is defined using the kinetic energy inner product $\ll \, , \, \gg$ defined in 118. Without loss of generality, we choose u as

$$u = \xi_Q^H = \gamma(\theta_{1,2})\sin\theta_{1,2}(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \delta(\theta_{1,2})\frac{\partial}{\partial\theta_1} - \Delta(\theta_{1,2})\frac{\partial}{\partial\theta_{1,2}} \ . \tag{148}$$

i) Let $d_2 \neq 0$. From 53, with u given by 148 and $\dot{v}$ given by 57, and since $\dot{v}_4 \equiv \ddot{\theta}_{1,2}$, we have

$$\begin{aligned}
&m_1 d_2\gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\nu}_1 \\
&\qquad + [I_{z_1}\delta(\theta_{1,2}) + I_{z_2}(\delta(\theta_{1,2}) - \Delta(\theta_{1,2}))]d_2\dot{\nu}_2 + I_{z_2}[\delta(\theta_{1,2}) - \Delta(\theta_{1,2})]\ddot{\theta}_{1,2} \\
&\qquad\qquad = \gamma(\theta_{1,2})\sin\theta_{1,2}(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\delta(\theta_{1,2}) - F_{\theta_{1,2}}\Delta(\theta_{1,2}) \ ,
\end{aligned} \tag{149}$$

for $\nu_1, \nu_2 \in I\!R$. Observe that $\Delta(\theta_{1,2}) - \delta(\theta_{1,2}) = I_{z_1}\sin^2\theta_{1,2} + m_1 d_1\cos\theta_{1,2}r(\theta_{1,2})$, then $I_{z_1}\delta(\theta_{1,2}) + I_{z_2}[\delta(\theta_{1,2}) - \Delta(\theta_{1,2})] = -m_1\gamma(\theta_{1,2})r(\theta_{1,2})$. Then, the LHS of 149 becomes

$$m_1 d_2\gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\nu}_1 - m_1 d_2\gamma(\theta_{1,2})r(\theta_{1,2})\dot{\nu}_2 + I_{z_2}[\delta(\theta_{1,2}) - \Delta(\theta_{1,2})]\ddot{\theta}_{1,2} \ . \tag{150}$$

By differentiating 92, the terms $\dot{\nu}_1$ and $\dot{\nu}_2$ above can be expressed as functions of $\theta_{1,2}$, $\dot{\theta}_{1,2}$, $\ddot{\theta}_{1,2}$, p, $\dot{p}$:

$$\begin{aligned}
\dot{\nu}_1 \;&=\; \frac{1}{d_2\Delta^2}\Big[r(\theta_{1,2})\Delta(\theta_{1,2})\dot{p} + (\frac{\partial r}{\partial\theta_{1,2}}\Delta(\theta_{1,2}) - r(\theta_{1,2})\frac{\partial\Delta}{\partial\theta_{1,2}})\dot{\theta}_{1,2}\, p \\
&\qquad - \gamma(\theta_{1,2})\Delta(\theta_{1,2})\sin\theta_{1,2}\ddot{\theta}_{1,2} \\
&\qquad + \Big(\gamma(\theta_{1,2})\sin\theta_{1,2}\frac{\partial\Delta}{\partial\theta_{1,2}} - \frac{\partial\gamma}{\partial\theta_{1,2}}\Delta(\theta_{1,2})\sin\theta_{1,2} - \gamma(\theta_{1,2})\Delta(\theta_{1,2})\cos\theta_{1,2}\Big)\dot{\theta}_{1,2}^2\Big], \\
\dot{\nu}_2 \;&=\; \frac{1}{d_2\Delta^2}\Big[\Delta(\theta_{1,2})\sin\theta_{1,2}\dot{p} + (\Delta(\theta_{1,2})\cos\theta_{1,2} - \frac{\partial\Delta}{\partial\theta_{1,2}}\sin\theta_{1,2})\dot{\theta}_{1,2}\, p \\
&\qquad - \Delta(\theta_{1,2})\delta(\theta_{1,2})\ddot{\theta}_{1,2} + (\frac{\partial\Delta}{\partial\theta_{1,2}}\delta(\theta_{1,2}) - \frac{\partial\delta}{\partial\theta_{1,2}}\Delta(\theta_{1,2}))\dot{\theta}_{1,2}^2\Big] \; .
\end{aligned} \tag{151}$$

Thus, the LHS of 149 becomes, after some calculations using 150 and 151:

$$\begin{aligned}
&\Big[m_1 d_2\gamma(\theta_{1,2}) + I_{z_2}(\delta(\theta_{1,2}) - \Delta(\theta_{1,2}))\Big]\ddot{\theta}_{1,2} \\
&\quad + \frac{m_1\gamma(\theta_{1,2})}{\Delta(\theta_{1,2})}\Big(\frac{\partial r}{\partial\theta_{1,2}}\sin\theta_{1,2} - r(\theta_{1,2})\cos\theta_{1,2}\Big)\dot{\theta}_{1,2}\, p \\
&\quad - \frac{m_1\gamma(\theta_{1,2})}{\Delta(\theta_{1,2})}\Big(d_2\frac{\partial\Delta}{\partial\theta_{1,2}} + \frac{\partial\gamma}{\partial\theta_{1,2}}\sin^2\theta_{1,2} + \gamma(\theta_{1,2})\sin\theta_{1,2}\cos\theta_{1,2} - r(\theta_{1,2})\frac{\partial\delta}{\partial\theta_{1,2}}\Big)\dot{\theta}_{1,2}^2 \; .
\end{aligned} \tag{152}$$

The parenthesis in the second term above can be shown to be equal to $-\lambda(\theta_{1,2})$, while the parenthesis of the third term can be shown to be equal to $-[\gamma(\theta_{1,2})\cos\theta_{1,2} + d_1\delta(\theta_{1,2})]\sin\theta_{1,2}$. Thus, equation 146 follows.

ii) Let $d_2 = 0$. From 53, with u given by 148 and $\dot{v}$ given by 64, we have

$$\begin{aligned}
&[(I_{z_1} + I_{z_2})\delta(\theta_{1,2}) - I_{z_2}\Delta(\theta_{1,2}) + m_1 d_1\gamma(\theta_{1,2})\cos\theta_{1,2}]\sin\theta_{1,2}\dot{\nu}_1 + I_{z_2}[\delta(\theta_{1,2}) - \Delta(\theta_{1,2})]\dot{\nu}_2 \\
&\quad + [(I_{z_1} + I_{z_2})\delta(\theta_{1,2})\cos\theta_{1,2} - I_{z_2}\Delta(\theta_{1,2})\cos\theta_{1,2} - m_1 d_1\gamma(\theta_{1,2})\sin^2\theta_{1,2}]\dot{\theta}_{1,2}\,\nu_1 \\
&\qquad = \gamma(\theta_{1,2})\sin\theta_{1,2}(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \delta(\theta_{1,2})F_{\theta_1} - \Delta(\theta_{1,2})F_{\theta_{1,2}} \; ,
\end{aligned} \tag{153}$$

for $\nu_1, \nu_2 \in \mathbb{R}$. Observe that the coefficient of $\dot{\nu}_1$ is zero, while $I_{z_2}[\delta(\theta_{1,2}) - \Delta(\theta_{1,2})] = -\Delta_1(\theta_{1,2})$ and $(I_{z_1} + I_{z_2})\delta(\theta_{1,2})\cos\theta_{1,2} - I_{z_2}\Delta(\theta_{1,2})\cos\theta_{1,2} - m_1 d_1\gamma(\theta_{1,2})\sin^2\theta_{1,2} = -m_1 I_{z_2} d_1^2\cos\theta_{1,2}$. Notice that from 64, we have $\ddot{\theta}_{1,2} \equiv \dot{v}_4 = \dot{\nu}_2$. Using this in 153 and rearranging terms, the result follows.

■

The following two results are special cases of Proposition 4.14.

Proposition 4.15 *(Reduced Dynamics without External Forces)*

In the absence of external forces or torques, other than the torque $\tau_{1,2}$ applied to the joint $O_{1,2}$, the reduced dynamics of the Roller Racer, take the form

$$\begin{aligned} \ddot{\theta}_{1,2} &= B_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + B_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 + B_5^4(\theta_{1,2})\tau_{1,2} \; , \\ \frac{dp}{dt} &= A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \; . \end{aligned} \tag{154}$$

Proposition 4.16 *(Reduced Dynamics with Friction)*

In the presence of friction, the reduced dynamics of the Roller Racer take the form

$$\begin{aligned} \ddot{\theta}_{1,2} &= \left[B_1^4(\theta_{1,2})\dot{\theta}_{1,2} + B_6^4(\theta_{1,2})\right] p + \left[B_2^4(\theta_{1,2})\dot{\theta}_{1,2} - B_7^4(\theta_{1,2})\right] \dot{\theta}_{1,2} + B_5^4(\theta_{1,2})\, \tau_{1,2} \; , \\ \frac{dp}{dt} &= \left[A_1^4(\theta_{1,2})\dot{\theta}_{1,2} - A_1^5(\theta_{1,2})\right] p + \left[A_2^4(\theta_{1,2})\dot{\theta}_{1,2} + A_2^5(\theta_{1,2})\right] \dot{\theta}_{1,2} \; , \end{aligned} \tag{155}$$

where B_1^4, B_2^4 and B_5^4 were defined previously in 147 and where

$$B_6^4(\theta_{1,2}) \overset{\text{def}}{=} \frac{A_2^5(\theta_{1,2})}{\Delta_1(\theta_{1,2})} \; , \;\; B_7^4(\theta_{1,2}) \overset{\text{def}}{=} \frac{A_3^5(\theta_{1,2})}{\Delta_1(\theta_{1,2})} \; , \tag{156}$$

with A_2^5 as defined in 99, with Δ_1 as defined in 145 and with

$$A_3^5(\theta_{1,2}) \overset{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})} \left[\eta_3(\theta_{1,2})\gamma(\theta_{1,2}) \sin\theta_{1,2} + \eta_4(\theta_{1,2})\delta(\theta_{1,2}) + \eta_5(\theta_{1,2})\Delta(\theta_{1,2})\right] \; , \tag{157}$$

where

$$\begin{aligned} \eta_3(\theta_{1,2}) &\overset{\text{def}}{=} 2\left[\left(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\right)\gamma(\theta_{1,2}) + \frac{k_2}{R_2^2} d_1 \delta(\theta_{1,2}) \cos\theta_{1,2}\right] \sin\theta_{1,2} \; , \\ \eta_4(\theta_{1,2}) &\overset{\text{def}}{=} 2\frac{k_2}{R_2^2} d_1 \gamma(\theta_{1,2}) \sin^2\theta_{1,2} \cos\theta_{1,2} + 2\left(\frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}\frac{L_2^2}{4} + \frac{k_2}{R_2^2} d_1^2 \sin^2\theta_{1,2}\right)\delta(\theta_{1,2}) \; , \\ \eta_5(\theta_{1,2}) &\overset{\text{def}}{=} 2\frac{k_2}{R_2^2}\frac{L_2^2}{4}\Delta(\theta_{1,2}) \; . \end{aligned} \tag{158}$$

If $d_1 \neq d_2$, then $A_3^5(\theta_{1,2}) > 0$, for all $\theta_{1,2}$.

Proof

From the reduced dynamics of the Roller Racer with external forces given by equation 146 and from the external force 1–form due to friction given by equation 68, we have for the last three terms of 146, using the definitions of 147:

$$\begin{aligned} & B_3^4(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + B_4^4(\theta_{1,2})F_{\theta_1} + B_5^4(\theta_{1,2})F_{\theta_{1,2}} = \\ & \qquad = \frac{1}{\Delta_1(\theta_{1,2})}\eta_3(\theta_{1,2})(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) + \frac{1}{\Delta_1(\theta_{1,2})}\eta_4(\theta_{1,2})\dot{\theta}_1 \\ & \qquad\qquad - \frac{1}{\Delta_1(\theta_{1,2})}\eta_5(\theta_{1,2})\dot{\theta}_{1,2} + \frac{\Delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})}\tau_{1,2} \; , \end{aligned} \tag{159}$$

where η_3, η_4 and η_5 are defined in 157.

i) Let $d_2 \neq 0$. For $v = (\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2}) \in \mathcal{D}_q$, we have from 55

$$\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1 = \nu_1 d_2 \ \text{ and } \ \dot{\theta}_1 = \nu_2 d_2 \ ,$$

for $\nu_1, \nu_2 \in I\!R$. From this and from 92 we get

$$\begin{aligned} B_3^4(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + B_4^4(\theta_{1,2})F_{\theta_1} + B_5^4(\theta_{1,2})F_{\theta_{1,2}} \\ = B_6^4(\theta_{1,2})\, p - B_7^4(\theta_{1,2})\, \dot{\theta}_{1,2} + B_5^4(\theta_{1,2})\, \tau_{1,2} \ . \end{aligned} \tag{160}$$

It is an easy calculation to show that

$$B_6^4(\theta_{1,2}) \overset{\text{def}}{=} \frac{1}{\Delta_1(\theta_{1,2})}\frac{1}{\Delta(\theta_{1,2})}[\eta_3(\theta_{1,2})r(\theta_{1,2}) + \eta_4(\theta_{1,2})\sin\theta_{1,2}] = \frac{1}{\Delta_1(\theta_{1,2})}A_2^5(\theta_{1,2}) \ ,$$

with $A_2^5(\theta_{1,2})$ as defined in 99.

ii) Let $d_2 = 0$. From equations 62 and 96, we get, similarly to case (i), the desired result.

∎

Remark 4.2 Consider the unforced Roller Racer dynamics, where $\alpha_e = 0$. From equation 146, the dynamics of the shape variable $\theta_{1,2}$ are $\ddot{\theta}_{1,2} = B_1^4(\theta_{1,2})\, \dot{\theta}_{1,2}\, p + B_2^4(\theta_{1,2})\, \dot{\theta}_{1,2}^2$. It can be easily seen that the reduced Lagrangian l_c, defined in 144, is *conserved* on the trajectories of this system.

5 Controllability and Motion Control of the Roller Racer

We are interested in controlling a nonlinear system where the number of controls is less than the dimension of its state space and whose tangent linearization is uncontrollable. Tools from nonlinear control theory are, then, necessary to analyze it. The discussion in this section follows Nijmeier and van der Schaft [22], unless otherwise noted.

Consider the smooth affine nonlinear control system

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j \ , \tag{161}$$

where x are local coordinates for the smooth manifold M with $\dim M = n$ and $u : [0, T] \to U \subset I\!R^m$ is the set of admissible controls. The unique solution of 161 at time $t \geq t_0$ with initial condition $x(t_0) = x_0$ and input function $u(\cdot)$ is denoted $x(t, t_0, x_0, u)$ or simply $x(t)$.

The *reachable set* $R^V(x_0, T)$ is the set of points in M which are reachable from $x_0 \in M$ at exactly time $T > 0$, following system trajectories which, for $t \leq T$, remain in the neighborhood V of x_0. Consider also $R^V_T(x_0) \overset{\text{def}}{=} \bigcup_{t \leq T} R^V(x_0, t)$, the set of points in M reachable from x_0 at time less or equal to T.

The system 161 is locally *accessible* from $x_0 \in M$, if, for any neighborhood V of x_0 and all $T > 0$, the set $R^V_T(x_0)$ contains a non–empty open set. If the system is locally accessible from any $x_0 \in M$, then it is locally accessible. The system 161 is locally *strongly accessible* from $x_0 \in M$, if, for any neighborhood V of x_0 and for any $T > 0$ sufficiently small, the set $R^V(x_0, T)$ contains a non–empty open set.

The *strong accessibility algebra* $\mathcal{C}_0$ is the smallest subalgebra of the Lie algebra of smooth vector fields on M containing the control vector fields $g_1, \ldots, g_m$, which is invariant under the drift vector field f, i.e. $[f, X] \in \mathcal{C}_0, \ \forall X \in \mathcal{C}_0$. The *strong accessibility distribution* C_0 is the corresponding involutive distribution $C_0(x) = \text{sp}\{X(x) \,|\, X \in \mathcal{C}_0\}$. Every element of the algebra $\mathcal{C}_0$ is a linear combination of repeated Lie brackets of the form $[X_k, [X_{k-1}, [\ldots, [X_1, g_j] \ldots]]]$, for $j \in \{1, \ldots, m\}$ and where $X_i, \ i \in \{1, \ldots k\}, \ k = 0, 1, \ldots$ belongs to $\{f, g_1, \ldots, g_m\}$. Observe that the drift vector field f is not contained explicitly in these expressions.

Proposition 5.1 *If the Strong Accessibility Rank Condition at $x_0 \in M$ is satisfied, i.e. if*

$$\dim C_0(x_0) = n \ , \tag{162}$$

then the system 161 is locally strongly accessible from x_0. If the Strong Accessibility Rank Condition is satisfied at every $x \in M$, then the system is locally strongly accessible. If the system 161 is locally strongly accessible, then $\dim C_0(x) = n$, *for x in an open and dense subset of M.*

The system 161 is *controllable*, if, for every $x_1, x_2 \in M$, there exists a finite time $T > 0$ and an admissible control $u : [0, T] \to U$ such that $x(T, 0, x_1, u) = x_2$.

For systems without drift (i.e. where $f = 0$ in 161), accessibility is equivalent to controllability. However, this is no longer true for systems with drift and various notions of controllability have been developed. Below we consider the notion of small–time local controllability, for which relatively simple verification tests have been established, as well as links to the closed–loop control of nonholonomic systems (Sussmann [27], [28], Coron [10]).

The system 161 is *small–time locally controllable (STLC)* from $x_0 \in M$, if, for any neighborhood V of x_0 and any $T > 0$, x_0 is an interior point of the set $R_T^V(x_0)$, i.e. a whole neighborhood of x_0 is reachable from x_0 at arbitrarily small time.

In Sussmann [27], a condition for *lack* of STLC is given for single–input systems (see also the discussion on single–input systems in [28]).

Proposition 5.2 *Consider an analytic affine nonlinear system with a single input, of the form*

$$\dot{x} = f(x) + g(x)\, u\,, \tag{163}$$

with $|u| \leq 1$, $f(x_0) = 0$ *and* $g(x_0) \neq 0$, *for some* $x_0 \in M$. *Assume that the bracket* $[g,[g,f]](x_0)$ *does not belong to the linear span of the vector fields* $\{ad_f^j\, g\ (x_0),\ \ j = 0, 1, \ldots\}$. *Then, the system is not STLC from* x_0.

Certain nonlinear systems can be transformed, at least locally, into a linear controllable system, via a state coordinate transformation and static state feedback. This process is called *static feedback linearization.* Other nonlinear systems can be transformed into a linear controllable system via dynamic state feedback and a coordinate transformation involving the extended state of the system. This process is called *dynamic feedback linearization.* As Pomet [25] remarks, dynamic feedback linearization, as defined above, is equivalent to the concept of *differential flatness* introduced by Fliess, Levine, Martin and Rouchon [11].

Remark 5.1 Charlet, Levine and Marino [8] and Pomet [25] show that differential flatness is equivalent, for single–input systems, to static feedback linearization, a necessary and sufficient condition for which is provided in Nijmeier and van der Schaft [22]:

Proposition 5.3 *Consider system 161 with* $f(x_0) = 0$. *Assume that the strong accessibility rank condition holds at* x_0. *This system is static feedback linearizable if and only if the distributions* $D_1, \ldots, D_n$ *defined by*

$$D_k(x) = sp\{ad_f^r g_1(x), \ldots, ad_f^r g_m(x) \mid r = 0, 1, \ldots, k-1\},\ k = 1, 2, \ldots \tag{164}$$

are all involutive and constant dimensional in a neighborhood of x_0.

Assume further that this is a single–input system. This system is static feedback linearizable around x_0 *if and only if* $\dim D_n(x_0) = n$ *and* D_{n-1} *is involutive around* x_0.

5.1 The Reduced Dynamics

In this section we only consider the reduced dynamics for the Roller Racer model without external forces, other than the torque $\tau_{1,2}$ applied to the rotary joint (equations 154).

For control purposes, we assume that p, $\theta_{1,2}$ and $\dot{\theta}_{1,2}$ are available from proprioceptive sensors. The dynamics of the base variable $\theta_{1,2}$ for the system without external forces (first of equations 154) can be transformed into the form of a double integrator by the nonlinear static state feedback

$$\tau_{1,2} = \frac{1}{B_5^4(\theta_{1,2})}[u - B_1^4(\theta_{1,2})\dot{\theta}_{1,2}p - B_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2] \ . \tag{165}$$

Thus, after feedback linearization, the reduced dynamics take the form

$$\frac{dp}{dt} = A_1^4(\theta_{1,2})\dot{\theta}_{1,2}\, p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \ , \quad \frac{d\theta_{1,2}}{dt} = \dot{\theta}_{1,2} \ , \quad \frac{d\dot{\theta}_{1,2}}{dt} = u \ . \tag{166}$$

Defining the state vector $z \overset{\text{def}}{=} (p,\ \theta_{1,2},\ \dot{\theta}_{1,2})^\top \in M$, where $M \overset{\text{def}}{=} I\!R^2 \times S^1$, the reduced dynamics 166 take the form of an affine nonlinear system with a single control $u \in I\!R$:

$$\dot{z} = f(z) + g(z)\, u \ , \tag{167}$$

and with

$$f(z) \overset{\text{def}}{=} \begin{pmatrix} A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \\ \dot{\theta}_{1,2} \\ 0 \end{pmatrix} \quad \text{and} \quad g(z) \overset{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ . \tag{168}$$

The equilibria of this system are states $z_e \in M$ where $f(z_e) = 0$. It can be easily seen that these are of the form $z_e = (p_e,\ \theta_{1,2_e},\ 0)^\top \in M$, with $p_e \in I\!R$ and $\theta_{1,2_e} \in S^1$. In particular, the origin $z_0 = (0,\ 0,\ 0)^\top \in M$ is an equilibrium.

The tangent linearization of the system 167 is *not* controllable at equilibria, since the matrix

$$\left[g \,\middle|\, \frac{\partial f}{\partial z} g \,\middle|\, \left(\frac{\partial f}{\partial z}\right)^2 g \right]\Bigg|_{z_e} = \begin{pmatrix} 0 & A_1^4(\theta_{1,2_e})p_e & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is singular.

Define

$$A_3^4(\theta_{1,2}) \overset{\text{def}}{=} A_1^4(\theta_{1,2})A_2^4(\theta_{1,2}) - \frac{\partial A_2^4(\theta_{1,2})}{\partial \theta_{1,2}} \tag{169}$$

and, iteratively,

$$A_{i+1}^4(\theta_{1,2}) \overset{\text{def}}{=} -A_1^4(\theta_{1,2})A_i^4(\theta_{1,2}) + \frac{\partial A_i^4(\theta_{1,2})}{\partial \theta_{1,2}} \ , \quad \text{for } i = 3, 4, \ldots \tag{170}$$

When $d_1 > d_2$, the roots $\theta_{1,2}^\star$ of $A_2^4(\theta_{1,2})$ correspond to the solutions of $\gamma(\theta_{1,2}) = 0$, i.e. to $\theta_{1,2}^\star = \cos^{-1} \frac{I_{z_1} d_2}{I_{z_2} d_1}$. Notice that at roots of $A_2^4(\theta_{1,2})$ such that $I_{z_1} d_2 \neq I_{z_2} d_1$, we have $\theta_{1,2}^\star \neq 0, \pi$ and $\frac{\partial A_2^4(\theta_{1,2}^\star)}{\partial \theta_{1,2}} = -\frac{m_1}{\Delta^2} I_{z_2} d_1 \lambda \sin\theta_{1,2}^\star \neq 0$. Thus, when $A_2^4(\theta_{1,2}) = 0$ and $I_{z_1} d_2 \neq I_{z_2} d_1$, we have from 169 that $A_3^4(\theta_{1,2}) \neq 0$.

Proposition 5.4 *Assume* $d_1 > d_2$ *and* $I_{z_1} d_2 \neq I_{z_2} d_1$. *The reduced dynamics 167 are locally strongly accessible from equilibria* $z_e = (p_e,\ \theta_{1,2_e},\ 0)^\top$.

Proof

If $\theta_{1,2_e}$ is such that $A_2^4(\theta_{1,2_e}) \neq 0$, then $\mathrm{sp}\Big\{g,\ [f,g],\ [[f,g],g]\Big\}(z_e) = I\!R^3$. If $\theta_{1,2_e}$ is such that $A_2^4(\theta_{1,2_e}) = 0$ and $I_{z_1}d_2 \neq I_{z_2}d_1$, then $\mathrm{sp}\Big\{g,\ [f,g],\ \Big[[f,g],\ [[f,g],g]\Big]\ \Big\}(z_e) = I\!R^3$. In both cases, the system satisfies the strong accessibility rank condition at z_e.

∎

However, since 167 is a system with drift, its accessibility does not imply its controllability. In particular, it is possible to show the following:

Proposition 5.5 *The reduced dynamics 167 are not STLC from equilibria* $z_e = (p_e,\ \theta_{1,2_e},\ 0)^\top$ *where* $A_2^4(\theta_{1,2_e}) \neq 0$.

Proof

Observe that at equilibria z_e we have $g(z_e) \neq 0$,

$$[g,[g,f]]\Big|_{z_e} = \begin{pmatrix} 2A_2^4(\theta_{1,2_e}) \\ 0 \\ 0 \end{pmatrix} \tag{171}$$

and

$$\mathrm{sp}\{\mathrm{ad}_f^j\ g\ (z_e),\ j=0,1,\ldots\} = \mathrm{sp}\{g\ (z_e),\ [f,g]\ (z_e)\} = \mathrm{sp}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} A_1^4(\theta_{1,2_e})p_e \\ 1 \\ 0 \end{pmatrix}\right\}. \tag{172}$$

Obviously, the bracket $[g,[g,f]]\ (z_e)$ does not belong to $\mathrm{sp}\{\mathrm{ad}_f^j\ g(z_e),\ j=0,1,\ldots\}$ when $A_2^4(\theta_{1,2_e}) \neq 0$. Thus, from Proposition 5.2, the result follows.

∎

Proposition 5.6 *The reduced dynamics 167 are not static feedback linearizable around equilibria* $z_e = (p_e, \theta_{1,2_e}, 0)^\top$.

Proof

The dimension of the state space is $n = 3$. At the equilibrium z_e, the distribution D_n is

$$D_n(z_e) = \mathrm{sp}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} A_1^4(\theta_{1,2_e})p_e \\ 1 \\ 0 \end{pmatrix}\right\}$$

and its dimension is strictly less than n. Thus, the result follows from Proposition 5.3.

∎

Remark 5.2 *In view of Remark 5.1, the reduced dynamics 167 are neither dynamic feedback linearizable, nor differentially flat.*

5.2 The Full System

The full dynamics of the Roller Racer (model without external forces other than the torque actuating the rotary joint) are given by the reduced dynamics (equations 154) and by the group equations $\dot{g}_1 = g_1 \xi_1$, where $g_1 \in SE(2)$, $\xi_1 \in se(2)$ and ξ_1 is given by 105. Consider local coordinates (x_1, y_1, θ_1) for $g_1 \in SE(2)$. Letting $z \overset{\text{def}}{=} (\theta_1,\ x_1,\ y_1,\ p,\ \theta_{1,2},\ \dot{\theta}_{1,2})^\top \in M = I\!R^6$ and after feedback linearization of the dynamics of the base variable (cf. equation 165), the dynamics take the form of an affine nonlinear control system, with the shape acceleration $\ddot{\theta}_{1,2}$ being the single control of the system:

$$\dot{z} = f(z) + g(z)\, u\ , \tag{173}$$

with $u \in I\!R$ and

$$f(z) \overset{\text{def}}{=} \begin{pmatrix} \frac{1}{\Delta(\theta_{1,2})}[\sin\theta_{1,2}\, p - \delta(\theta_{1,2})\, \dot{\theta}_{1,2}] \\ \frac{\cos\theta_1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2}) \sin\theta_{1,2}\, \dot{\theta}_{1,2}] \\ \frac{\sin\theta_1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2}) \sin\theta_{1,2}\, \dot{\theta}_{1,2}] \\ A^4_1(\theta_{1,2})\dot{\theta}_{1,2} p + A^4_2(\theta_{1,2})\dot{\theta}^2_{1,2} \\ \dot{\theta}_{1,2} \\ 0 \end{pmatrix} \quad \text{and} \quad g(z) \overset{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\ . \tag{174}$$

The *equilibria* of the system are states $z_e \in M$ of the form $z_e = (\theta_{1_e},\ x_{1_e},\ y_{1_e},\ 0,\ \theta_{1,2_e},\ 0)^\top \in M$, i.e. states where, not only the shape $\theta_{1,2}$ is constant, but also the nonholonomic momentum p is zero.

Remark 5.3 *Based in part on the results for the reduced dynamics in section 5.1, it can be shown that for $d_1 > d_2$, the full dynamics of the Roller Racer (equation 173) are locally accessible from equilibria z_e, but are not STLC from equilibria where $A^4_2(\theta_{1,2_e}) \neq 0$ and are not differentially flat.* See details in (Krishnaprasad and Tsakiris [18]).

Remark 5.4 *The above properties still hold for $d_2 = 0$. This is due to the nonzero inertia I_{z_2} of the second platform.*

Other undulatory locomotors, like the snakeboard, are known to be STLC (Ostrowski and Burdick [24]). The nontrivial second term in the momentum equation 76 of the Roller Racer ($A^4_2(\theta_{1,2})\ \dot{\theta}^2_{1,2}$) plays a crucial role in its property of being accessible, but not being STLC.

It is interesting to observe that, even though the Roller Racer resembles a unicycle with one trailer which is hitched to a point displaced from the center of the unicycle's wheel axis (also referred to as kingpin hitch) and this last system has been shown to be differentially flat (Rouchon, Fliess, Levine and Martin [26]), the peculiar actuation scheme of the Roller Racer makes it non–flat.

6 Simulation and Experimental Results

A computer–controlled prototype of the Roller Racer was built at the Intelligent Servosystems Laboratory (ISL) of the University of Maryland (fig. 5). The assumption of our models that the only feature of the body motion of a Roller Racer rider which is crucial to the propulsion of this mechanism is the swinging of the steering arm around the pivot axis, was verified using this and other similar prototypes.

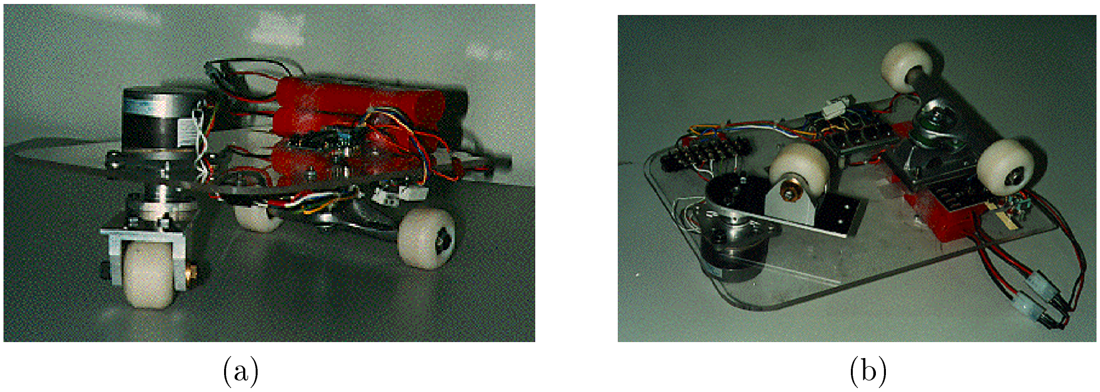

(a) (b)

Figure 5: Roller Racer Prototype

The models of the dynamics of the Roller Racer, which were developed in the previous sections, were used in computer simulations of the system on Silicon Graphics workstations and in Mathematica and Simparc [3] simulations on SUN SPARCstations.

A periodic shape trajectory of period $T_{1,2}$ of the form

$$\theta_{1,2}(t) = \theta_{1,2}(0) + \alpha_{1,2} \sin(\omega_{1,2} t + \phi_{1,2}) \ , \tag{175}$$

with $\omega_{1,2} = \frac{2\pi}{T_{1,2}}$ is used in the simulations. The average value of $\theta_{1,2}$ is $\theta_{1,2}(0)$. Setting this average to π, as in fig. 6, generates a "straight–line" motion. Setting $\theta_{1,2}(0)$ to a value other than π or zero, as in fig. 9 (where $\theta_{1,2}(0) = 1.7721542$ rad), generates a rotation around the point where the axes of the platforms intersect when the system is in the configuration corresponding to this average value. Once momentum has built up through periodic shape variations, we can stop varying the shape periodically and use $\theta_{1,2}$ just to steer the system. In what follows, we give only a sampling of our simulation results. For further details, see Krishnaprasad and Tsakiris [18]. Movies of experiments with Roller Racer prototypes can be seen in the home page of the Intelligent Servosystems Laboratory (URL: *http://www.isr.umd.edu/Labs/ISL/isl.html*) and in the second author's home page at FORTH (URL: *http://www.ics.forth.gr/~tsakiris*).

6.1 Gaits

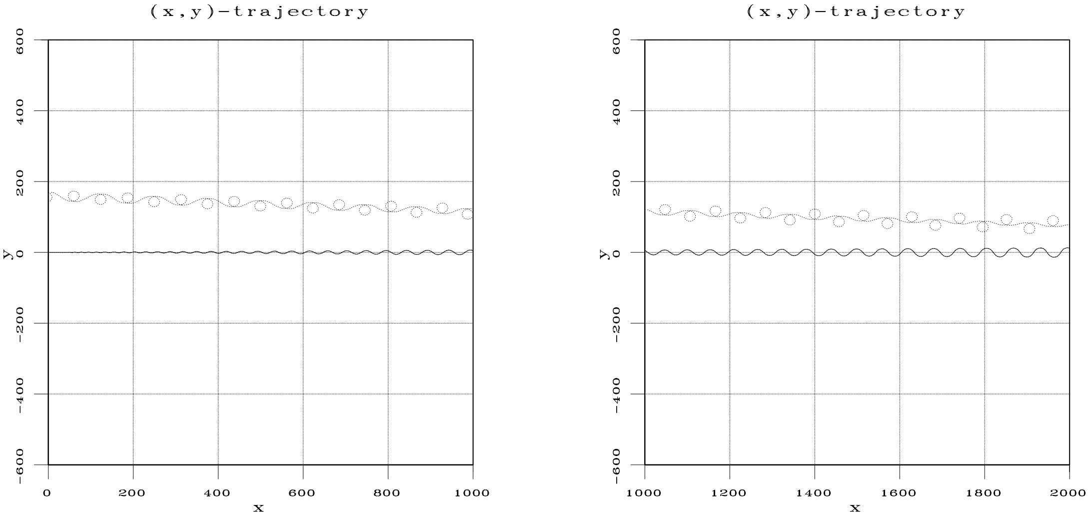
Consider the Roller Racer model without friction or external forces, except for the joint torque needed to actuate $\theta_{1,2}$. The model parameters used in these simulations are $m_1 = 1$, $d_1 = 5$, $d_2 = 1$, $I_{z_1} = 10$, $I_{z_2} = 1$ ($I_{z_1} d_2 > I_{z_2} d_1$).

In all the (x_1, y_1) plots that follow, the system starts at $(0,0)$ and is initially oriented towards the positive x_1–axis ($\theta_1 = 0$).

a. Forward Translation

When the initial shape angle $\theta_{1,2}(0)$ is equal to π, the system translates forward. In the present simulations, the system starts at $(x_1, y_1) = (0,0)$ pointing towards the positive x_1–axis and the shape control has amplitude of oscillation $\alpha_{1,2} = 0.3$ and frequency $\omega_{1,2} = 1$.

In fig. 6, the (x_1, y_1)–trajectory is shown. Fig. 6.a shows the initial part of the trajectory: the system initially translates to the right, while oscillating about the x_1–axis. These oscillations become more pronounced as the momentum increases, giving rise to elastica–like trajectory segments (cf. §3.2.6 of Tsakiris [29]), which at some point reverse direction and the system starts moving to the left, creating the upper branch of the trajectory of fig. 6. As we move from fig. 6.a to 6.e, we are moving to the right of the x_1–axis. The lower branch of the trajectory, the one that corresponds to a translation of the Roller Racer to the right, is shown as a solid line. The upper branch of the trajectory, the one that corresponds to a translation to the left, is shown as a dotted line. Notice that the scales of the x and y axes are different.



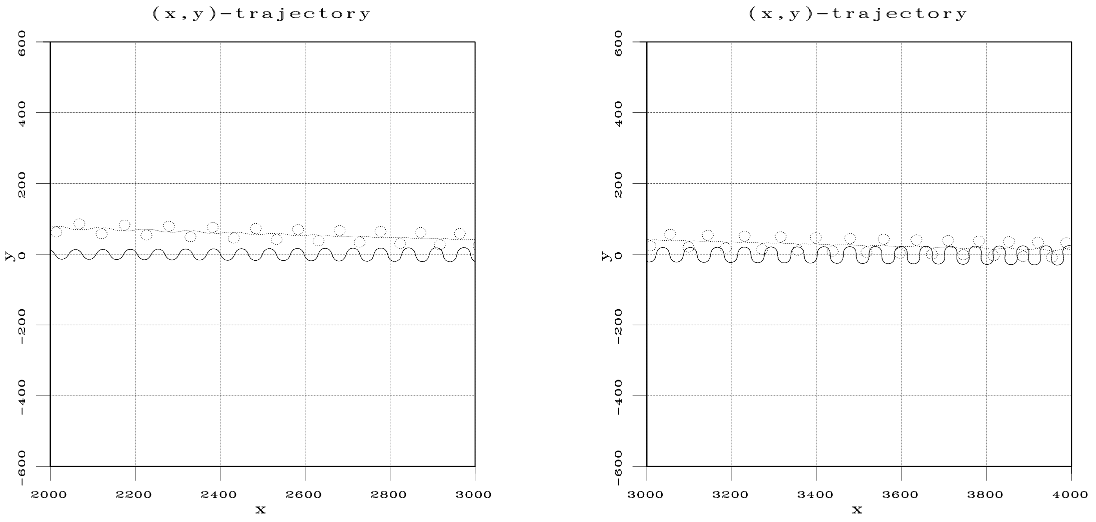
(a) $x_1 \in [0, 1000]$

(b) $x_1 \in [1000, 2000]$

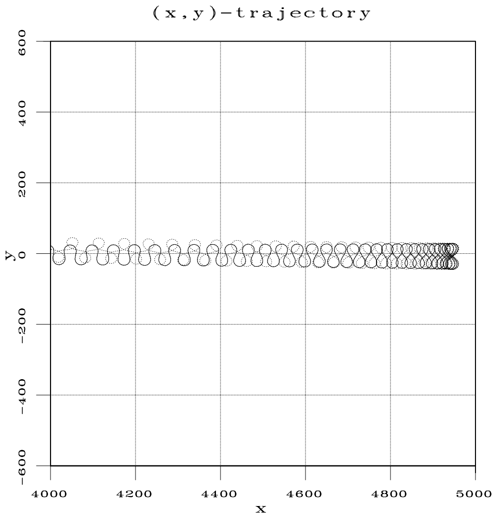
The group variable θ_1 is shown in fig. 7.a, showing that the system oscillates with increasing amplitude as the nonholonomic momentum increases, but that the average of this oscillation is zero. Thus, the system translates on a more or less straight–line trajectory. The corresponding nonholonomic momentum p, which increases on the average, is shown in fig. 7.b.

b. Backward Translation

When $\theta_{1,2}(0) = 0$, the system translates backwards. In the present simulations, the system starts at $(x_1, y_1) = (0,0)$ pointing towards the positive x_1–axis and the shape oscillation has amplitude $\alpha_{1,2} = 0.1$ and frequency $\omega_{1,2} = 1$. The corresponding group variables $(x_1,\ y_1,\ \theta_1)$ are shown in fig. 8. Observe that y_1 and θ_1 merely oscillate around zero, while the magnitude of x_1 increases.

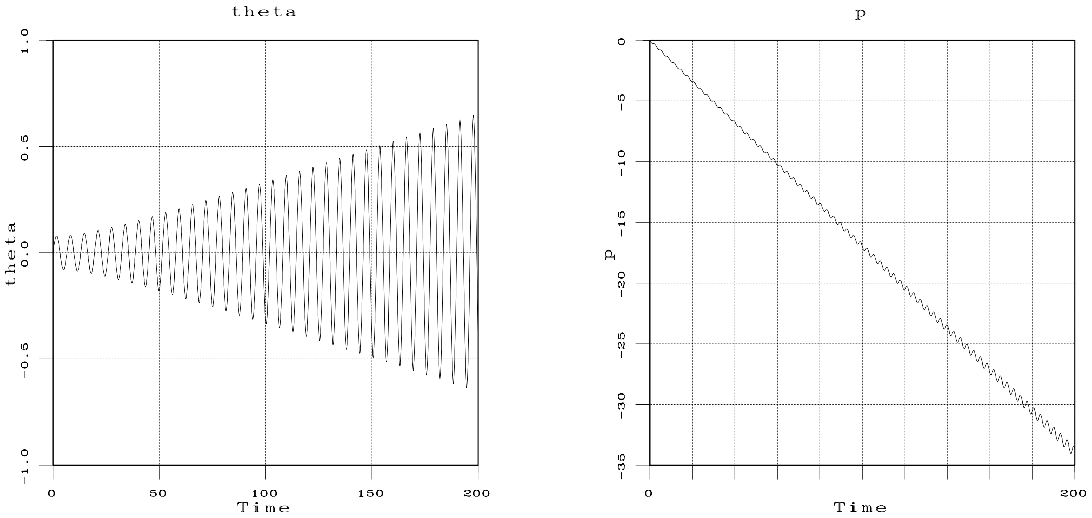


(c) $x_1 \in [2000, 3000]$ (d) $x_1 \in [3000, 4000]$



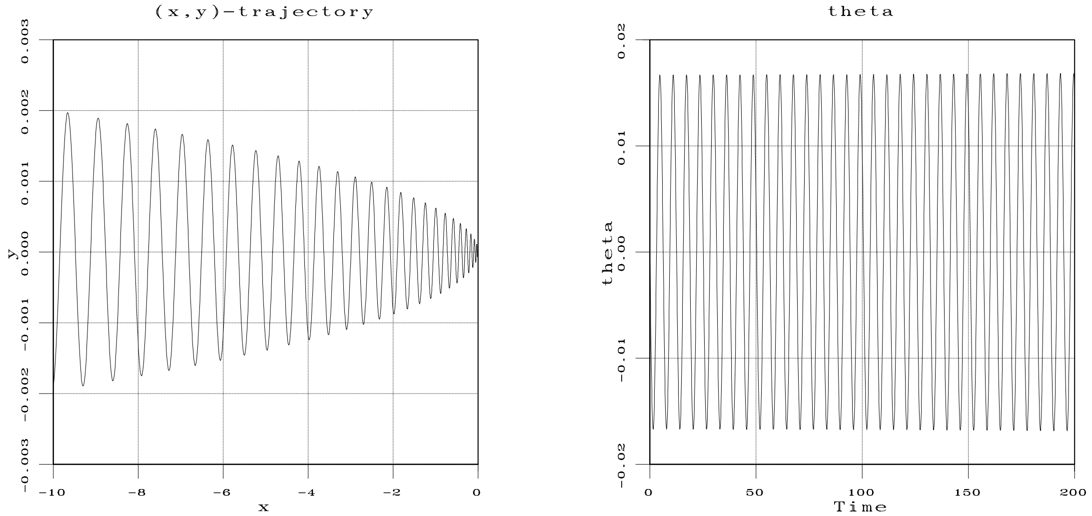
(e) $x_1 \in [4000, 5000]$

Figure 6: Forward Translation



(a) Angle θ_1 (b) Nonholonomic Momentum p

Figure 7: Forward Translation



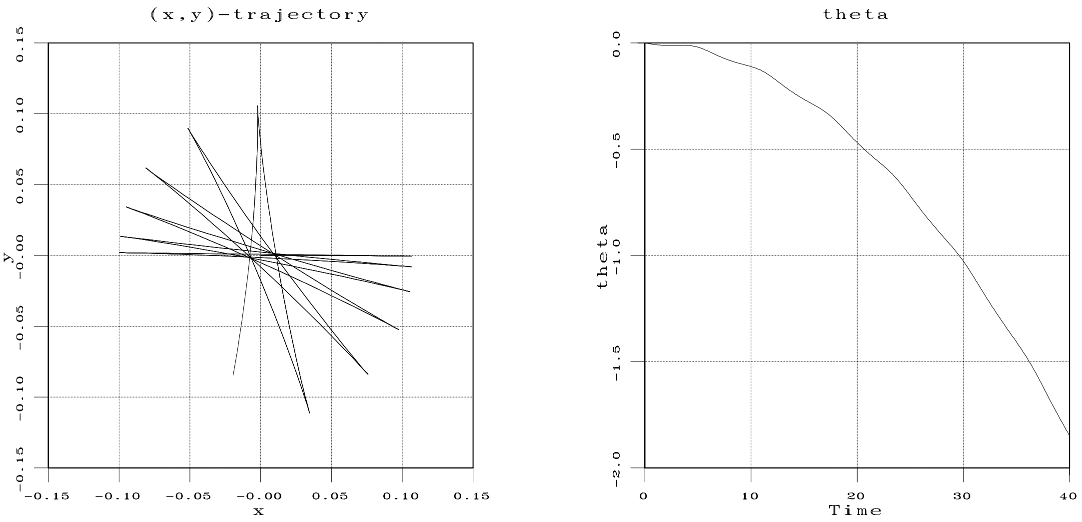
(a) (x_1, y_1)–trajectory (b) θ_1–trajectory

Figure 8: Backward Translation

c. Pure Rotation

When the instantaneous center of rotation of the system is, on the average, at the middle of the rear wheel axis, i.e. when $\theta_{1,2}(0)$ is a root of $r(\theta_{1,2}) = 0$, which we denote as $\theta_{1,2}^{r=0}$, the Roller Racer rotates without translating (on the average). This can be seen in fig. 9 for a clockwise rotation with $\theta_{1,2}(0) = \theta_{1,2}^{r=0} = 1.7721542$ rad.

When the average of the shape oscillation ($\theta_{1,2}(0)$) is not set to $0, \pi$ or $\pm\ \theta_{1,2}^{r=0}$, the system rotates around the average position of the instantaneous center of rotation.



(a) (x_1, y_1)–trajectory (b) θ_1–trajectory

Figure 9: Clockwise Rotation by $\frac{\pi}{2}$

6.2 Geometric and Dynamic Phase

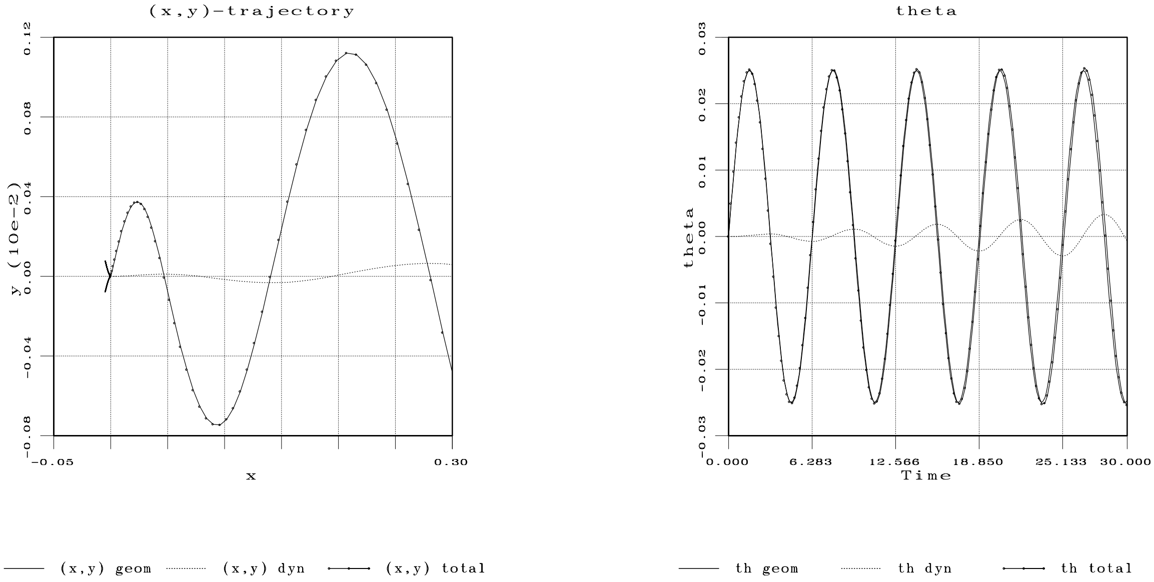
Consider a periodic shape variation of the type of equation 175 corresponding to forward translation of the system with $\theta_{1,2}(0) = \pi$, $\alpha_{1,2} = 0.1$ and $\omega_{1,2} = 1$. We consider the momentum equation without friction (equation 76).

The notions of *geometric* and *dynamic phase* ([20], [5]) describe how much the system moved after one period of the oscillatory controls. The group velocity, given by the reconstructed group motion equations 142

$$\xi_1 = g_1^{-1}\dot{g}_1 = A_{loc}(\theta_{1,2})\dot{\theta}_{1,2} + \mathbb{I}_{loc}^{-1}(\theta_{1,2})\ p \tag{176}$$

is composed of two parts: the system motion due to the first term $A_{loc}(\theta_{1,2})\ \dot{\theta}_{1,2}$ (where the nonholonomic momentum plays no role) is called the *geometric phase*, while the system motion due to the second term $\mathbb{I}_{loc}^{-1}(\theta_{1,2})\ p$ is called the *dynamic phase*. Thus, the geometric phase is $\int_0^{T_{1,2}} \dot{g}_1(t)dt$, with $\dot{g}_1(t) = g_1(t)A_{loc}(\theta_{1,2}(t))\dot{\theta}_{1,2}(t)$ and the dynamic phase is $\int_0^{T_{1,2}} \dot{g}_1(t)dt$, with $\dot{g}_1(t) = g_1(t)\mathbb{I}_{loc}^{-1}(\theta_{1,2}(t))\ p(t)$. The dynamic phase obviously depends on the initial value of the momentum $p(0)$. In the simulation results presented below, we suppose that the system starts at rest, i.e. that $p(0) = 0$.

The components of x_1, y_1 and θ_1, that are due to each of the above two terms, are shown in fig. 10. It is easy to see from this figure that the geometric phase, over one period of the periodic shape controls, is zero (this is shown by the curves marked "(x,y) geom" and "th geom" in the figures). In fig. 10.a, the contribution of "(x,y) geom" is the swallowtail to the left of point $(0, 0)$. In fig. 10.b, the curve "th geom", an oscillation around zero, initially overlaps the curve "th total". These simulation results show that the geometric phase in this case is zero. However, the dynamic phase is not zero. In fig. 10.a, the curve "(x,y) dyn" has an evident non–zero component in the x_1–direction, while the motions in the y_1 and θ_1 directions are, again, oscillations around zero.



(a) (x_1, y_1)–trajectory

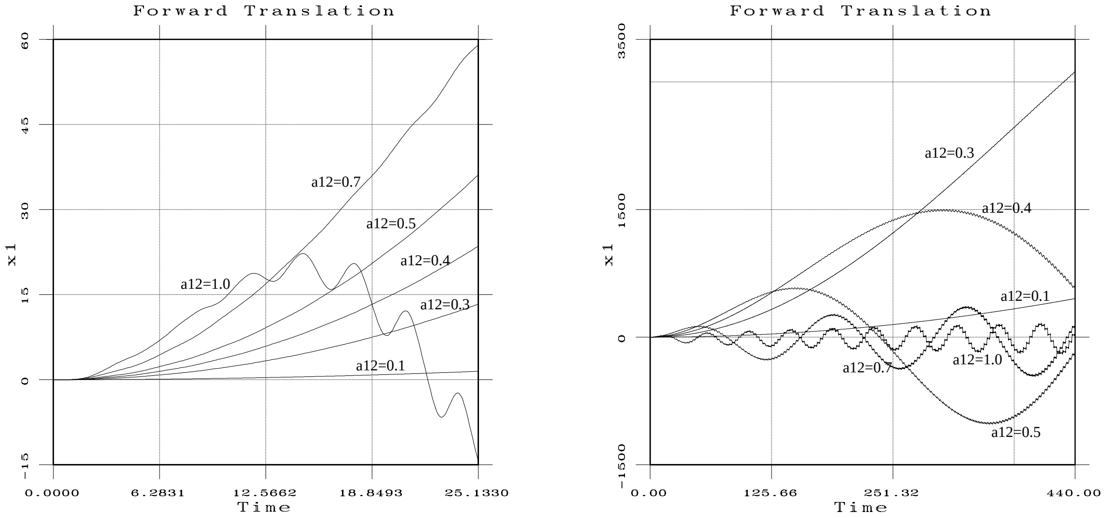
(b) θ_1–trajectory

Figure 10: Geometric and Dynamic Phase

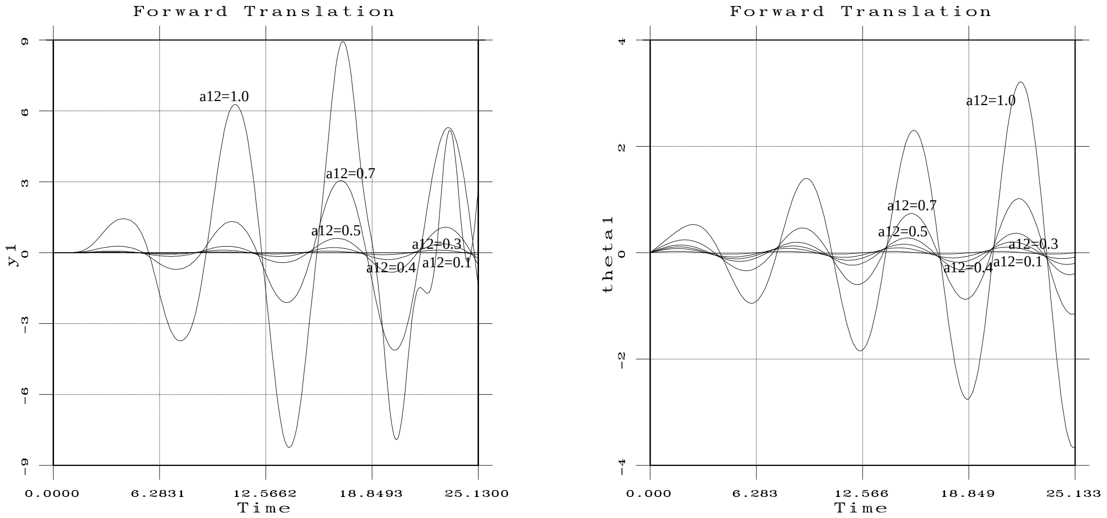
6.3 Parametric Study of the System

In the present section, we study the dependence of the motion of the Roller Racer, for the model without external forces, on the amplitude and the frequency of the sinusoidal shape controls (equation 175).

Fig. 11 shows the evolution of the group variables x_1, y_1, θ_1 for a forward translation of the system ($\theta_{1,2} = \pi$, $\omega_{1,2} = 1.0$) and for control oscillation amplitude $\alpha_{1,2}$ varying from 0.1 to 1.0. Fig. 11.a shows the evolution of x_1, fig. 11.c shows this of y_1 and fig. 11.d shows this of θ_1, all for a time duration of four time periods of the controls, while fig. 11.b shows the evolution of x_1 for a bigger time duration of twenty time periods of the controls. It is obvious that y_1 and θ_1 merely oscillate around zero. This is not the case for x_1. Fig. 11.a shows that for short times (of 1–2 periods of the controls), the bigger $\alpha_{1,2}$ is, the bigger the system's forward motion. However, as can be seen in fig. 11.b, this is no longer true for longer time periods. From the above it appears that small–amplitude motion gives forward translation without too much oscillation in the group variables, which closely approximates a straight–line motion.



(a) x_1 for a duration of 4 $T_{1,2}$ (b) x_1 for a duration of 20 $T_{1,2}$

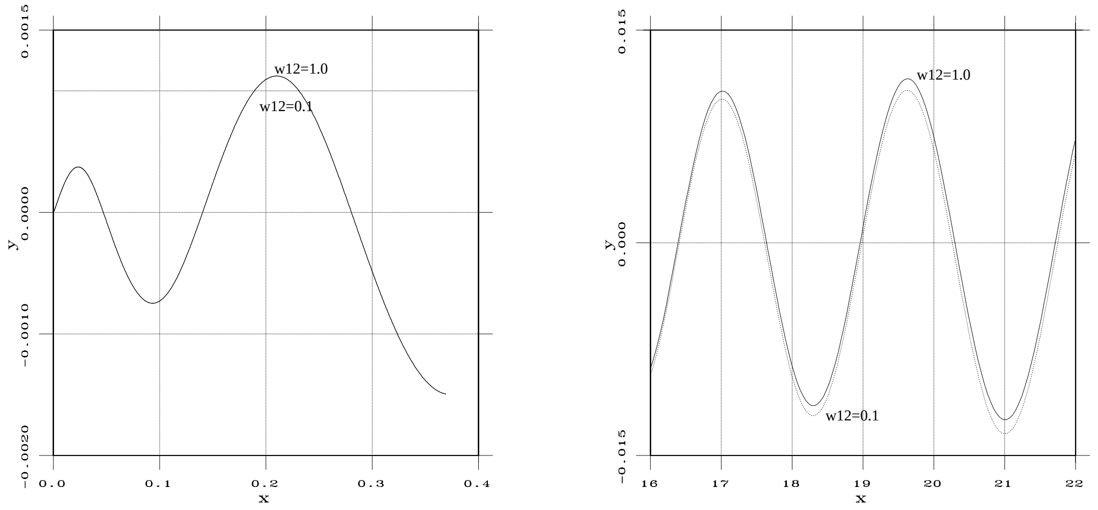


(c) y_1 for a duration of 4 $T_{1,2}$ (d) θ_1 for a duration of 4 $T_{1,2}$

Figure 11: Effect of the amplitude $\alpha_{1,2}$ on x_1, y_1, θ_1

Consider now the effect of the frequency $\omega_{1,2}$ on the group variables x_1, y_1, θ_1. We vary the frequency from 0.1 to 1.0, while $\theta_{1,2}(0) = \pi$ and $\alpha_{1,2} = 0.1$

Fig. 12 shows the (x_1, y_1)–trajectory of the system for $\omega_{1,2} = 0.1$, superimposed to the corresponding trajectory for $\omega_{1,2} = 1.0$, for a time duration of 2 $T_{1,2}$ (40 π and 4 π *sec* respectively). Fig. 12.a shows the initial part of the trajectory, where nonholonomic momentum is low, and fig. 12.b shows a later part of it, where nonholonomic momentum is higher. The trajectory corresponding to $\omega_{1,2} = 1.0$ appears as a solid line, while the one for $\omega_{1,2} = 0.1$ appears as a dotted line. When nonholonomic momentum is low, the trajectories for $\omega_{1,2} = 1.0$ and $\omega_{1,2} = 0.1$ are geometrically almost the same (fig. 12.a); it is the time traversal of the trajectory that becomes faster as $\omega_{1,2}$ increases. However, as nonholonomic momentum increases, both the geometry of the trajectory and its time traversal become different (fig. 12.b).



(a) Low p (b) High p

Figure 12: Effect of the frequency $\omega_{1,2}$ on (x_1, y_1)–trajectory

6.4 Model with Friction

a. Forward Translation:

We consider the Roller Racer model with friction (momentum equation 98) with the following parameters (in addition to the ones mentioned earlier): $k_1 = k_2 = 0.01$, $R_1 = R_2 = 0.5$, $L_1 = 1, L_2 = 0.25$. The control input 175 is considered with $\theta_{1,2}(0) = \pi$, $\alpha_{1,2} = 0.1$ and $\omega_{1,2} = 1.0$. The nonholonomic momentum corresponding to this control is shown in fig. 13. Comparing this with fig. 7.b, we observe that, contrary to the continuously increasing, on the average, momentum p of fig. 7.b, here, each shape oscillation pumps just enough energy into the system to overcome friction. This is similar to the real system's behavior observed by the prototypes built at ISL.

b. Parallel Parking:

We now set the friction coefficients to $k_1 = k_2 = 0.1$, leaving the rest of the parameters as before. In order to create a "parallel parking" behavior, the idea is to generate a motion in the Lie–bracket direction by first translating forward, then rotating clockwise, then translating backwards and finally rotating counter–clockwise. The first step corresponds to a shape oscillation with average π, for a few periods, the second step corresponds to a shape oscillation

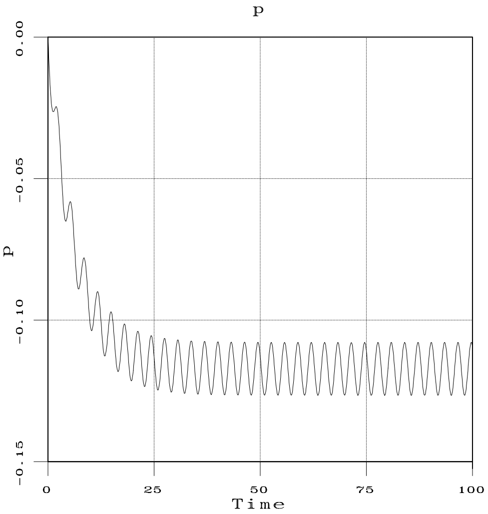
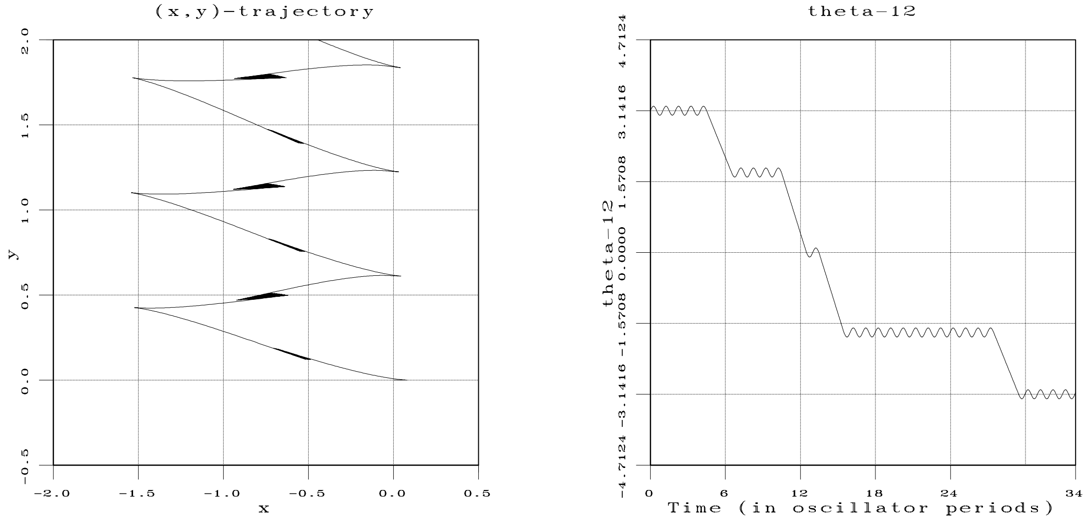


Figure 13: Model with Friction: Forward Translation: Nonholonomic Momentum p

with average $\theta^{r=0}_{1,2}$, the third step corresponds to a shape oscillation with average 0 and the final step corresponds to a shape oscillation with average $-\theta^{r=0}_{1,2}$. This sequence of shape controls is shown in fig. 14.b, where, starting with a basic shape oscillation of the type of equation 175 with amplitude $\alpha_{1,2} = 0.1$, frequency $\omega_{1,2} = 1.0$ and period $T_{1,2} = \frac{2\,\pi}{\omega_{1,2}}$, we reset its average as was described above. The whole cycle lasts $30\,T_{1,2}$, after which we restart at π (shown as $-\pi$ in fig. 14.b). The corresponding (x_1, y_1)–trajectory is shown in fig. 14.a.



(a) (x_1, y_1)–trajectory

(b) $\theta_{1,2}$–trajectory

Figure 14: Model with Friction: Parallel Parking Maneuver

7 Conclusions

The present work is aimed at revealing some of the rich mathematical and physical structure associated with a specific mechanical system that is underactuated. Part of our fascination with this system derives from the drive to understand how it works at all! As shown in this paper, the interplay between the symmetries and the constraints is crucial to this understanding. Additionally, Lie algebraic analysis reveals both the capabilities and the limitations of such an underactuated system. The first draft of this paper was provided to the organizers of a workshop at the IEEE Conference on Decision and Control in Kobe in December 1996. (After the first draft of this paper was completed, in the Summer of 1997, we received a preprint of Zenkov, Bloch and Marsden [35], which investigates the stability of relative equilibria of the *unforced* Roller Racer as an application of a general theory of stability of nonholonomic systems.)

The present paper also explores via simulation certain motion control questions: specifically, controls for generating translation and curved motions, as well as parking maneuvers. The influence of dissipation is also considered in some detail. Much remains to be done to understand the problem of constructive control for the Roller Racer. Models of the type used here may prove to be of interest in understanding problems of locomotion in biology and in bio-mimetic robotic systems.

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List of Symbols

$\mathcal{A}_i$	Basis element of Lie algebra $\mathcal{G}$, 8
$\mathcal{A}_i^\flat$	Basis element of dual space $\mathcal{G}^*$ of Lie algebra $\mathcal{G}$, 8
$A_{loc}(\theta_{1,2})$	Local form of nonholonomic connection, 34
$A_i^4(\theta_{1,2})$	Auxiliary function ($i = 1, 2, 3, \ldots$), 22, 42
$A_i^5(\theta_{1,2})$	Auxiliary function ($i = 1, 2, 3$), 26, 38
$B_i^4(\theta_{1,2})$	Auxiliary function ($i = 1, \ldots, 5$), 36, 38
$\mathcal{D}_q$	Constraint distribution, 10
$D_k(x)$	Distributions related to static feedback linearization, 41
d_i	Distance of center of platform i from the rotary joint, 9
$(F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}})$	Components of external forcing one–form α_e, 17
$f(x)$	Drift vector field of affine nonlinear control system, 40, 42, 44
G	Lie group, 4, 8
$\mathcal{G}$	Lie algebra of Lie group G, 8
$\mathcal{G}^*$	Dual space of Lie algebra $\mathcal{G}$, 8
$\mathcal{G}^q$	Set of Lie algebra elements with infinitesimal generators in $\mathcal{S}_q$, 34
g	Element of Lie group G, 8
$g_j(x)$	Control vector field of affine nonlinear control system, 40, 42, 44
H_q	Horizontal subspace of principal bundle $(Q, \mathcal{S}, \pi, G)$, 31, 33
H_q	Orthogonal complement of $\mathcal{S}_q$ with respect to $\mathcal{D}_q$, 31
$\mathbb{I}(q)$	Locked inertia tensor relative to $\mathcal{G}^q$, 34
I_{z_i}	Moment of inertia of platform i, 16
k_i	Friction coefficient of wheels of platform i, 20
L	Lagrangian, 16
L_i	Length of wheel axis for platform i, 11
m_i	Mass of platform i, 16
$\mathrm{Orb}(q)$	Orbit of $q \in Q$ under action Φ, 12
p	Nonholonomic momentum, 21
Q	Configuration space, 5
q	Element of configuration space Q, 10
$(Q, \mathcal{S}, \pi, G)$	Principal fiber bundle, 30, 33
$\mathcal{R}$	Rayleigh dissipation function, 20
R_i	Wheel radius for platform i, 11
$r(\theta_{1,2})$	Auxiliary function, 17, 22
$\mathcal{S}$	Shape space, 5
$\mathcal{S}_q$	Intersection of constraint distribution $\mathcal{D}_q$ with $T_q\mathrm{Orb}(q)$, 13
$SE(2)$	Special Euclidean group of rigid planar motions, 4, 8
$se(2)$	Lie algebra of the Special Euclidean group $SE(2)$, 8
T_qQ	Tangent space to $q \in Q$, 10

$T_{1,2}$	Period of shape control, 45
$\mathcal{U}_q$	Orthogonal complement of $\mathcal{S}_q$ with respect to $T_qOrb(q)$, 32
V_q	Vertical subspace of principal bundle $(Q, \mathcal{S}, \pi, G)$, 30, 33
(x, y, θ)	Coordinates of $g \in SE(2)$, 8
$(x_1, y_1, \theta_1, \theta_{1,2})$	Coordinates of Roller Racer configuration space $Q = SE(2) \times S^1$, 9
z	State of affine nonlinear control system, 42
z_e	Equilibrium of affine nonlinear control system, 42

Greek Symbols

$\alpha_{1,2}$	Amplitude of shape control, 45
α_e	External forcing one–form, 17
$\beta(\theta_{1,2})$	Auxiliary function, 22
$\Gamma^k_{i,j}$	Structure constants of Lie algebra $\mathcal{G}$, 8
$\gamma(\theta_{1,2})$	Auxiliary function, 22
$\Delta(\theta_{1,2})$	Auxiliary function, 21
$\Delta_1(\theta_{1,2})$	Auxiliary function, 35
$\delta(\theta_{1,2})$	Auxiliary function, 21
$\eta_i(\theta_{1,2})$	Auxiliary function ($i = 1, \ldots, 5$), 26, 38
$\theta_{1,2}$	Angle of Roller Racer rotary joint, 9
$\lambda(\theta_{1,2})$	Auxiliary function, 22
ν_i	Auxiliary velocity ($i = 1, 2$), 17
ξ	Element of Lie algebra $\mathcal{G}$, 8
$\xi_Q(q)$	Infinitesimal generator of action Φ corresponding to $\xi \in \mathcal{G}$, 12
ξ^i_Q	Basis element of $\mathcal{D}_q$, 10
ξ^H_Q	Basis element of H_q, 31
$\xi^{\mathcal{U}_i}_Q$	Basis element of $\mathcal{U}_q$, 32
ξ^q_Q	Basis element of $\mathcal{S}_q$, 13
$\pi : Q \to \mathcal{S} = Q/G$	Canonical bundle projection, 30, 33
$\tau_{1,2}$	Torque applied by motor to the rotary joint, 20
Φ	Action of a Lie group on a manifold, 12
$\Phi(t, t_0)$	State transition matrix, 23
$\phi_{j,i}$	Angle of j–th wheel of platform i, 11
$\omega_{1,2}$	Frequency of shape control, 45
ω^i_q	Nonholonomic constraint one–forms, 10

Miscellaneous Symbols

$\blacksquare$	End of proof, 14
$[\ ,\ ]$	Lie bracket of Lie algebra $\mathcal{G}$, 8
$\ll\ ,\ \gg$	Kinetic energy inner product, 31

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