OSCILLATIONS, SE(2)–SNAKES AND MOTION CONTROL: A STUDY OF THE ROLLER RACER

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Abstract

This report is concerned with the problem of motion generation via cyclic variations in selected degrees of freedom (usually referred to as shape variables) in mechanical systems subject to nonholonomic constraints (here the classical one of a disk rolling without sliding on a flat surface). In earlier work, we identified an interesting class of such problems arising in the setting of Lie groups, and investigated these under a hypothesis on constraints, that naturally led to a purely kinematic approach. In the present work, the hypothesis on constraints does not hold, and as a consequence, it is necessary to take into account certain dynamical phenomena. Specifically we concern ourselves with the group $SE(2)$ of rigid motions in the plane and a concrete mechanical realization dubbed the 2–node, 1–module $SE(2)$–snake. In a restricted version, it is also known as the Roller Racer (a patented ride/toy).

Based on the work of Bloch, Krishnaprasad, Marsden and Murray, one recognizes in the example of this report a balance law called the momentum equation, which is a direct consequence of the interaction of the $SE(2)$–symmetry of the problem with the constraints. The systematic use of this type of balance law results in certain structures in the example of this report. We exploit these structures to demonstrate that the single shape freedom in this problem can be cyclically varied to produce a rich variety of motions of the $SE(2)$–snake.

In their study of the snakeboard, a patented modification of the skateboard that also admits the group $SE(2)$ as a symmetry group, Lewis, Ostrowski, Burdick and Murray, exploited the same type of balance law as the one discussed here to generate motions. A key difference however is that, in the present report, we have only one control variable and thus controllability considerations become somewhat more delicate.

In the present report, we give a self–contained treatment of the geometry, mechanics and motion control of the Roller Racer.

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1 Introduction

The idea of using periodic driving signals to produce rectified movement appears in a number of settings in engineering. Some of the more inventive examples are associated with the design and operation of novel actuators exploiting vibratory transduction (Ueha & Tomikawa [1993]; Venkataraman et al. [1995]). In his paper (Brockett [1989]), Brockett develops a mathematical basis for understanding such devices. Elsewhere, in the context of robotic machines with many degrees of freedom designed to mimic snake–like movements (Hirose [1993]), periodic variations in the shape parameters are used in an essential way to generate global movements. In (Krishnaprasad & Tsakiris [1994b]) and (Krishnaprasad & Tsakiris [1994a]), we have developed a general mathematical formulation to study systems of this type. The study of periodic signal generators (also called central pattern generators), as sources of timing signals to compose movements has a long history in the neurophysiology of movement dating back to the early work of Sherrington, Brown and Bernstein.

Recent studies by neurophysiologists (Carling, Bowtell & Williams [1994]) have attempted to bring together principles of motion control based on pattern generation in the spinal cord of the *lamprey*, its compliant body dynamics, and the fluid dynamics of its environment to achieve a comprehensive understanding of the swimming behavior of such anguilliform animals. These efforts have in part relied on continuum mechanical models of the body, and computational fluid dynamical (CFD) calculations. There appear to be some unifying themes that underlie this type of neural–mechanical approach to biological locomotion, and the work of the authors and others involving the study of land–based robotic machines subject to the constraint of 'no sliding'. As pointed out in (Krishnaprasad [1995]), the connecting links between these two streams of research appear to be related to the manner in which systems of coupled oscillators are used to generate finite dimensional shape variations of the bodies of specialized robot designs, and the associated geometric–mechanical descriptions of the constraints to produce effective motion control strategies (see also the work of Collins and Stewart (Collins & Stewart [1993]) for another dynamical systems perspective).

In the present paper, we report on a complete study of an interesting example, the (single module) $SE(2)$–snake, with a view towards deeper appreciation of the above–mentioned connections. In section 3, we present the basic geometry of the configuration space, and the applicable constraints. We also discuss a simplification that reduces the shape freedom to one variable, leading to the Roller Racer. The constraints of 'no sliding' are *insufficient* to determine the movement of the Roller Racer from shape variations alone. In section 5, a model Lagrangian and the action of $SE(2)$, the rigid motion group in the plane as a symmetry group (of the Lagrangian and the constraints) are discussed. A balance law associated to the $SE(2)$–symmetry, the momentum equation, is derived, which is a consequence of the Lagrange–D'Alembert principle (The basic results behind momentum equations are to be found in (Bloch, Krishnaprasad, Marsden & Murray [1994])). This momentum equation is the key additional data that, together with the constraints, allows us to generate motion control laws. In section 6, we consider controllability issues.

G–snakes are kinematic chains with configurations taking values in products of several copies of a Lie group G, and subject to nonholonomic constraints (Krishnaprasad & Tsakiris [1994a]; Tsakiris [1995]). The group G acts on the chain by diagonal action as a symmetry group. The shape space is the quotient by this action. Fig. 1.1 illustrates an $SE(2)$–snake composed of two modules and three nodes and the configuration space Q is $SE(2)\times SE(2)\times SE(2)$.

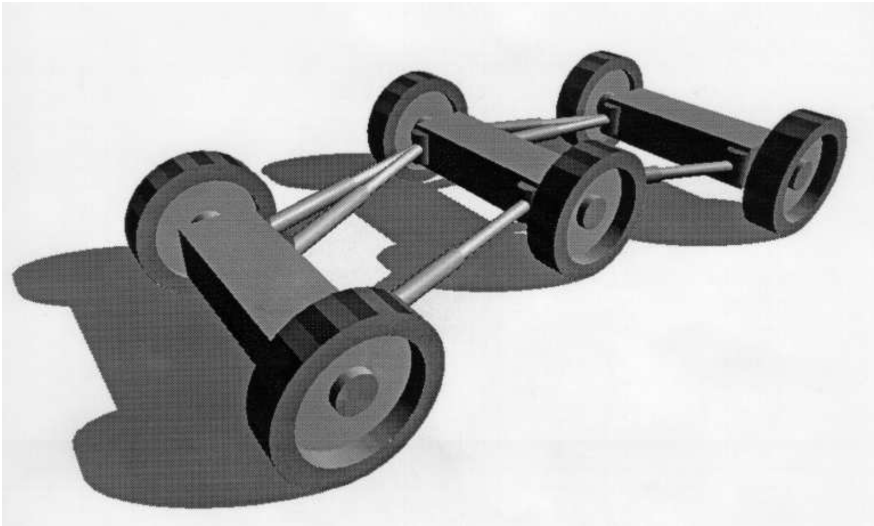

Fig. 1.1: The 2–module $SE(2)$–Snake

The machine in fig. 1.1 is composed of three axles and linearly actuated linkages connecting each adjacent pair of axles, resulting in an assembly of two identical modules. Altering the lengths of the connecting linkages leads to changes in the shapes of component modules. The wheels mounted on each axle are independent and are *not* actuated but subject to the constraint of 'no sliding'. In this case, there are three constraints, the shape space S is $SE(2)\times SE(2)$, the constraints define a principal connection on the bundle $(Q, SE(2), S)$, away from a set of nonholonomic singularities, and it is possible to generate global movement of the assembly by periodic variations in the module shapes.

The entire situation can be understood at a kinematic level as long as the shapes are control variables ((Krishnaprasad & Tsakiris [1994a]; Krishnaprasad & Tsakiris [1994b]; Tsakiris [1995])).

When one of the modules is removed from the machine in fig. 1.1, leaving us with two axles connected by linkages and two nonholonomic constraints, the resulting problem is kinematically under–constrained. It is no longer possible to define a connection without using additional information. It is this type of 1–module $SE(2)$–snake that is of interest here. Matters can be simplified by limiting the extent of shape freedom. In 1972, W.E. Hendricks was awarded U.S. patent no. $3,663,038$ for a toy illustrated in fig. 1.2 and dubbed the Roller Racer, that serves as one such simplification. The rider, on the seat shown, has to merely oscillate the handle–bars from side–to–side to generate forward propulsion, a behavior for which Hendricks did not claim to have an explanation.

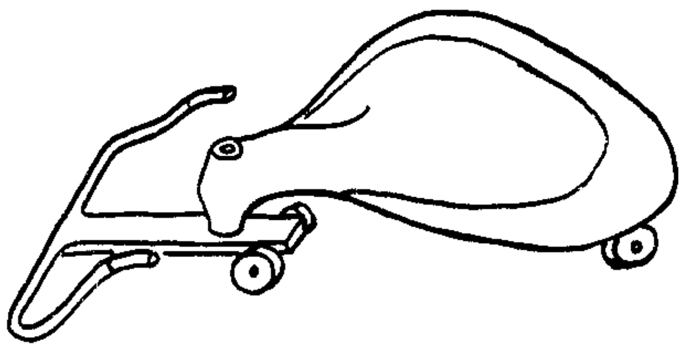

Fig. 1.2: The Roller Racer

The model of fig. 1.3 will be used in our analysis. Two planar platforms with centers of mass (c.o.m.) located at points O_1 and O_2 are connected with a rotary joint at $O_{1,2}$. A pair of idler wheels is attached on each of the platforms, with the axis of the wheels perpendicular to the line connecting the c.o.m. with the joint. A coordinate frame centered at the c.o.m. and with its x–axis along the line $O_iO_{1,2}$ for $i = 1, 2$ connecting the c.o.m. with the joint, will be used to describe the configuration of each platform with respect to a global coordinate system at some reference point O. For simplicity, it will be assumed that the axis of the wheels passes through the c.o.m. of each platform.

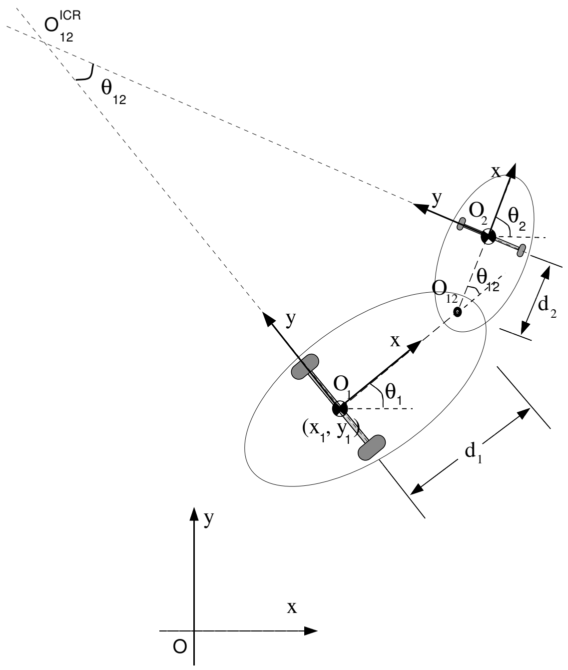

Fig. 1.3: The Roller Racer Model

The effects of the rider's body motion will be ignored at first approximation. Experiments with the Roller Racer and our analysis below, show that, even though these body motions may amplify the resulting motion of the system, the fundamental means of its propulsion is the pivoting of the steering arm around the joint axis and the nonholonomic constraints coming from the wheels' rolling–without–slipping on the plane supporting the vehicle. In this respect, this system is very different from the Snakeboard, a variation of the skateboard, where the motion of the rider is essential for the propulsion of the system (Lewis et al. [1994]). Riderless prototypes of the Roller Racer built at the Intelligent Servosystems Laboratory verified this. The propulsion and steering mechanism in these vehicles comes from a rotary motor at the joint $O_{1,2}$, whose torque can be considered as the control of our system. As discussed in (Krishnaprasad & Tsakiris [1993]; Krishnaprasad & Tsakiris [1994b]), the purely kinematic analysis of such a system does not allow us to determine the global motion of the system by just the shape variations (the joint velocity in this case), since (unlike the 2-module case) it does not possess a sufficient number of nonholonomic constraints for this to happen. Our goal here is to complement this kinematic analysis with the dynamics of the system, which will provide the necessary information. Thus, certain fundamental behaviors of the system ("straight–line" motion, "turning" motion) can be achieved by proper oscillatory relative motions of the two platforms. In both numerical simulations and experiments with prototypes, we observed such behaviors.

In order to study the dynamics of this system, an alternative to the usual approach of solving the full Lagrange–d' Alembert equations of motion of the system is considered here. In (Bloch, Krishnaprasad, Marsden & Murray [1994]), the notion of the momentum map is examined for systems with nonholonomic constraints and symmetries and its evolution law, the momentum equation, is derived from the Lagrange–d' Alembert equations. By applying this method to the problem at hand, a useful decomposition of the equations of motion is obtained: Given a shape–space trajectory (which corresponds to the controls of our system), first we compute the nonholonomic momentum from the momentum equation. *This only involves the solution of a linear ordinary differential equation.* Subsequently, we use the momentum to reconstruct the group trajectory, which corresponds to the global motion of the system. The corresponding velocities depend linearly on the momentum. This process is very useful for the derivation of motion control laws for this system and can be extended to 1–module $SE(2)$–snakes with more general shape–changing mechanisms.

2 Preliminaries

2.1 Group Actions, Principal Fiber Bundles and Connections

Definition 2.1 (Action of Lie Group on Manifold)

Let Q be a smooth manifold. A (left) *action* of a Lie group G on Q is a smooth mapping $\Phi : G \times Q \longrightarrow Q$ such that:

i) For all $q \in Q$, $\Phi(e, x) = x$.

ii) For every $g, h \in G$, $\Phi(g, \Phi(h, q)) = \Phi(gh, q)$, for all $q \in Q$.

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For every $g \in G$, define $\Phi_g : Q \longrightarrow Q : q \longrightarrow \Phi(g, q)$. Then from property (i) above, we have $\Phi_e = id_Q$ and from (ii), $\Phi_{gh} = \Phi_g \circ \Phi_h$. Moreover, $(\Phi_g)^{-1} = \Phi_{g^{-1}}$. Thus, Φ_g is a diffeomorphism (i.e. one–to–one, onto and both Φ_g and $(\Phi_g)^{-1}$ are smooth).

Definitions 2.2

Let Φ be an action of G on Q.

i) For $q \in Q$, the *orbit* (or Φ–orbit) of q is $\mathrm{Orb}(q) = \{\Phi_g(q) | g \in G\}$.

ii) An action is *transitive* if there is only one orbit.

iii) An action Φ is *effective* (or *faithful*) if $g \mapsto \Phi_g$ is one–to–one.

iv) An action Φ is *free* if, for each $q \in Q$, $g \mapsto \Phi_g(q)$ is one–to–one, i.e. the identity e is the only element of G with a fixed point.

v) An action Φ is *proper* if and only if the map $\Phi : G \times Q \longrightarrow Q \times Q : (g, q) \mapsto \big(q, \Phi(g, q)\big) = \tilde{\Phi}(g, q)$ is proper, i.e. if $K \subset Q \times Q$ is compact, then $\tilde{\Phi}^{-1}(K)$ is compact.

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Definition 2.3 (Infinitesimal Generator)

Let $\Phi : G \times Q \longrightarrow Q$ be a smooth action. If $\xi \in \mathcal{G}$, then $\Phi^{\xi} : I\!R \times Q \longrightarrow Q : (t, q) \mapsto \Phi(\exp t\xi, q)$ is an $I\!R$–action on Q, i.e. is a flow on Q. The corresponding vector field on Q is called the *infinitesimal generator* of Φ corresponding to ξ and is given by

$$\xi_Q(q) = \frac{d}{dt}\Phi(\exp t\xi, q)\Big|_{t=0} \; . \tag{2.1}$$

Then, the tangent space to the orbit $\mathrm{Orb}(q)$ of q is

$$T_q\mathrm{Orb}(q) = \{\xi_Q(q) | \xi \in \mathcal{G}\} \; . \tag{2.2}$$

∎

Examples 2.4

i) Let Φ be the left translation of a matrix Lie group G, considered as an action of G on G, i.e. $\Phi : G \times G \longrightarrow G : (g, h) \mapsto gh = L_g h$. Then the infinitesimal generator $\xi_{_G}$ corresponding to $\xi \in \mathcal{G}$ is $\xi_{_G}(q) = T_e R_g \cdot \xi = \xi g$, which is a right invariant vector field.

ii) Consider the *adjoint action* of G on its Lie algebra $\mathcal{G}$, defined as $\Phi : G \times \mathcal{G} \longrightarrow \mathcal{G} : (g, \eta) \mapsto Ad_g \eta = T_e(R_{g^{-1}} L_g)\eta$. If G is a matrix group, then

$$Ad_g \eta = g \eta g^{-1} \ , \tag{2.3}$$

for $\eta \in \mathcal{G}$. If $\xi \in \mathcal{G}$, then

$$\xi_{_\mathcal{G}} \eta = ad_\xi \eta \stackrel{\text{def}}{=} [\xi, \eta] \ . \tag{2.4}$$

Further define $ad_\xi^k \eta \stackrel{\text{def}}{=} ad_\xi(ad_\xi^{k-1}\eta) = [\xi, ad_\xi^{k-1}\eta]$.
The following properties of the adjoint action of a matrix Lie group are easily checked:
i) $Ad_g(a_1\eta_1 + a_2\eta_2) = a_1 Ad_g \eta_1 + a_2 Ad_g \eta_2$, for $a_1, a_2 \in I\!R$ and $\eta_1, \eta_2 \in \mathcal{G}$.
ii) $Ad_{g_1 g_2}(\eta) = Ad_{g_1}(Ad_{g_2}\eta)$, for $g_1, g_2 \in G$.
iii) $Ad_g(\eta_1\eta_2) = Ad_g\eta_1 Ad_g\eta_2$, for $\eta_1, \eta_2 \in \mathcal{G}$.

■

Consider a left–invariant dynamical system on a matrix Lie group G with n–dimensional Lie algebra $\mathcal{G}$. Consider a curve $g(.) \subset G$. Then, there exists a curve $\xi(.) \in \mathcal{G}$ such that:

$$\dot{g} = T_e L_g \cdot \xi = L_g \xi = g \ \xi \ . \tag{2.5}$$

Let $\{\mathcal{A}_i, \ i = 1, \ldots, n\}$ be a basis of $\mathcal{G}$ and let $[.,.]$ be the usual Lie bracket on $\mathcal{G}$ defined by: $[\mathcal{A}_i, \mathcal{A}_j] = \mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i$. Then, there exist constants $\Gamma_{i,j}^k$, called *structure constants,* such that:

$$[\mathcal{A}_i, \mathcal{A}_j] = \sum_{k=1}^{n} \Gamma_{i,j}^k \mathcal{A}_k \ , \quad i, j = 1, \ldots, n. \tag{2.6}$$

Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$, i.e. the space of linear functions from $\mathcal{G}$ to $I\!R$. Let $\{\mathcal{A}_i^\flat, \ i = 1, \ldots, n\}$ be the basis of $\mathcal{G}^*$ such that

$$\mathcal{A}_i^\flat(\mathcal{A}_j) = \delta_i^j, \quad i, j = 1, \ldots, n, \tag{2.7}$$

where δ_i^j is the Kronecker symbol. Then the curve $\xi(.) \subset \mathcal{G}$ can be represented as:

$$\xi = \sum_{i=1}^{n} \xi_{_i} \mathcal{A}_i = \sum_{i=1}^{n} \mathcal{A}_i^\flat(\xi)\mathcal{A}_i \ , \tag{2.8}$$

for $\xi_i \stackrel{\text{def}}{=} \mathcal{A}_i^\flat(\xi) \in I\!R,\ i = 1, \ldots, n$.

To obtain a solution of the dynamical system (2.5) we use the following product–of–exponentials representation.

Proposition 2.5 (Wei & Norman [1964])

Let $g(0) = I$, the identity of G and let $g(t)$ be the solution of (2.5). Then, locally around $t = 0$, g is of the form:

$$g(t) = e^{\gamma_1(t)\mathcal{A}_1} e^{\gamma_2(t)\mathcal{A}_2} \cdots e^{\gamma_n(t)\mathcal{A}_n} \,, \tag{2.9}$$

where the coefficients γ_i are determined by differentiating (2.9) and using (2.5). For the coordinates ξ_i of $\xi \in \mathcal{G}$ defined in (2.8), we get:

$$\begin{pmatrix} \dot\gamma_1 \\ \vdots \\ \dot\gamma_n \end{pmatrix} = M(\gamma_1, \ldots, \gamma_n) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} . \tag{2.10}$$

The matrix M is analytic in γ and depends only on the Lie algebra $\mathcal{G}$ and its structure constants in the given basis. If $\mathcal{G}$ is solvable, then there exists a basis of $\mathcal{G}$ and an ordering of this basis, for which (2.9)is global. Then the γ_i's can be found by quadratures.

∎

From (2.8) and from the properties of the adjoint action, we have:

$$Ad_g\xi = \sum_{i=1}^{n} \xi_i Ad_g\mathcal{A}_i = \sum_{i=1}^{n} \mathcal{A}_i^\flat(\xi) Ad_g\mathcal{A}_i \,. \tag{2.11}$$

From (2.9) we have:

$$Ad_g\mathcal{A}_i = g\mathcal{A}_i g^{-1} = e^{\gamma_1\mathcal{A}_1} \cdots e^{\gamma_n\mathcal{A}_n} \mathcal{A}_i e^{-\gamma_n\mathcal{A}_n} \cdots e^{-\gamma_1\mathcal{A}_1} \tag{2.12}$$

and

$$Ad_{g^{-1}}\mathcal{A}_i = g^{-1}\mathcal{A}_i g = e^{-\gamma_n\mathcal{A}_n} \cdots e^{-\gamma_1\mathcal{A}_1} \mathcal{A}_i e^{\gamma_1\mathcal{A}_1} \cdots e^{\gamma_n\mathcal{A}_n} \,. \tag{2.13}$$

Equation (2.12) can be made more explicit by the Baker–Campbell–Hausdorff formula (Wei & Norman [1964]), which for $\xi, \eta \in \mathcal{G}$ states that (c.f. the definitions (2.3) and (2.4)):

$$Ad_{\exp\xi}\eta = e^{\xi}\eta e^{-\xi} = \left(e^{ad_\xi}\right)\eta = \eta + [\xi,\eta] + \frac{1}{2!}[\xi,[\xi,\eta]] + \cdots \,. \tag{2.14}$$

Thus

$$Ad_g\mathcal{A}_i = e^{ad(\gamma_1\mathcal{A}_1)} \cdots e^{ad(\gamma_n\mathcal{A}_n)} \mathcal{A}_i \tag{2.15}$$

and

$$Ad_{g^{-1}}\mathcal{A}_i = e^{ad(-\gamma_n \mathcal{A}_n)} \cdots e^{ad(-\gamma_1 \mathcal{A}_1)} \mathcal{A}_i \ , \tag{2.16}$$

for $i = 1, \ldots, n$.

The following material on principal fiber bundles and connections is based on (Bleecker [1981]; Nomizu [1956]). These references consider principal fiber bundles where the group action is a right action. Here we consider left actions and modify appropriately the definition of a principal fiber bundle, as for instance done in (Yang [1992]).

Definition 2.6 (Principal Fiber Bundle)

Let $\mathcal{S}$ be a differentiable manifold and G a Lie group. A differentiable manifold Q is called a (differentiable) *principal fiber bundle*, if the following conditions are satisfied:
1) G acts on Q to the left, freely and differentiably:

$$\Phi : \ G \times Q \to Q : \ (g, q) \mapsto g \cdot q \stackrel{\text{def}}{=} \Phi_g \cdot q \ .$$

2) $\mathcal{S}$ is the quotient space of Q by the equivalence relation induced by G, i.e. $\mathcal{S} = Q/G$ and the canonical projection $\pi : Q \to \mathcal{S}$ is differentiable.
3) Q is locally trivial, i.e. every point $s \in \mathcal{S}$ has a neighborhood U such that $\pi^{-1}(U) \subset Q$ is isomorphic with $U \times G$, in the sense that $q \in \pi^{-1}(U) \mapsto \big(\pi(q), \phi(q)\big) \in U \times G$ is a diffeomorphism such that $\phi : \pi^{-1}(U) \to G$ satisfies $\phi(g \cdot q) = g\phi(q), \forall g \in G$.
For $s \in \mathcal{S}$, the *fiber over* s is a closed submanifold of Q which is differentiably isomorphic with G. For any $q \in Q$, the *fiber through* q is the fiber over $s = \pi(q)$. When $Q = \mathcal{S} \times G$, then Q is said to be a *trivial* principal fiber bundle (fig. 2.1).

∎

Definition 2.7 (Connection on a Principal Fiber Bundle)

Let $(Q, \mathcal{S}, \pi, G)$ be a principal fiber bundle. A *connection* on the principal fiber bundle is a choice of a tangent subspace $H_q \subset T_qQ$ at each point $q \in Q$ (horizontal subspace) such that, if $V_q \stackrel{\text{def}}{=} \{v \in T_qQ | T_q\pi(v) = 0\}$ is the subspace of T_qQ tangent to the fiber through q (vertical subspace), we have:
1) $T_qQ = H_q \oplus V_q$.
2) For every $g \in G$ and $q \in Q$, $T_q\Phi_g \cdot H_q = H_{g \cdot q}$.
3) H_q depends differentiably on q.

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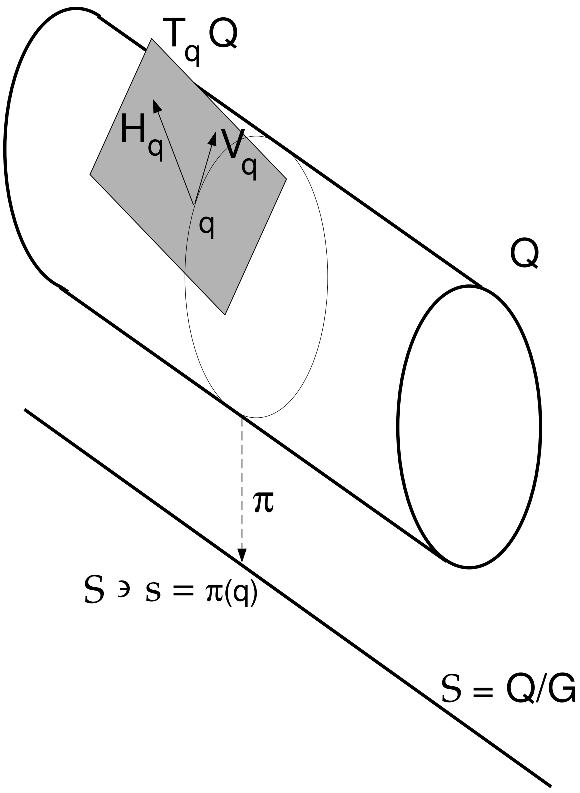


Fig. 2.1: Connection on a Principal Fiber Bundle

2.2 The Special Euclidean Group $SE(2)$

Let $G = SE(2)$ be the Special Euclidean group of rigid motions on the plane and $\mathcal{G} = se(2)$ be the corresponding Lie algebra with the following basis:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.17}$$

Observe that

$$[\mathcal{A}_1 \ , \ \mathcal{A}_2] = \mathcal{A}_3 \ , \ [\mathcal{A}_1 \ , \ \mathcal{A}_3] = -\mathcal{A}_2 \ , \ [\mathcal{A}_2 \ , \ \mathcal{A}_3] = 0 \ . \tag{2.18}$$

From (2.8), an element $\xi \in \mathcal{G}$ is represented as

$$\xi = \sum_{i=1}^{3} \xi_i \mathcal{A}_i = \begin{pmatrix} 0 & -\xi_1 & \xi_2 \\ \xi_1 & 0 & \xi_3 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.19}$$

For the proofs of the subsequent propositions, refer to (Tsakiris [1995]).

The Wei–Norman representation of a curve $g(.)$ on $SE(2)$ is as follows:

Proposition 2.8

Let $g(0) = I$, where I is the identity of $G = SE(2)$. There exists a *global* representation of the curve $g(.) \subset G$ of the form

$$g(t) = e^{\gamma_1(t)\mathcal{A}_1} e^{\gamma_2(t)\mathcal{A}_2} e^{\gamma_3(t)\mathcal{A}_3} \ . \tag{2.20}$$

The coefficients $\gamma_i \in I\!R$ are related to the components of ξ by

$$\begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \\ \dot\gamma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \ . \tag{2.21}$$

Equation (2.21) is solvable by quadratures:

$$\begin{aligned}
\gamma_1(t) &= \gamma_1(0) + \int_0^t \xi_1(\tau) d\tau \ , \\
\gamma_2(t) &= \gamma_2(0) \cos\left(\int_0^t \xi_1(\sigma)d\sigma\right) + \gamma_3(0) \sin\left(\int_0^t \xi_1(\sigma)d\sigma\right) \\
&\quad + \int_0^t \xi_2(\tau) \cos\left(\int_\tau^t \xi_1(\sigma)d\sigma\right) d\tau + \int_0^t \xi_3(\tau) \sin\left(\int_\tau^t \xi_1(\sigma)d\sigma\right) d\tau \ , \\
\gamma_3(t) &= -\gamma_2(0) \sin\left(\int_0^t \xi_1(\sigma)d\sigma\right) + \gamma_3(0) \cos\left(\int_0^t \xi_1(\sigma)d\sigma\right) \\
&\quad - \int_0^t \xi_2(\tau) \sin\left(\int_\tau^t \xi_1(\sigma)d\sigma\right) d\tau + \int_0^t \xi_3(\tau) \cos\left(\int_\tau^t \xi_1(\sigma)d\sigma\right) d\tau \ .
\end{aligned} \tag{2.22}$$

For the initial condition $g(0) = I$, we have $\gamma_i(0) = 0, \ i = 1, 2, 3$.

∎

Lemma 2.9

Consider the Wei–Norman representation (2.20) of $g \in SE(2)$ determined by (2.21) and (2.22). Then:

$$\begin{aligned}
Ad_g\mathcal{A}_1 &= \mathcal{A}_1 + (\gamma_2 \sin\gamma_1 + \gamma_3 \cos\gamma_1)\mathcal{A}_2 + (-\gamma_2 \cos\gamma_1 + \gamma_3 \sin\gamma_1)\mathcal{A}_3 \ , \\
Ad_g\mathcal{A}_2 &= \cos\gamma_1\mathcal{A}_2 + \sin\gamma_1\mathcal{A}_3 \ , \\
Ad_g\mathcal{A}_3 &= -\sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3 \ .
\end{aligned} \tag{2.23}$$

Moreover,

$$\begin{aligned}
Ad_{g^{-1}}\mathcal{A}_1 &= \mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3 \ , \\
Ad_{g^{-1}}\mathcal{A}_2 &= \cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\mathcal{A}_3 \ , \\
Ad_{g^{-1}}\mathcal{A}_3 &= \sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3 \ .
\end{aligned} \tag{2.24}$$

■

Using the definition of the basis $\{\mathcal{A}_i\}$ from (2.17), we have:

$$e^{\gamma_1\mathcal{A}_1} = \begin{pmatrix} \cos\gamma_1 & -\sin\gamma_1 & 0 \\ \sin\gamma_1 & \cos\gamma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \quad e^{\gamma_2\mathcal{A}_2} = \begin{pmatrix} 1 & 0 & \gamma_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \quad e^{\gamma_3\mathcal{A}_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma_3 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{2.25}$$

Then, from (2.20):

$$g = e^{\gamma_1\mathcal{A}_1}e^{\gamma_2\mathcal{A}_2}e^{\gamma_3\mathcal{A}_3} = \begin{pmatrix} \cos\gamma_1 & -\sin\gamma_1 & \gamma_2\cos\gamma_1 - \gamma_3\sin\gamma_1 \\ \sin\gamma_1 & \cos\gamma_1 & \gamma_2\sin\gamma_1 + \gamma_3\cos\gamma_1 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{2.26}$$

Define:

$$\begin{aligned}
\phi &\stackrel{\text{def}}{=} \gamma_1 \\
x &\stackrel{\text{def}}{=} \gamma_2\cos\gamma_1 - \gamma_3\sin\gamma_1 \\
&= x_0 + \int_0^t \xi_2(\tau)\cos\phi(\tau)\ d\tau - \int_0^t \xi_3(\tau)\sin\phi(\tau)\ d\tau \ , \\
y &\stackrel{\text{def}}{=} \gamma_2\sin\gamma_1 + \gamma_3\cos\gamma_1 \\
&= y_0 + \int_0^t \xi_2(\tau)\sin\phi(\tau)\ d\tau + \int_0^t \xi_3(\tau)\cos\phi(\tau)\ d\tau \ ,
\end{aligned} \tag{2.27}$$

where $x_0 \stackrel{\text{def}}{=} \gamma_2(0)\cos\gamma_1(0) - \gamma_3(0)\sin\gamma_1(0)$ and $y_0 \stackrel{\text{def}}{=} \gamma_2(0)\sin\gamma_1(0) + \gamma_3(0)\cos\gamma_1(0)$. Then we take from (2.26) the usual homogeneous matrix representation of the elements of $SE(2)$

$$g = \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{2.28}$$

Differentiating and using (2.19) and (2.21), we have:

$$\begin{aligned}
\dot{x} &= \xi_2\cos\gamma_1 - \xi_3\sin\gamma_1 \ , \\
\dot{y} &= \xi_2\sin\gamma_1 + \xi_3\cos\gamma_1 \ , \\
\dot{\phi} &= \xi_1 \ .
\end{aligned} \tag{2.29}$$

Observe that

$$
\begin{aligned}
\gamma_1 &= \phi \ , \\
\gamma_2 &= x \cos \phi + y \sin \phi \ , \\
\gamma_3 &= -x \sin \phi + y \cos \phi
\end{aligned}
\tag{2.30}
$$

and

$$
\begin{aligned}
\xi_1 &= \dot{\phi} \ , \\
\xi_2 &= \dot{x} \cos \phi + \dot{y} \sin \phi \ , \\
\xi_3 &= -\dot{x} \sin \phi + \dot{y} \cos \phi \ .
\end{aligned}
\tag{2.31}
$$

3 Kinematics of the Roller Racer

Let $g_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & x_i \\ \sin\theta_i & \cos\theta_i & y_i \\ 0 & 0 & 1 \end{pmatrix} \in SE(2)$, for $i = 1, 2$, be the configuration of platform i with respect to the global coordinate frame at O, where x_i, y_i and θ_i are indicated in fig. 1.3. Let $g_{1,2} = \begin{pmatrix} \cos\theta_{1,2} & -\sin\theta_{1,2} & x_{1,2} \\ \sin\theta_{1,2} & \cos\theta_{1,2} & y_{1,2} \\ 0 & 0 & 1 \end{pmatrix} \in SE(2)$ be the configuration of platform 2 with respect to the coordinate frame of platform 1 at O_1. Because of the special structure of the joint, we have

$$x_{1,2} = d_1 + d_2 \cos\theta_{1,2}\ , \quad y_{1,2} = d_2 \sin\theta_{1,2}\ , \tag{3.1}$$

where $\theta_{1,2}$ is the relative angle of the two platforms and d_i is the distance of O_i from the joint $O_{1,2}$, as indicated in fig. 1.3. We consider non–negative d_1 and d_2. In fact, we assume $d_1 > 0$. However, we allow for the case $d_2 = 0$ and we examine it in detail.

Since the platforms form a kinematic chain, we have

$$g_2 = g_1 g_{1,2}\ , \tag{3.2}$$

thus

$$\begin{aligned} &\theta_2 = \theta_1 + \theta_{1,2}\ , \\ &x_2 = x_1 + x_{1,2}\cos\theta_1 - y_{1,2}\sin\theta_1 = x_1 + d_1\cos\theta_1 + d_2\cos\theta_2\ , \\ &y_2 = y_1 + x_{1,2}\sin\theta_1 + y_{1,2}\cos\theta_1 = y_1 + d_1\sin\theta_1 + d_2\sin\theta_2\ . \end{aligned} \tag{3.3}$$

The system kinematics are a special case of the n–module $SE(2)$–snake (n–VGT) assembly (Krishnaprasad & Tsakiris [1994b]; Tsakiris [1995]), i.e. for $\xi_i = \begin{pmatrix} 0 & -\xi^i_1 & \xi^i_2 \\ \xi^i_1 & 0 & \xi^i_3 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{G} = se(2)$ we have:

$$\dot g_i = g_i \xi_i\ , \ i = 1, 2 \tag{3.4}$$

and

$$\dot g_{1,2} = g_{1,2}\xi_{1,2}\ , \tag{3.5}$$

where $\xi_{1,2} = \begin{pmatrix} 0 & -\xi^{1,2}_1 & \xi^{1,2}_2 \\ \xi^{1,2}_1 & 0 & \xi^{1,2}_3 \\ 0 & 0 & 0 \end{pmatrix} = \dot\theta_{1,2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & d_2 \\ 0 & 0 & 0 \end{pmatrix}$. From (3.4) and (3.5), we can show that

$$\xi_2 = Ad_{g^{-1}_{1,2}}\xi_1 + \xi_{1,2}\ . \tag{3.6}$$

Equivalently, using (2.11), this can be written as

$$
\begin{aligned}
\xi_1^2 &= \xi_1^1 + \xi_1^{1,2} \ , \\
\xi_2^2 &= -\xi_1^1 \gamma_3^{1,2} + \xi_2^1 \cos\gamma_1^{1,2} + \xi_3^1 \sin\gamma_1^{1,2} + \xi_2^{1,2} \ , \\
\xi_3^2 &= \xi_1^1 \gamma_2^{1,2} - \xi_2^1 \sin\gamma_1^{1,2} + \xi_3^1 \cos\gamma_1^{1,2} + \xi_3^{1,2} \ ,
\end{aligned}
\tag{3.7}
$$

where $\gamma_i^{1,2}$ are the Wei–Norman parameters of $g_{1,2}$. From this and from (2.29), or directly by differentiating (3.3), we get

$$
\begin{aligned}
\dot\theta_2 &= \dot\theta_1 + \dot\theta_{1,2} \ , \\
\dot x_2 &= \dot x_1 + \dot x_{1,2}\cos\theta_1 - \dot y_{1,2}\sin\theta_1 - (x_{1,2}\sin\theta_1 + y_{1,2}\cos\theta_1)\dot\theta_1 \\
&= \dot x_1 - \dot\theta_1[d_1\sin\theta_1 + d_2\sin(\theta_1+\theta_{1,2})] - \dot\theta_{1,2} d_2 \sin(\theta_1+\theta_{1,2}) \ , \\
\dot y_2 &= \dot y_1 + \dot x_{1,2}\sin\theta_1 + \dot y_{1,2}\cos\theta_1 + (x_{1,2}\cos\theta_1 - y_{1,2}\sin\theta_1)\dot\theta_1 \\
&= \dot y_1 + \dot\theta_1[d_1\cos\theta_1 + d_2\cos(\theta_1+\theta_{1,2})] + \dot\theta_{1,2} d_2 \cos(\theta_1+\theta_{1,2}) \ .
\end{aligned}
\tag{3.8}
$$

From (3.3) we see that the configuration space for the Roller Racer system is $Q = SE(2)\times S^1$.

Consider the usual bases $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ for $\mathcal{G} = se(2)$ and $\{\mathcal{A}_1^\flat, \mathcal{A}_2^\flat, \mathcal{A}_3^\flat\}$ for its dual space $\mathcal{G}^*$. The nonholonomic constraints on the wheels of the two platforms can be expressed as:

$$
\xi_3^1 = \mathcal{A}_3^\flat(\xi_1) = -\dot x_1 \sin\theta_1 + \dot y_1\cos\theta_1 = 0 \ , \tag{3.9}
$$

$$
\xi_3^2 = \mathcal{A}_3^\flat(\xi_2) = -\dot x_2 \sin\theta_2 + \dot y_2\cos\theta_2 = 0 \ , \tag{3.10}
$$

From (3.8) and (3.10), we get

$$
\xi_3^2 = \mathcal{A}_3^\flat(\xi_2) = -\dot x_1\sin(\theta_1+\theta_{1,2}) + \dot y_1\cos(\theta_1+\theta_{1,2}) + \dot\theta_1(d_1\cos\theta_{1,2} + d_2) + \dot\theta_{1,2} d_2 = 0 \ . \tag{3.11}
$$

Observe that for $d_2 = 0$, neither one of the constraints (3.9) and (3.11) involves $\dot\theta_{1,2}$. From (3.9) and (3.11), we get:

$$
\xi_3^2 = \mathcal{A}_3^\flat(\xi_2) = -(\dot x_1\cos\theta_1 + \dot y_1\sin\theta_1)\sin\theta_{1,2} + (d_1\cos\theta_{1,2} + d_2)\dot\theta_1 + d_2\dot\theta_{1,2} = 0 \ . \tag{3.12}
$$

Proposition 3.1

The nonholonomic constraints (3.9) and (3.11) are linearly independent for all $q \in Q$.

Proof

We can rewrite the constraints in the form:

$$\begin{pmatrix} -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ -\sin(\theta_1+\theta_{1,2}) & \cos(\theta_1+\theta_{1,2}) & d_1\cos\theta_{1,2}+d_2 & d_2 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\theta}_{1,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Consider all possible 2×2 minors of the constraint matrix above:

$$\begin{aligned}
\Delta_1 &= -\sin\theta_1\cos(\theta_1+\theta_{1,2}) + \cos\theta_1\sin(\theta_1+\theta_{1,2}) = \sin\theta_{1,2} \ , \\
\Delta_2 &= -\sin\theta_1(d_1\cos\theta_{1,2}+d_2) \ , \\
\Delta_3 &= -\sin\theta_1 \ d_2 \ , \\
\Delta_4 &= \cos\theta_1(d_1\cos\theta_{1,2}+d_2) \ , \\
\Delta_5 &= \cos\theta_1 \ d_2 \ , \\
\Delta_6 &= 0 \ .
\end{aligned}$$

When $d_2 \neq 0$, consider $\Delta_1, \ldots, \Delta_5$. If $\sin\theta_{1,2} \neq 0$, then $\Delta_1 \neq 0$ establishes linear independence. If $\sin\theta_{1,2} = 0$, then $\Delta_3 \neq 0$ or $\Delta_5 \neq 0$. When $d_2 = 0$, consider Δ_1, Δ_2 and Δ_4. If $\sin\theta_{1,2} \neq 0$, then $\Delta_1 \neq 0$. If $\sin\theta_{1,2} = 0$, then $\cos\theta_{1,2} \neq 0$ and we have either $\Delta_2 \neq 0$ or $\Delta_4 \neq 0$. Thus, the null space of this matrix is always of dimension 2.

∎

Define the *constraint one–forms:*

$$\begin{aligned}
\omega_q^1 &= -\sin\theta_1 dx_1 + \cos\theta_1 dy_1 \ , \\
\omega_q^2 &= -\sin(\theta_1+\theta_{1,2})dx_1 + \cos(\theta_1+\theta_{1,2})dy_1 + (d_1\cos\theta_{1,2}+d_2)d\theta_1 + d_2 d\theta_{1,2} \ .
\end{aligned} \tag{3.13}$$

The *constraint distribution* $\mathcal{D}_q$ is the subspace of T_qQ which is the intersection of the kernels of the constraint one–forms, i.e.

$$\mathcal{D}_q = \mathrm{Ker}\ \omega_q^1 \cap \mathrm{Ker}\ \omega_q^2 \ . \tag{3.14}$$

Since the constraints are linearly independent, we know that $\mathcal{D}_q$ is always 2–dimensional. Next, we will specify a basis for it.

Proposition 3.2

The constraint distribution $\mathcal{D}_q \subset T_qQ$ is

$$\mathcal{D}_q = \mathrm{sp}\{\xi^1_Q, \xi^2_Q\} \ , \tag{3.15}$$

where in the case $d_2 \neq 0$:

$$\begin{aligned}
\xi_Q^1 &= d_2(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \sin\theta_{1,2} \frac{\partial}{\partial \theta_{1,2}} \ , \\
\xi_Q^2 &= d_2 \frac{\partial}{\partial \theta_1} - (d_1 \cos\theta_{1,2} + d_2) \frac{\partial}{\partial \theta_{1,2}} \ ,
\end{aligned} \tag{3.16}$$

while in the case $d_2 = 0$:

$$\begin{aligned}
\xi_Q^1 &= d_1 \cos\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \sin\theta_{1,2} \frac{\partial}{\partial \theta_1} \ , \\
\xi_Q^2 &= \frac{\partial}{\partial \theta_{1,2}} \ .
\end{aligned} \tag{3.17}$$

Proof

Let $X_q = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial y_1} + v_3 \frac{\partial}{\partial \theta_1} + v_4 \frac{\partial}{\partial \theta_{1,2}} \in \mathcal{D}_q$, with $v_i \in I\!R$, be a vector field in the constraint distribution. By definition it should annihilate both constraint one–forms. Consider first the case $d_2 \neq 0$. From the constraints we have:

$$\begin{aligned}
\omega_q^1(X_q) &= -v_1 \sin\theta_1 + v_2 \cos\theta_1 = 0 \ , \\
\omega_q^2(X_q) &= -(v_1 \cos\theta_1 + v_2 \sin\theta_1) \sin\theta_{1,2} + (d_1 \cos\theta_{1,2} + d_2) v_3 + d_2 v_4 = 0 \ .
\end{aligned}$$

Let $v_1 = d_2 \cos\theta_1 v_5, \ v_2 = d_2 \sin\theta_1 v_5$ and $v_4 = \sin\theta_{1,2} v_5 - \frac{d_1 \cos\theta_{1,2} + d_2}{d_2} v_3$. Then:

$$\begin{aligned}
X_q = v_5 \big[d_2 \cos\theta_1 \frac{\partial}{\partial x_1} + d_2 \sin\theta_1 \frac{\partial}{\partial y_1} + \sin\theta_{1,2} \frac{\partial}{\partial \theta_{1,2}} \big] \\
+ v_3 \big[d_2 \frac{\partial}{\partial \theta_1} - (d_1 \cos\theta_{1,2} + d_2) \frac{\partial}{\partial \theta_{1,2}} \big] \ ,
\end{aligned}$$

for arbitrary v_3, v_5, satisfies both constraints. The two vector fields are obviously linearly independent.

Consider the case $d_2 = 0$. The constraints now become:

$$\begin{aligned}
\omega_q^1(X_q) &= -v_1 \sin\theta_1 + v_2 \cos\theta_1 = 0 \ , \\
\omega_q^2(X_q) &= -(v_1 \cos\theta_1 + v_2 \sin\theta_1) \sin\theta_{1,2} + d_1 \cos\theta_{1,2} v_3 = 0 \ .
\end{aligned}$$

Let $v_1 = d_1 \cos\theta_1 v_5, \ v_2 = d_1 \sin\theta_1 v_5$ and $v_3 = \sin\theta_{1,2} v_5$. Then:

$$X_q = v_5 \big[d_1 \cos\theta_1 \frac{\partial}{\partial x_1} + d_1 \sin\theta_1 \frac{\partial}{\partial y_1} + \sin\theta_{1,2} \frac{\partial}{\partial \theta_1} \big] + v_4 \frac{\partial}{\partial \theta_{1,2}} \ ,$$

for arbitrary v_4, v_5, satisfies both constraints. The two vector fields are obviously linearly independent.

∎

The relationship of the angular velocities $\dot\phi_{l,i}, \dot\phi_{r,i},\ i=1,2$, of the left and right wheel of the wheel assembly of platform i with the configuration velocities $\dot q = (\dot x_1, \dot y_1, \dot\theta_1, \dot\theta_{1,2})^\top$ of the system, is as follows:

$$
\begin{aligned}
\dot\phi_{l,1} &= \frac{1}{R_1}\big[-\frac{L_1}{2}\dot\theta_1 + \dot x_1\cos\theta_1 + \dot y_1\sin\theta_1\big]\ ,\\
\dot\phi_{r,1} &= \frac{1}{R_1}\big[\frac{L_1}{2}\dot\theta_1 + \dot x_1\cos\theta_1 + \dot y_1\sin\theta_1\big]\ ,\\
\dot\phi_{l,2} &= \frac{1}{R_2}\big[-\frac{L_2}{2}(\dot\theta_1 + \dot\theta_{1,2}) + (\dot x_1\cos\theta_1 + \dot y_1\sin\theta_1)\cos\theta_{1,2} + \dot\theta_1 d_1\sin\theta_{1,2}\big]\ ,\\
\dot\phi_{r,2} &= \frac{1}{R_2}\big[\frac{L_2}{2}(\dot\theta_1 + \dot\theta_{1,2}) + (\dot x_1\cos\theta_1 + \dot y_1\sin\theta_1)\cos\theta_{1,2} + \dot\theta_1 d_1\sin\theta_{1,2}\big]\ ,
\end{aligned}
\tag{3.18}
$$

where R_i and L_i are the wheel radius and the axis length of platform i.

These formulas are used later in section 5 in accounting for friction in the bearings of the wheels.

4 Symmetry of the Roller Racer

Consider now the effect of symmetries on this system. In particular, consider the action Φ of the group $G = SE(2)$ on the configuration space $Q = SE(2) \times S^1$ defined by:

$$\begin{aligned}\Phi :& G \times Q \to Q \\ & (g, (g_1, \theta_{1,2})) \mapsto (gg_1, \theta_{1,2}) \\ & ((x, y, \theta), (x_1, y_1, \theta_1, \theta_{1,2})) \;\mapsto \\ & \qquad (x_1 \cos\theta - y_1 \sin\theta + x, x_1 \sin\theta + y_1 \cos\theta + y, \theta_1 + \theta, \theta_{1,2}) ,\end{aligned} \tag{4.1}$$

where $g = g(x, y, \theta) \in G$. The tangent space at $q \in Q$ to the orbit of Φ is given by

$$T_q\mathrm{Orb}(q) = \mathrm{sp}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial \theta_1}\} . \tag{4.2}$$

Notice that the sum of the subspaces $\mathcal{D}_q$ and $T_q\mathrm{Orb}(q)$ gives the entire T_qQ.

$$\mathcal{D}_q + T_q\mathrm{Orb}(q) = T_qQ .$$

In (Bloch, Krishnaprasad, Marsden & Murray [1994]), this is referred to as the *principal* case. Our goal is to show that the nonholonomic constraints, together with a momentum equation, can specify a connection on the principal fiber bundle $Q \longrightarrow Q/G$.

An important observation, that we prove below, is that the intersection $\mathcal{S}_q$ of $\mathcal{D}_q$ and $T_q\mathrm{Orb}(q)$ is non–trivial. Contrast this with the $(n-1)$–module G–snake with $\dim G = n$, where $T_qQ = \mathcal{D}_q \oplus T_q\mathrm{Orb}(q)$, thus the intersection of $\mathcal{D}_q$ and $T_q\mathrm{Orb}(q)$ is trivial (Krishnaprasad & Tsakiris [1994a]; Tsakiris [1995]); this is referred to as the *purely kinematic* case. We specify a basis for $\mathcal{S}_q$ as follows:

Proposition 4.1

Consider the intersection

$$\mathcal{S}_q \stackrel{\mathrm{def}}{=} \mathcal{D}_q \cap T_q\mathrm{Orb}(q) . \tag{4.3}$$

In the case $d_1 \neq d_2$, the distribution $\mathcal{S}_q$ is 1–dimensional and is given by:

$$\mathcal{S}_q = \mathrm{sp}\{\xi^q_Q\} , \tag{4.4}$$

where

$$\xi^q_Q = (d_1 \cos\theta_{1,2} + d_2)(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \sin\theta_{1,2} \frac{\partial}{\partial \theta_1} . \tag{4.5}$$

Proof

Consider $X_q \in \mathcal{S}_q = \mathcal{D}_q \cap T_q\mathrm{Orb}(q)$. Because $X_q \in \mathcal{D}_q$, we have $X_q = u_1\xi^1_Q + u_2\xi^2_Q$, for $u_i \in I\!R$. Because $X_q \in T_q\mathrm{Orb}(q)$, we have $X_q = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial \theta_1}$, for $v_i \in I\!R$. In order for X_q to lie in the intersection of the two spaces, we should have:

$$u_1\xi^1_Q + u_2\xi^2_Q = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial \theta_1} \ . \tag{4.6}$$

In the case $d_2 \neq 0$, we have from (3.16):

$$\begin{aligned}\big[d_2(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial \theta_{1,2}}\big]u_1 + \big[d_2\frac{\partial}{\partial \theta_1} - (d_1\cos\theta_{1,2} + d_2)\frac{\partial}{\partial \theta_{1,2}}\big]u_2 \\ = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial \theta_1} \ .\end{aligned}$$

This corresponds to a system of four equations:

$$\begin{pmatrix} d_2\cos\theta_1 & 0 & -1 & 0 & 0 \\ d_2\sin\theta_1 & 0 & 0 & -1 & 0 \\ 0 & d_2 & 0 & 0 & -1 \\ \sin\theta_{1,2} & -(d_1\cos\theta_{1,2} + d_2) & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ .$$

When $d_1 \neq d_2$, the 4×5 matrix above is always of maximal rank, thus $\dim\mathcal{S}_q = 5 - 4 = 1$, for all $q \in Q$. Pick $u_1 = (d_1\cos\theta_{1,2} + d_2)u_5$ and $u_2 = \sin\theta_{1,2}u_5$. Then, $v_1 = d_2\cos\theta_1(d_1\cos\theta_{1,2} + d_2)u_5$, $v_2 = d_2\sin\theta_1(d_1\cos\theta_{1,2} + d_2)u_5$ and $v_3 = d_2\sin\theta_{1,2}u_5$. Thus

$$X_q = \big[(d_1\cos\theta_{1,2} + d_2)(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial \theta_1}\big]d_2u_5 \ ,$$

for arbitrary u_5. Observe that when $d_1 \neq d_2$, the vector field X_q is nontrivial for all $q \in Q$.

In the case $d_2 = 0$, we have from (3.17):

$$\Big[d_1\cos\theta_{1,2}(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial \theta_1}\Big]u_1 + u_2\frac{\partial}{\partial \theta_{1,2}} = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial \theta_1} \ .$$

From this, we get:

$$\begin{aligned} u_2 &= 0 \ , \\ v_1 &= d_1\cos\theta_{1,2}\cos\theta_1 u_1 \ , \\ v_2 &= d_1\cos\theta_{1,2}\sin\theta_1 u_1 \ , \\ v_3 &= \sin\theta_{1,2}u_1 \ . \end{aligned}$$

Therefore,

$$X_q = \big[d_1 \cos\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \sin\theta_{1,2} \frac{\partial}{\partial \theta_1}\big] u_1 \ ,$$

for arbitrary u_1. Thus, $\mathcal{S}_q$ is again a 1–dimensional distribution.
The two cases can be unified in the expression (4.5).

∎

Proposition 4.2

For the action Φ of $SE(2)$ on Q defined in (4.1), the infinitesimal generators corresponding to the basis elements of $\mathcal{G} = se(2)$, defined in (2.17), at the point $q \in Q$, are:

$$\begin{aligned} \mathcal{A}_{1}{}_Q^q &= -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} + \frac{\partial}{\partial \theta_1} \ , \\ \mathcal{A}_{2}{}_Q^q &= \frac{\partial}{\partial x_1} \ , \\ \mathcal{A}_{3}{}_Q^q &= \frac{\partial}{\partial y_1} \ . \end{aligned} \tag{4.7}$$

The infinitesimal generator corresponding to $\xi^q = \xi_1 \mathcal{A}_1 + \xi_2 \mathcal{A}_2 + \xi_3 \mathcal{A}_3 \in \mathcal{G}$ is

$$\xi_Q^q = (\xi_2 - y_1 \xi_1) \frac{\partial}{\partial x_1} + (\xi_3 + x_1 \xi_1) \frac{\partial}{\partial y_1} + \xi_1 \frac{\partial}{\partial \theta_1} \ . \tag{4.8}$$

A given vector field $\xi_Q^q = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial y_1} + v_3 \frac{\partial}{\partial \theta_1}$ can be considered as the infinitesimal generator of an element $\xi^q \in \mathcal{G} = se(2)$, under the action Φ. Then, ξ^q is:

$$\xi^q = v_3 \mathcal{A}_1 + (v_1 + y_1 v_3) \mathcal{A}_2 + (v_2 - x_1 v_3) \mathcal{A}_3 \ . \tag{4.9}$$

Proof

From (2.1), (2.25) and (4.1):

$$\begin{aligned} \mathcal{A}_{1}{}_Q^q &= \frac{d}{dt}\bigg|_{t=0} \Phi(e^{t\mathcal{A}_1}, q) = \frac{d}{dt}\bigg|_{t=0} (x_1 \cos t - y_1 \sin t, x_1 \sin t + y_1 \cos t, \theta_1 + t, \theta_{1,2}) \\ &= (-y_1, x_1, 1, 0) \longleftrightarrow -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} + \frac{\partial}{\partial \theta_1} \ . \end{aligned}$$

The infinitesimal generators for the other elements of the basis are specified similarly.

∎

The vector field ξ^q_Q in (4.5) corresponds to the following element ξ^q of $se(2)$:

$$\begin{aligned}\xi^q = \sin\theta_{1,2}\mathcal{A}_1 + [(d_1\cos\theta_{1,2} + d_2)\cos\theta_1 + y_1\sin\theta_{1,2}]\mathcal{A}_2 \\ + [(d_1\cos\theta_{1,2} + d_2)\sin\theta_1 - x_1\sin\theta_{1,2}]\mathcal{A}_3 \ .\end{aligned} \tag{4.10}$$

By differentiating (4.10), we get

$$\begin{aligned}\frac{d\xi^q}{dt} &= \cos\theta_{1,2}\dot{\theta}_{1,2}\mathcal{A}_1 \\ &+ [-d_1\sin\theta_{1,2}\cos\theta_1\dot{\theta}_{1,2} - (d_1\cos\theta_{1,2} + d_2)\sin\theta_1\dot{\theta}_1 + \dot{y}_1\sin\theta_{1,2} + y_1\cos\theta_{1,2}\dot{\theta}_{1,2}]\mathcal{A}_2 \\ &+ [-d_1\sin\theta_{1,2}\sin\theta_1\dot{\theta}_{1,2} + (d_1\cos\theta_{1,2} + d_2)\cos\theta_1\dot{\theta}_1 - \dot{x}_1\sin\theta_{1,2} - x_1\cos\theta_{1,2}\dot{\theta}_{1,2}]\mathcal{A}_3 \ .\end{aligned} \tag{4.11}$$

The corresponding infinitesimal generator is

$$\begin{aligned}\left[\frac{d\xi^q}{dt}\right]_Q = \big[- d_1\sin\theta_{1,2}\cos\theta_1\dot{\theta}_{1,2} - (d_1\cos\theta_{1,2} + d_2)\sin\theta_1\dot{\theta}_1 + \dot{y}_1\sin\theta_{1,2}\big]\frac{\partial}{\partial x_1} \\ + \big[- d_1\sin\theta_{1,2}\sin\theta_1\dot{\theta}_{1,2} + (d_1\cos\theta_{1,2} + d_2)\cos\theta_1\dot{\theta}_1 - \dot{x}_1\sin\theta_{1,2}\big]\frac{\partial}{\partial y_1} \\ + \cos\theta_{1,2}\dot{\theta}_{1,2}\frac{\partial}{\partial\theta_1} \ .\end{aligned} \tag{4.12}$$

By differentiating ξ^q_Q in (4.5), we get

$$\begin{aligned}\frac{d\xi^q_Q}{dt} = \big[- d_1\sin\theta_{1,2}\cos\theta_1\dot{\theta}_{1,2} - (d_1\cos\theta_{1,2} + d_2)\sin\theta_1\dot{\theta}_1\big]\frac{\partial}{\partial x_1} \\ + \big[- d_1\sin\theta_{1,2}\sin\theta_1\dot{\theta}_{1,2} + (d_1\cos\theta_{1,2} + d_2)\cos\theta_1\dot{\theta}_1\big]\frac{\partial}{\partial y_1} \\ + \cos\theta_{1,2}\dot{\theta}_{1,2}\frac{\partial}{\partial\theta_1} \ .\end{aligned} \tag{4.13}$$

5 Dynamics of the Roller Racer

5.1 The Lagrange–d' Alembert Equations of Motion

Consider the following Lagrangian for the Roller Racer:

$$
\begin{aligned}
L(\dot{q}) &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}I_{z_1}\dot{\theta}_1^2 + \frac{1}{2}I_{z_2}(\dot{\theta}_1 + \dot{\theta}_{1,2})^2 \\
&= \frac{1}{2}\begin{pmatrix} \dot{x}_1 & \dot{y}_1 & \dot{\theta}_1 & \dot{\theta}_{1,2} \end{pmatrix} \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & I_{z_1} + I_{z_2} & I_{z_2} \\ 0 & 0 & I_{z_2} & I_{z_2} \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\theta}_{1,2} \end{pmatrix} ,
\end{aligned}
\tag{5.1}
$$

for $q = (x_1, y_1, \theta_1, \theta_{1,2})$, where m_i and I_{z_i} is respectively the mass and moment of inertia of platform i. The choice of Lagrangian reflects our assumption that the mass and linear momentum of platform 2 are much smaller than those of platform 1 and can be ignored. However, the inertia of platform 2 is not ignored. From (5.1):

$$
\frac{\partial L}{\partial \dot{q}} = \begin{pmatrix} m_1\dot{x}_1 \\ m_1\dot{y}_1 \\ (I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2} \\ I_{z_2}\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2} \end{pmatrix} . \tag{5.2}
$$

The equations of motion of the Roller Racer are derived using the *Lagrange–d'Alembert principle* for a system with nonholonomic constraints.

Proposition 5.1 (Lagrange–d' Alembert Principle) (Vershik & Faddeev [1981]; Yang [1992])

In the case of linear constraints on the velocities, the Lagrange–d' Alembert principle for our system and for the Lagrangian $L(q,v)$ given by (5.1), with $q = (x_1, y_1, \theta_1, \theta_{1,2}) \in Q$ and $v = (v_1, v_2, v_3, v_4) \in T_qQ$, takes the form:

$$
\left(\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial q}\right) \cdot u = \alpha_e \cdot u , \tag{5.3}
$$

where (q, v) satisfy the nonholonomic constraints:

$$
\begin{aligned}
\omega_q^1(v) &= -\sin\theta_1 v_1 + \cos\theta_1 v_2 = 0 , \\
\omega_q^2(v) &= -\sin(\theta_1 + \theta_{1,2})v_1 + \cos(\theta_1 + \theta_{1,2})v_2 + (d_1\cos\theta_{1,2} + d_2)v_3 + d_2 v_4 = 0 ,
\end{aligned}
\tag{5.4}
$$

and the test vector $u = (u_1, u_2, u_3, u_4) \in T_qQ$ satisfies:

$$
\begin{aligned}
\frac{\partial}{\partial v}\omega_q^1(v) \cdot u &= -\sin\theta_1 u_1 + \cos\theta_1 u_2 = 0 , \\
\frac{\partial}{\partial v}\omega_q^2(v) \cdot u &= -\sin(\theta_1 + \theta_{1,2})u_1 + \cos(\theta_1 + \theta_{1,2})u_2 + (d_1\cos\theta_{1,2} + d_2)u_3 + d_2 u_4 = 0 ,
\end{aligned}
\tag{5.5}
$$

while α_e is the 1–form describing the external forcing to the system.

Using Lagrange multipliers, the Lagrange–d' Alembert principle for the case of a system with two linear (in the velocity) nonholonomic constraints, takes the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial q} = \alpha_e + \lambda_1 \frac{\partial \omega_q^1}{\partial v} + \lambda_2 \frac{\partial \omega_q^2}{\partial v} \ , \tag{5.6}$$

for functions λ_1 and λ_2 on TQ and for (q, v) such that the nonholonomic constraints (5.4) are satisfied.

■

Consider external forcing to the system described by the 1–form

$$\alpha_e = (F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}}) \ , \tag{5.7}$$

where $F_{\theta_{1,2}}$ may be the torque applied by the motor that actuates the rotary joint $O_{1,2}$ and $F_{x_1}, F_{y_1}, F_{\theta_1}$ may be the result of friction in the bearings of the wheels. The equations of motion of the Roller Racer are given below.

Proposition 5.2 (Lagrange–d' Alembert Equations of Motion)

i) In the case $d_2 \neq 0$, the equations of motion for the Roller Racer are:

$$\begin{aligned}
&(I_{z_2} \sin^2\theta_{1,2} + m_1 d_2^2)\dot{\nu}_1 - I_{z_2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}\dot{\nu}_2 \\
&\qquad + I_{z_2}\sin^2\theta_{1,2}\cos\theta_{1,2}\nu_1^2 \\
&\qquad - I_{z_2}\sin\theta_{1,2}\big[d_1(\cos^2\theta_{1,2} - \sin^2\theta_{1,2}) + d_2\cos\theta_{1,2}\big]\nu_1\nu_2 \\
&\qquad - I_{z_2} d_1 \sin^2\theta_{1,2}(d_1\cos\theta_{1,2} + d_2)\nu_2^2 \\
&\qquad\qquad = d_2\big(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1\big) + \sin\theta_{1,2} F_{\theta_{1,2}} \ , \\
&- I_{z_2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}\dot{\nu}_1 + (I_{z_1} d_2^2 + I_{z_2} d_1^2\cos^2\theta_{1,2})\dot{\nu}_2 \\
&\qquad - I_{z_2} d_1 \sin\theta_{1,2}\cos^2\theta_{1,2}\nu_1^2 \\
&\qquad + I_{z_2} d_1 \cos\theta_{1,2}\big[d_1(\cos^2\theta_{1,2} - \sin^2\theta_{1,2}) + d_2\cos\theta_{1,2}\big]\nu_1\nu_2 \\
&\qquad + I_{z_2} d_1^2 (d_1\cos\theta_{1,2} + d_2)\sin\theta_{1,2}\cos\theta_{1,2}\nu_2^2 \\
&\qquad\qquad = d_2 F_{\theta_1} - r(\theta_{1,2}) F_{\theta_{1,2}} \ ,
\end{aligned} \tag{5.8}$$

where $\nu_1 \stackrel{\text{def}}{=} \frac{1}{d_2}(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)$, $\nu_2 \stackrel{\text{def}}{=} \frac{1}{d_2}\dot{\theta}_1$ and $r(\theta_{1,2}) \stackrel{\text{def}}{=} d_1\cos\theta_{1,2} + d_2$.

ii) In the case $d_2 = 0$, the equations of motion are

$$\begin{aligned}
&\big[(I_{z_1} + I_{z_2})\sin^2\theta_{1,2} + m_1 d_1^2\cos^2\theta_{1,2}\big]\dot{\nu}_1 + I_{z_2}\sin\theta_{1,2}\dot{\nu}_2 \\
&\qquad + (I_{z_1} + I_{z_2} - m_1 d_1^2)\sin\theta_{1,2}\cos\theta_{1,2}\nu_1\nu_2 \\
&\qquad\qquad = d_1\cos\theta_{1,2}(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \sin\theta_{1,2}F_{\theta_1} \ ,
\end{aligned} \tag{5.9}$$

$$I_{z_2}\sin\theta_{1,2}\dot{\nu}_1 + I_{z_2}\dot{\nu}_2 + I_{z_2}\cos\theta_{1,2}\nu_1\nu_2 = F_{1,2} \ ,$$

where $\nu_1 \stackrel{\text{def}}{=} \frac{1}{d_1 \cos\theta_{1,2}}(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1)$, in the case $\cos\theta_{1,2} \neq 0$, or $\nu_1 \stackrel{\text{def}}{=} \frac{1}{\sin\theta_{1,2}}\dot{\theta}_1$ otherwise and $\nu_2 \stackrel{\text{def}}{=} \dot{\theta}_{1,2}$.

Proof

The Lagrange–d'Alembert principle (5.3) for the Lagrangian given by (5.1), for $u = (u_1, u_2, u_3, u_4), v = (v_1, v_2, v_3, v_4) \in \mathcal{D}_q$ and for $\alpha_e = (F_{x_1}, F_{y_1}, F_{\theta_1}, F_{\theta_{1,2}})$, takes the form:

$$\begin{aligned} m_1\dot{v}_1 u_1 + m_1\dot{v}_2 u_2 + \left[(I_{z_1} + I_{z_2})\dot{v}_3 + I_{z_2}\dot{v}_4\right]u_3 + I_{z_2}(\dot{v}_3 + \dot{v}_4)u_4 \\ = F_{x_1}u_1 + F_{y_1}u_2 + F_{\theta_1}u_3 + F_{\theta_{1,2}}u_4 \ . \end{aligned} \tag{5.10}$$

i) Let $d_2 \neq 0$:

The constraint distribution $\mathcal{D}_q$ is spanned, as shown in (3.16), by

$$\begin{aligned} \xi_Q^1 &= d_2(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_{1,2}} \ , \\ \xi_Q^2 &= d_2\frac{\partial}{\partial\theta_1} - (d_1\cos\theta_{1,2} + d_2)\frac{\partial}{\partial\theta_{1,2}} \ . \end{aligned} \tag{5.11}$$

Any $u \in \mathcal{D}_q$ can be represented as $u = \alpha_1\xi_Q^1 + \alpha_2\xi_Q^2$, for $\alpha_1, \alpha_2 \in I\!R$. Then its components are:

$$\begin{aligned} u_1 &= \alpha_1 d_2 \cos\theta_1 \ , \\ u_2 &= \alpha_1 d_2 \sin\theta_1 \ , \\ u_3 &= \alpha_2 d_2 \ , \\ u_4 &= \alpha_1 \sin\theta_{1,2} - \alpha_2 r(\theta_{1,2}) \ . \end{aligned} \tag{5.12}$$

Similarly, any $v \in \mathcal{D}_q$ can be represented as $v = \nu_1\xi_Q^1 + \nu_2\xi_Q^2$, for $\nu_1, \nu_2 \in I\!R$. Then its components are:

$$\begin{aligned} v_1 &= \nu_1 d_2 \cos\theta_1 \ , \\ v_2 &= \nu_1 d_2 \sin\theta_1 \ , \\ v_3 &= \nu_2 d_2 \ , \\ v_4 &= \nu_1 \sin\theta_{1,2} - \nu_2 r(\theta_{1,2}) \ . \end{aligned} \tag{5.13}$$

These relationships can be used to define ν_1 and ν_2 as follows:

$$\begin{aligned} \nu_1 &\stackrel{\text{def}}{=} \frac{1}{d_2}(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) \ , \\ \nu_2 &\stackrel{\text{def}}{=} \frac{1}{d_2}\dot{\theta}_1 \ . \end{aligned} \tag{5.14}$$

By differentiating (5.13) we get

$$\begin{aligned}
\dot v_1 &= \dot\nu_1 d_2\cos\theta_1 - \nu_1 d_2\sin\theta_1\dot\theta_1 \ , \\
\dot v_2 &= \dot\nu_1 d_2\sin\theta_1 + \nu_1 d_2\cos\theta_1\dot\theta_1 \ , \\
\dot v_3 &= \dot\nu_2 d_2 \ , \\
\dot v_4 &= \dot\nu_1\sin\theta_{1,2} - \dot\nu_2 r(\theta_{1,2}) + \nu_1\cos\theta_{1,2}\dot\theta_{1,2} + \nu_2 d_1\sin\theta_{1,2}\dot\theta_{1,2} \ .
\end{aligned} \tag{5.15}$$

Introducing (5.12) and (5.15) in (5.10), we get

$$\begin{aligned}
& m_1\dot\nu_1 d_2^2\alpha_1 \\
&\quad + \big[(I_{z_1}+I_{z_2})\dot\nu_2 d_2 + I_{z_2}(\dot\nu_1\sin\theta_{1,2} - \dot\nu_2 r(\theta_{1,2}) + \nu_1\cos\theta_{1,2}\dot\theta_{1,2} + \nu_2 d_1\sin\theta_{1,2}\dot\theta_{1,2})\big]\cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot d_2\alpha_2 \\
&\quad + I_{z_2}\big(\dot\nu_2 d_2 + \dot\nu_1\sin\theta_{1,2} - \dot\nu_2 r(\theta_{1,2}) + \nu_1\cos\theta_{1,2}\dot\theta_{1,2} + \nu_2 d_1\sin\theta_{1,2}\dot\theta_{1,2}\big)\cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot\big(\sin\theta_{1,2}\alpha_1 - r(\theta_{1,2})\alpha_2\big) \\
&\qquad = F_{x_1} d_2\cos\theta_1\alpha_1 + F_{y_1} d_2\sin\theta_1\alpha_1 + F_{\theta_1} d_2\alpha_2 + F_{\theta_{1,2}}\big(\sin\theta_{1,2}\alpha_1 - r(\theta_{1,2})\alpha_2\big) \ ,
\end{aligned} \tag{5.16}$$

for arbitrary $\alpha_1, \alpha_2 \in I\!R$ and, from (5.13), we have:

$$\dot\theta_{1,2} = v_4 = \nu_1\sin\theta_{1,2} - \nu_2 r(\theta_{1,2}) \ . \tag{5.17}$$

Since α_1, α_2 are arbitrary, (5.16) splits (after using (5.17)) in the following two equations:

$$\begin{aligned}
& (I_{z_2}\sin^2\theta_{1,2} + m_1 d_2^2)\dot\nu_1 - I_{z_2} d_1\sin\theta_{1,2}\cos\theta_{1,2}\dot\nu_2 \\
&\qquad + I_{z_2}\sin\theta_{1,2}(\cos\theta_{1,2}\nu_1 + d_1\sin\theta_{1,2}\nu_2)\big(\nu_1\sin\theta_{1,2} - \nu_2 r(\theta_{1,2})\big) \\
&\qquad\qquad = (F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1)d_2 + F_{\theta_{1,2}}\sin\theta_{1,2} \ , \\
& - I_{z_2} d_1\sin\theta_{1,2}\cos\theta_{1,2}\dot\nu_1 + (I_{z_1} d_2^2 + I_{z_2} d_1^2\cos^2\theta_{1,2})\dot\nu_2 \\
&\qquad - I_{z_2} d_1\cos\theta_{1,2}\big(\nu_1\sin\theta_{1,2} - \nu_2 r(\theta_{1,2})\big) \\
&\qquad\qquad = F_{\theta_1} d_2 - F_{\theta_{1,2}} r(\theta_{1,2}) \ .
\end{aligned}$$

By rearranging terms, we obtain (5.8). Obviously, the first of the equations (5.8) is equation (5.3) with $u = \xi_Q^1$, while the second is (5.3) with $u = \xi_Q^2$.

ii) Let $d_2 = 0$:

The constraint distribution $\mathcal{D}_q$ is spanned, as shown in (3.17), by

$$\begin{aligned}
\xi_Q^1 &= d_1\cos\theta_{1,2}(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1} \ , \\
\xi_Q^2 &= \frac{\partial}{\partial\theta_{1,2}} \ .
\end{aligned} \tag{5.18}$$

Any $u \in \mathcal{D}_q$ can be represented as $u = \alpha_1 \xi_Q^1 + \alpha_2 \xi_Q^2$, for $\alpha_1, \alpha_2 \in I\!R$. Then its components are:

$$
\begin{aligned}
u_1 &= \alpha_1 d_1 \cos\theta_{1,2} \cos\theta_1 \ , \\
u_2 &= \alpha_1 d_1 \cos\theta_{1,2} \sin\theta_1 \ , \\
u_3 &= \alpha_1 \sin\theta_{1,2} \ , \\
u_4 &= \alpha_2 \ .
\end{aligned}
\tag{5.19}
$$

Similarly, any $v \in \mathcal{D}_q$ can be represented as $v = \nu_1 \xi_Q^1 + \nu_2 \xi_Q^2$, for $\nu_1, \nu_2 \in I\!R$. Then its components are:

$$
\begin{aligned}
v_1 &= \nu_1 d_1 \cos\theta_{1,2} \cos\theta_1 \ , \\
v_2 &= \nu_1 d_1 \cos\theta_{1,2} \sin\theta_1 \ , \\
v_3 &= \nu_1 \sin\theta_{1,2} \ , \\
v_4 &= \nu_2 \ .
\end{aligned}
\tag{5.20}
$$

These relationships can be used to define ν_1 and ν_2 as follows:

$$
\begin{aligned}
\nu_1 &\stackrel{\text{def}}{=} \frac{1}{d_1 \cos\theta_{1,2}} (\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) \ , \quad \text{in the case } \cos\theta_{1,2} \neq 0 \ , \\
&\stackrel{\text{def}}{=} \frac{1}{\sin\theta_{1,2}} \dot{\theta}_1 \ , \quad \text{otherwise} \ , \\
\nu_2 &\stackrel{\text{def}}{=} \dot{\theta}_{1,2} \ .
\end{aligned}
\tag{5.21}
$$

By differentiating (5.20) we get

$$
\begin{aligned}
\dot{v}_1 &= \dot{\nu}_1 d_1 \cos\theta_{1,2} \cos\theta_1 - \nu_1 d_1 (\cos\theta_{1,2} \sin\theta_1 \dot{\theta}_1 + \sin\theta_{1,2} \cos\theta_1 \dot{\theta}_{1,2}) \ , \\
\dot{v}_2 &= \dot{\nu}_1 d_1 \cos\theta_{1,2} \sin\theta_1 + \nu_1 d_1 (\cos\theta_{1,2} \cos\theta_1 \dot{\theta}_1 - \sin\theta_{1,2} \sin\theta_1 \dot{\theta}_{1,2}) \ , \\
\dot{v}_3 &= \dot{\nu}_1 \sin\theta_{1,2} + \nu_1 \cos\theta_{1,2} \dot{\theta}_{1,2} \ , \\
\dot{v}_4 &= \dot{\nu}_2 \ .
\end{aligned}
\tag{5.22}
$$

Introducing (5.19) and (5.22) in (5.10), we get

$$
\begin{aligned}
& m_1 (\dot{\nu}_1 d_1 \cos\theta_{1,2} \cos\theta_1 - \nu_1 d_1 \sin\theta_{1,2} \cos\theta_1 \dot{\theta}_{1,2}) d_1 \cos\theta_{1,2} \cos\theta_1 \alpha_1 \\
&\quad + m_1 (\dot{\nu}_1 d_1 \cos\theta_{1,2} \sin\theta_1 - \nu_1 d_1 \sin\theta_{1,2} \sin\theta_1 \dot{\theta}_{1,2}) d_1 \cos\theta_{1,2} \sin\theta_1 \alpha_1 \\
&\quad + \left[(I_{z_1} + I_{z_2})(\dot{\nu}_1 \sin\theta_{1,2} + \nu_1 \cos\theta_{1,2} \dot{\theta}_{1,2}) + I_{z_2} \dot{\nu}_2 \right] \sin\theta_{1,2} \alpha_1 \\
&\quad + I_{z_2} \left(\dot{\nu}_1 \sin\theta_{1,2} + \nu_1 \cos\theta_{1,2} \dot{\theta}_{1,2} + \dot{\nu}_2 \right) \alpha_2 \\
&\qquad = F_{x_1} d_1 \cos\theta_{1,2} \cos\theta_1 \alpha_1 + F_{y_1} d_1 \cos\theta_{1,2} \sin\theta_1 \alpha_1 + F_{\theta_1} \sin\theta_{1,2} \alpha_1 + F_{\theta_{1,2}} \alpha_2 \ ,
\end{aligned}
\tag{5.23}
$$

for arbitrary $\alpha_1, \alpha_2 \in I\!R$. Since α_1, α_2 are arbitrary, (5.23) splits in the following two equations:

$$\begin{aligned}
&\big[(I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1 d_1^2 \cos^2\theta_{1,2}\big]\dot\nu_1 + I_{z_2}\sin\theta_{1,2}\dot\nu_2 \\
&\qquad\qquad + (I_{z_1} + I_{z_2} - m_1 d_1^2)\sin\theta_{1,2}\cos\theta_{1,2}\nu_1\dot\theta_{1,2} \\
&\qquad\qquad\qquad\qquad = d_1\cos\theta_{1,2}(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + \sin\theta_{1,2}F_{\theta_1} \ , \\
&I_{z_2}\sin\theta_{1,2}\dot\nu_1 + I_{z_2}\dot\nu_2 + I_{z_2}\cos\theta_{1,2}\nu_1\dot\theta_{1,2} = F_{1,2} \ .
\end{aligned}$$

By using $\dot\theta_{1,2} = v_4 = \nu_2$ from (5.20) and by rearranging terms, we obtain (5.9). Again, the first of the equations (5.9) is equation (5.3) with $u = \xi_Q^1$, while the second is (5.3) with $u = \xi_Q^2$.

∎

Suppose *friction* is present in the joints of the Roller Racer wheels with their axes.

We consider a simple viscous friction model, where the frictional forces are introduced in the Lagrange–d'Alembert equations through the following Rayleigh dissipation function that involves the angular velocities $\dot\phi_{l,i}, \dot\phi_{r,i}, \ i = 1, 2$:

$$\begin{aligned}
\mathcal{R} &= \frac{1}{2}k_1\dot\phi_{l,1}^2 + \frac{1}{2}k_1\dot\phi_{r,1}^2 + \frac{1}{2}k_2\dot\phi_{l,2}^2 + \frac{1}{2}k_2\dot\phi_{r,2}^2 \\
&\overset{(3.18)}{=} \frac{k_1}{R_1^2}\left[\frac{L_1^2}{4}\dot\theta_1^2 + (\dot x_1\cos\theta_1 + \dot y_1\sin\theta_1)^2\right] \\
&\qquad\qquad + \frac{k_2}{R_2^2}\left[\frac{L_2^2}{4}(\dot\theta_1 + \dot\theta_{1,2})^2 + \big((\dot x_1\cos\theta_1 + \dot y_1\sin\theta_1)\cos\theta_{1,2} + \dot\theta_1 d_1\sin\theta_{1,2}\big)^2\right] \\
&= \dot q^\top R\, \dot q \ ,
\end{aligned} \tag{5.24}$$

where $k_1 > 0$ and $k_2 > 0$ are friction coefficients, $\dot q = (\dot x_1, \dot y_1, \dot\theta_1, \dot\theta_{1,2})^\top$ and

$$R \overset{\text{def}}{=} \begin{pmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{12} & R_{22} & R_{23} & 0 \\ R_{13} & R_{23} & R_{33} & 0 \\ 0 & 0 & 0 & R_{44} \end{pmatrix} , \tag{5.25}$$

with

$$
\begin{aligned}
R_{11} &\stackrel{\text{def}}{=} \left(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\right)\cos^2\theta_1 \ , \\
R_{12} &\stackrel{\text{def}}{=} \left(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\right)\sin\theta_1\cos\theta_1 \ , \\
R_{13} &\stackrel{\text{def}}{=} \frac{k_2}{R_2^2} d_1 \cos\theta_1 \sin\theta_{1,2}\cos\theta_{1,2} \ , \\
R_{22} &\stackrel{\text{def}}{=} \left(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\right)\sin^2\theta_1 \ , \\
R_{23} &\stackrel{\text{def}}{=} \frac{k_2}{R_2^2} d_1 \sin\theta_1 \sin\theta_{1,2}\cos\theta_{1,2} \ , \\
R_{33} &\stackrel{\text{def}}{=} \frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}\left(\frac{L_2^2}{4} + d_1^2\sin^2\theta_{1,2}\right) \ , \\
R_{44} &\stackrel{\text{def}}{=} \frac{k_2}{R_2^2}\frac{L_2^2}{4} \ .
\end{aligned}
\tag{5.26}
$$

Proposition 5.3 (Lagrange–d' Alembert Equations of Motion with Friction)

The Lagrange–d'Alembert equations for the Roller Racer in the presence of friction take the form:

$$
\left(\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial q}\right)\cdot u = \alpha_e \cdot u \ , \tag{5.27}
$$

where (q, v) satisfy the nonholonomic constraints and $u \in \mathcal{D}_q$, the constraint distribution, and where the components of the external force 1–form α_e are:

$$
\begin{aligned}
F_{x_1} &= -2\Bigg[\left(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\right)(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) \\
&\qquad\qquad + \frac{k_2}{R_2^2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}\dot{\theta}_1\Bigg]\cos\theta_1 \ , \\
F_{y_1} &= -2\Bigg[\left(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\right)(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) \\
&\qquad\qquad + \frac{k_2}{R_2^2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}\dot{\theta}_1\Bigg]\sin\theta_1 \ , \\
F_{\theta_1} &= -2\Bigg[\frac{k_2}{R_2^2} d_1 \sin\theta_{1,2}\cos\theta_{1,2}(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) \\
&\qquad\qquad + \left(\frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}\left(\frac{L_2^2}{4} + d_1^2\sin^2\theta_{1,2}\right)\right)\dot{\theta}_1\Bigg] \ , \\
F_{\theta_{1,2}} &= \tau_{1,2} - 2\frac{k_2}{R_2^2}\frac{L_2^2}{4}\dot{\theta}_{1,2} \ ,
\end{aligned}
\tag{5.28}
$$

with $\tau_{1,2}$ being the torque applied by the motor at the joint $O_{1,2}$.

Proof

By direct calculation of the external force 1–form α_e

$$\alpha_e = \begin{pmatrix} F_{x_1} \\ F_{y_1} \\ F_{\theta_1} \\ F_{\theta_{1,2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tau_{1,2} \end{pmatrix} - \frac{\partial \mathcal{R}}{\partial \dot{q}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tau_{1,2} \end{pmatrix} - 2R\,\dot{q}\ , \tag{5.29}$$

using (5.25) and (5.26), we get the result.

∎

5.2 Nonholonomic Momentum and the Momentum Equation

Proposition 5.4

The constraints (3.9) and (3.11) and the Lagrangian (5.1) are invariant under the action Φ given by (4.1).

Proof

Fix $g(b,c,a) \in G = SE(2)$. Consider $q = (x_1, y_1, \theta_1, \theta_{1,2}) \in Q$. Under the action Φ of G on Q, $\bar{q} = (\bar{x}_1, \bar{y}_1, \bar{\theta}_1, \bar{\theta}_{1,2}) = \Phi(g,q)$:

$$\begin{aligned}
\bar{x}_1 &\overset{\text{def}}{=} x_1\cos a - y_1 \sin a + b \Longrightarrow \dot{\bar{x}}_1 = \dot{x}_1 \cos a - \dot{y}_1 \sin a\ ,\\
\bar{y}_1 &\overset{\text{def}}{=} x_1\sin a + y_1 \cos a + c \Longrightarrow \dot{\bar{y}}_1 = \dot{x}_1 \sin a + \dot{y}_1 \cos a\ ,\\
\bar{\theta}_1 &\overset{\text{def}}{=} \theta_1 + a \Longrightarrow \dot{\bar{\theta}}_1 = \dot{\theta}_1\ ,\\
\bar{\theta}_{1,2} &\overset{\text{def}}{=} \theta_{1,2} \Longrightarrow \dot{\bar{\theta}}_{1,2} = \dot{\theta}_{1,2}\ .
\end{aligned}$$

Consider the effect of Φ on the constraints (3.9) and (3.11):

$$\begin{aligned}
&-\dot{\bar{x}}_1 \sin\bar{\theta}_1 + \dot{\bar{y}}_1 \cos\bar{\theta}_1 = \\
&\qquad = -\sin(\theta_1 + a)(\dot{x}_1\cos a - \dot{y}_1 \sin a) + \cos(\theta_1+a)(\dot{x}_1 \sin a + \dot{y}_1\cos a)\\
&\qquad = \dot{x}_1 \sin\theta_1 + \dot{y}_1\cos\theta_1\ ,\\
&-\dot{\bar{x}}_1 \sin(\bar{\theta}_1 + \bar{\theta}_{1,2}) + \dot{\bar{y}}_1\cos(\bar{\theta}_1+\bar{\theta}_{1,2}) + \dot{\bar{\theta}}_1(d_1\cos\bar{\theta}_{1,2} + d_2) + \dot{\bar{\theta}}_{1,2}d_2 = \\
&\qquad = -\sin(\theta_1+\theta_{1,2}+a)(\dot{x}_1\cos a - \dot{y}_1\sin a)\\
&\qquad\qquad + \cos(\theta_1+\theta_{1,2}+a)(\dot{x}_1\sin a + \dot{y}_1\cos a)\\
&\qquad\qquad\qquad + \dot{\theta}_1(d_1\cos\theta_{1,2} + d_2) + \dot{\theta}_{1,2}d_2\\
&\qquad = -\dot{x}_1\sin(\theta_1+\theta_{1,2}) + \dot{y}_1\cos(\theta_1+\theta_{1,2}) + \dot{\theta}_1(d_1\cos\theta_{1,2}+d_2) + \dot{\theta}_{1,2}d_2\ .
\end{aligned}$$

Consider the effect of Φ on the Lagrangian (5.1):

$$
\begin{aligned}
L(\dot{\bar{x}}_1, \dot{\bar{y}}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_{1,2}) = \\
&= \frac{1}{2}m_1(\dot{\bar{x}}_1^2 + \dot{\bar{y}}_1^2) + \frac{1}{2}I_{z_1}\dot{\bar{\theta}}_1^2 + \frac{1}{2}I_{z_2}(\dot{\bar{\theta}}_1 + \dot{\bar{\theta}}_{1,2})^2 \\
&= \frac{1}{2}m_1\left[(\dot{x}_1\cos a - \dot{y}_1\sin a)^2 + (\dot{x}_1\sin a + \dot{y}_1\cos a)^2\right] + \frac{1}{2}I_{z_1}\dot{\theta}_1^2 + \frac{1}{2}I_{z_2}(\dot{\theta}_1 + \dot{\theta}_{1,2})^2 \\
&= L(\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2}) \ .
\end{aligned}
$$

∎

Momentum–like quantities can be defined for a constrained system by

$$p = \frac{\partial L}{\partial v} \cdot u \ ,$$

where $v \in T_qQ$ and $u \in \mathcal{D}_q$, the constraint distribution. In the present case, it is particularly advantageous (see Theorem 5.7 below) to restrict u to $\mathcal{S}_q$, the intersection of the subspaces $\mathcal{D}_q$ and $T_qOrb(q)$.

We define, then, the *nonholonomic momentum* as:

$$p = \sum_i \frac{\partial L}{\partial \dot{q}^i}(\xi_Q^q)^i \ , \tag{5.30}$$

where $\xi_Q^q \in \mathcal{S}_q$.

Proposition 5.5 (Nonholonomic Momentum)

The nonholonomic momentum for the Roller Racer system is

$$p = m_1(d_1\cos\theta_{1,2} + d_2)(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) + [(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}]\sin\theta_{1,2} \ . \tag{5.31}$$

Proof

From (5.2) and (4.5), we get (5.31).

∎

The nonholonomic momentum p given by equation (5.31) is (up to a scale factor) the angular momentum about the point of intersection O_{12}^{ICR} of the two wheel axles (c.f. fig. 1.3). It can be easily seen that $O_1O_{12}^{\mathrm{ICR}} = \frac{d_1\cos\theta_{1,2} + d_2}{\sin\theta_{1,2}}$, when $\sin\theta_{1,2} \neq 0$.

Let

$$\Delta(\theta_{1,2}) \stackrel{\mathrm{def}}{=} (I_{z_1} + I_{z_2})\sin^2\theta_{1,2} + m_1(d_1\cos\theta_{1,2} + d_2)^2 \ . \tag{5.32}$$

For $d_1 \neq d_2$, we have $\Delta > 0$ for all $q \in Q$.

Proposition 5.6

The angular velocity $\dot\theta_1$ is an affine function of the nonholonomic momentum

$$\dot\theta_1 = \frac{1}{\Delta(\theta_{1,2})}\big[\sin\theta_{1,2}\ \ p - \delta(\theta_{1,2})\dot\theta_{1,2}\big]\ , \tag{5.33}$$

where $\delta(\theta_{1,2}) \stackrel{\mathrm{def}}{=} I_{z_2}\sin^2\theta_{1,2} + m_1 d_2(d_1\cos\theta_{1,2} + d_2)$.

Proof

Multiplying both sides of (5.31) by $\sin\theta_{1,2}$ we get

$$\begin{aligned}\sin\theta_{1,2}\ \ p = m_1(d_1\cos\theta_{1,2} + d_2)(\dot x_1\cos\theta_1 + \dot y_1\sin\theta_1)\sin\theta_{1,2}\\ + [(I_{z_1} + I_{z_2})\dot\theta_1 + I_{z_2}\dot\theta_{1,2}]\sin^2\theta_{1,2}\ .\end{aligned}$$

From (3.11) we get

$$\begin{aligned}\sin\theta_{1,2}\ \ p &= m_1(d_1\cos\theta_{1,2} + d_2)[(d_1\cos\theta_{1,2} + d_2)\dot\theta_1 + d_2\dot\theta_{1,2}]\\ &\quad + [(I_{z_1} + I_{z_2})\dot\theta_1 + I_{z_2}\dot\theta_{1,2}]\sin^2\theta_{1,2}\\ &= \big[(I_{z_1} + I_{z_2})\sin^2\theta_{1,2} + m_1(d_1\cos\theta_{1,2} + d_2)^2\big]\dot\theta_1\\ &\quad + \big[I_{z_2}\sin^2\theta_{1,2} + m_1 d_2(d_1\cos\theta_{1,2} + d_2)\big]\dot\theta_{1,2}\ .\end{aligned}$$

Solving for $\dot\theta_1$, we get (5.33).

∎

Note that for $d_2 = 0$

$$\dot\theta_1 = \frac{\sin\theta_{1,2}}{\Delta(\theta_{1,2})}(p - I_{z_2}\sin\theta_{1,2}\dot\theta_{1,2})\ . \tag{5.34}$$

The momentum equation presented in the next result is derived from the Lagrange–d'Alembert principle by considering only variations that satisfy the constraints and that depend on the symmetry, as it is expressed by a free group action. The equation does not depend on internal torques and depends only on the shape variables and not on the group variables. It is given below for the case where external torques are not present.

Theorem 5.7 (Bloch, Krishnaprasad, Marsden & Murray [1994])

Consider a Lagrangian L which is invariant under the action Φ of a group G on a configuration space Q. Let $\mathcal{D}_q$ be a constraint distribution on T_qQ and consider the intersection $\mathcal{S}_q$ of $\mathcal{D}_q$ with the tangent space to the orbit of Φ at q. Let $\xi^q_Q \in \mathcal{S}_q$ and let ξ^q be the corresponding element of the Lie algebra $\mathcal{G}$. The evolution of the nonholonomic momentum p, defined as in equation (5.30), satisfies the equation:

$$\frac{dp}{dt} = \sum_i \frac{\partial L}{\partial \dot q^i}\left[\frac{d\xi^q}{dt}\right]^i_Q\ . \tag{5.35}$$

$\blacksquare$

This result generalizes the classical Noether Theorem, which specifies conserved quantities for solutions of the Euler–Lagrange equations (Abraham & Marsden [1985]; Arnold [1978]; Marsden & Ratiu [1994]).

Proposition 5.8 (Momentum Equation without External Forces)

The momentum equation for the Roller Racer system is

$$\frac{dp}{dt} = A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \ , \tag{5.36}$$

where

$$A_1^4(\theta_{1,2}) \overset{\text{def}}{=} \frac{(I_{z_1}+I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2}+d_2)}{(I_{z_1}+I_{z_2})\sin^2\theta_{1,2} + m_1(d_1\cos\theta_{1,2}+d_2)^2}\sin\theta_{1,2} = \frac{1}{\Delta(\theta_{1,2})}\beta(\theta_{1,2})\sin\theta_{1,2} \tag{5.37}$$

and

$$A_2^4(\theta_{1,2}) \overset{\text{def}}{=} \frac{m_1(d_1+d_2\cos\theta_{1,2})(-I_{z_1}d_2 + I_{z_2}d_1\cos\theta_{1,2})}{(I_{z_1}+I_{z_2})\sin^2\theta_{1,2} + m_1(d_1\cos\theta_{1,2}+d_2)^2} = \frac{m_1}{\Delta(\theta_{1,2})}\lambda(\theta_{1,2})\gamma(\theta_{1,2}),$$

where

$$\begin{aligned}
\beta(\theta_{1,2}) &\overset{\text{def}}{=} (I_{z_1}+I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2}+d_2) = I\cos\theta_{1,2} - m_1 d_1 r \ , \\
\gamma(\theta_{1,2}) &\overset{\text{def}}{=} -I_{z_1}d_2 + I_{z_2}d_1\cos\theta_{1,2} \ , \\
r(\theta_{1,2}) &\overset{\text{def}}{=} d_1\cos\theta_{1,2} + d_2 \ , \\
\lambda(\theta_{1,2}) &\overset{\text{def}}{=} d_1 + d_2\cos\theta_{1,2} \ , \\
I &\overset{\text{def}}{=} I_{z_1} + I_{z_2} \ .
\end{aligned}$$

Proof

From (5.35), (5.2) and (4.12) we get

$$\begin{aligned}
\frac{dp}{dt} &= m_1\dot{x}_1\big[-d_1\sin\theta_{1,2}\cos\theta_1\dot{\theta}_{1,2} - (d_1\cos\theta_{1,2}+d_2)\sin\theta_1\dot{\theta}_1 + \dot{y}_1\sin\theta_{1,2}\big] \\
&\qquad + m_1\dot{y}_1\big[-d_1\sin\theta_{1,2}\sin\theta_1\dot{\theta}_{1,2} + (d_1\cos\theta_{1,2}+d_2)\cos\theta_1\dot{\theta}_1 - \dot{x}_1\sin\theta_{1,2}\big] \\
&\qquad + \big[(I_{z_1}+I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}\big]\cos\theta_{1,2}\dot{\theta}_{1,2} \\
&= -m_1 d_1(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\sin\theta_{1,2}\dot{\theta}_{1,2} \\
&\qquad + m_1(d_1\cos\theta_{1,2}+d_2)(-\dot{x}_1\sin\theta_1 + \dot{y}_1\cos\theta_1)\dot{\theta}_1 \\
&\qquad + \big[(I_{z_1}+I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}\big]\cos\theta_{1,2}\dot{\theta}_{1,2}
\end{aligned}$$

Using the nonholonomic constraints (3.9) and (3.11) we get

$$\begin{aligned}\frac{dp}{dt} &= -m_1 d_1 \big[(d_1\cos\theta_{1,2}+d_2)\dot\theta_1 + d_2\dot\theta_{1,2}\big] + \big[(I_{z_1}+I_{z_2})\dot\theta_1 + I_{z_2}\dot\theta_{1,2}\big]\cos\theta_{1,2}\dot\theta_{1,2} \\ &= \big[(I_{z_1}+I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2}+d_2)\big]\dot\theta_1\dot\theta_{1,2} + \big[I_{z_2}\cos\theta_{1,2} - m_1 d_1 d_2\big]\dot\theta_{1,2}^2 .\end{aligned}$$

Using (5.33), we substitute $\dot\theta_1$ in the above expression and we get (5.36).

∎

Proposition 5.9

The solution of the momentum equation (5.36), where no external forces act on the system, is:

$$p(t) = \Phi(t,t_0)p(t_0) + \int_{t_0}^{t} \Phi(t,\tau)A_2^4(\theta_{1,2}(\tau))\dot\theta_{1,2}^2(\tau)d\tau \ , \tag{5.38}$$

where

$$\Phi(t,t_0) = \exp\Big[\int_{t_0}^{t} A_1^4(\theta_{1,2}(\tau))\dot\theta_{1,2}(\tau)d\tau\Big] = \exp\Big[\int_{\theta_{1,2}(t_0)}^{\theta_{1,2}(t)} A_1^4(\theta_{1,2})d\theta_{1,2}\Big] = \sqrt{\frac{\Delta\big(\theta_{1,2}(t)\big)}{\Delta\big(\theta_{1,2}(t_0)\big)}} \tag{5.39}$$

is the state transition matrix of (5.36) and where $\Delta(\theta_{1,2})$ is defined in equation (5.32).

Proof

Equation (5.36) is a first–order linear time–varying ODE with state transition matrix $\Phi(t,t_0)$. Thus, (5.38) is obvious. To compute the state transition matrix $\Phi(t,t_0)$, observe that we get from (5.32):

$$\frac{d\Delta}{d\theta_{1,2}} = 2\sin\theta_{1,2}\big[(I_{z_1}+I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2}+d_2)\big] = 2\beta(\theta_{1,2})\sin\theta_{1,2} .$$

From this and the definition of A_1^4 in (5.36) we get

$$A_1^4(\theta_{1,2}) = \frac{\beta(\theta_{1,2})}{\Delta(\theta_{1,2})}\sin\theta_{1,2} = \frac{1}{2\Delta}\frac{d\Delta}{d\theta_{1,2}} .$$

Thus

$$\begin{aligned}\Phi(t,t_0) &= \exp\Big[\int_{t_0}^{t} A_1^4(\theta_{1,2}(\tau))\dot\theta_{1,2}(\tau)d\tau\Big] = \exp\Big[\int_{\theta_{1,2}(t_0)}^{\theta_{1,2}(t)} A_1^4(\theta_{1,2})d\theta_{1,2}\Big] \\ &= \exp\Big[\int_{\Delta(\theta_{1,2}(t_0))}^{\Delta(\theta_{1,2}(t))} \frac{1}{2}\frac{d\Delta}{\Delta}\Big] = \exp\Big[\ln\Big(\sqrt{\frac{\Delta\big(\theta_{1,2}(t)\big)}{\Delta\big(\theta_{1,2}(t_0)\big)}}\Big)\Big] = \sqrt{\frac{\Delta\big(\theta_{1,2}(t)\big)}{\Delta\big(\theta_{1,2}(t_0)\big)}}\end{aligned} \tag{5.40}$$

■

Equation (5.38) can be used to derive qualitative information about the momentum, which can be useful in motion control.

Proposition 5.10 (Sign of the Nonholonomic Momentum)

Assume $d_1 > d_2$.

a) Let $I_{z_1} d_2 > I_{z_2} d_1$.

Assume further that the initial momentum of the system is non–positive. Then, the momentum p is negative at all times.

b1) Let $I_{z_1} d_2 < I_{z_2} d_1$.

Suppose, in addition, that the angle $\theta_{1,2}$ remains in an $\tilde{\epsilon}$–neighborhood of $\theta_{1,2} = 0$, with $\tilde{\epsilon} \leq \cos^{-1}(\frac{I_{z_1} d_2}{I_{z_2} d_1})$. Assume further that the initial momentum of the system is non–negative. Then, the momentum p is positive at all times.

b2) Let $I_{z_1} d_2 < I_{z_2} d_1$.

Suppose, in addition, that the angle $\theta_{1,2}$ remains *outside* an $\tilde{\epsilon}$–neighborhood of $\theta_{1,2} = 0$, with $\tilde{\epsilon} \leq \cos^{-1}(\frac{I_{z_1} d_2}{I_{z_2} d_1})$. Assume further that the initial momentum of the system is non–positive. Then, the momentum p is negative at all times.

Proof

Since $d_1 > d_2$, we know that $\Delta > 0$ and $\lambda = d_1 + d_2 \cos\theta_{1,2} > 0$, for all $\theta_{1,2}$.

a) In the case $I_{z_1} d_2 > I_{z_2} d_1$ we have $\gamma = -I_{z_1} d_2 + I_{z_2} d_1 \cos\theta_{1,2} < 0$, for all $\theta_{1,2}$. Thus $A_2^4 = \frac{m_1}{\Delta}\lambda\gamma < 0$, for all $\theta_{1,2}$ and, thus, the second term of (5.38) is negative. If $p(t_0) \leq 0$, then $p < 0$.

b1) In the case $I_{z_1} d_2 < I_{z_2} d_1$, by our choice of the $\tilde{\epsilon}$–neighborhood we have $\gamma > 0$, for all $\theta_{1,2}$ in this neighborhood. Then, $A_2^4 = \frac{m_1}{\Delta}\lambda\gamma > 0$ and the second term of (5.38) is positive. If $p(t_0) \geq 0$, then $p > 0$.

b2) In the case $I_{z_1} d_2 < I_{z_2} d_1$, by our choice of the $\tilde{\epsilon}$–neighborhood we have $\gamma < 0$, for all $\theta_{1,2}$ outside this neighborhood. Then, $A_2^4 = \frac{m_1}{\Delta}\lambda\gamma < 0$ and the second term of (5.38) is negative. If $p(t_0) \leq 0$, then $p < 0$.

■

The momentum equation (5.36) can be derived directly from the Lagrange–d'Alembert Principle (5.3), by considering a test vector u in the space $\mathcal{S}_q \subset \mathcal{D}_q \subset T_qQ$. This approach is used below to derive the momentum equation for the case when external forces, of the type considered in equation (5.8), are acting on the Roller Racer.

Proposition 5.11 (Momentum Equation with External Forces)

In the case $d_2 \neq 0$, the nonholonomic momentum evolves according to the equation below:

$$\frac{dp}{dt} = A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 + r(\theta_{1,2})\big(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1\big) + \sin\theta_{1,2}F_{\theta_1} \ , \tag{5.41}$$

Proof

Consider the Lagrange–d' Alembert principle (5.10) with u restricted to $\mathcal{S}_q \subset \mathcal{D}_q$, instead of the whole $\mathcal{D}_q$. For $d_2 \neq 0$, the vectors $v \in \mathcal{D}_q$ and $\dot{v}$ in this equation are given by (5.13) and (5.15), while $u \in \mathcal{S}_q \subset \mathcal{D}_q$ is given by (5.12), where $\alpha_1 = \frac{r(\theta_{1,2})}{d_2}$ and $\alpha_2 = \frac{\sin\theta_{1,2}}{d_2}$ (c.f. equation (4.5)). Thus, (5.10) takes the form (5.16), with α_1 and α_2 as specified above, which gives:

$$\begin{aligned}\big(I_{z_2}\sin^2\theta_{1,2} + m_1 d_2 r(\theta_{1,2})\big)\dot{\nu}_1 + \big(I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2}\big)\sin\theta_{1,2}\dot{\nu}_2 \\ + I_{z_2}(\nu_1\cos\theta_{1,2} + \nu_2 d_1\sin\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2} \\ = r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} \ ,\end{aligned}$$

from which we get

$$\begin{aligned}\delta(\theta_{1,2})\dot{\nu}_1 - \gamma(\theta_{1,2})\sin\theta_{1,2}\ \dot{\nu}_2 = -I_{z_2}(\nu_1\cos\theta_{1,2} + \nu_2 d_1\sin\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2} \\ + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} \ .\end{aligned} \tag{5.42}$$

Consider the nonholonomic momentum defined in (5.31), i.e.

$$p = m_1 r(\theta_{1,2})(v_1\cos\theta_1 + v_2\sin\theta_1) + \big[(I_{z_1} + I_{z_2})v_3 + I_{z_2}v_4\big]\sin\theta_{1,2} \ ,$$

with $v = (v_1,\ v_2,\ v_3,\ v_4) \in T_qQ$. By restricting v to $\mathcal{D}_q$, we get for p (using, for $d_2 \neq 0$, the expression (5.13)):

$$\begin{aligned}p &= m_1 r(\theta_{1,2})d_2\nu_1 + (I_{z_1} + I_{z_2})d_2\sin\theta_{1,2}\nu_2 + I_{z_2}\sin\theta_{1,2}\big(\sin\theta_{1,2}\nu_1 - r(\theta_{1,2})\nu_2\big) \\ &= \delta(\theta_{1,2})\nu_1 - \gamma(\theta_{1,2})\sin\theta_{1,2}\nu_2 \ .\end{aligned} \tag{5.43}$$

The last of the equations (5.13) (the one for $v_4 \equiv \dot{\theta}_{1,2}$) and equation (5.43) are linear in ν_1 and ν_2. By inverting them, we get

$$\begin{aligned}\nu_1 &= \frac{1}{d_2\Delta(\theta_{1,2})}\big[r(\theta_{1,2})p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\big] \ , \\ \nu_2 &= \frac{1}{d_2\Delta(\theta_{1,2})}\big[\sin\theta_{1,2}p - \delta(\theta_{1,2})\dot{\theta}_{1,2}\big] \ .\end{aligned} \tag{5.44}$$

By differentiating (5.43), we get

$$\frac{dp}{dt} = \delta(\theta_{1,2})\dot{\nu}_1 - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\nu}_2 + \left(\frac{\partial\delta}{\partial\theta_{1,2}}\nu_1 - \frac{\partial\gamma}{\partial\theta_{1,2}}\sin\theta_{1,2}\nu_2 - \gamma(\theta_{1,2})\cos\theta_{1,2}\nu_2\right)\dot{\theta}_{1,2} \, .$$

Replacing the first two terms of the RHS above with their expression from (5.42) and using (5.44), we get

$$\begin{aligned}
\frac{dp}{dt} &= \Big[\big(- I_{z_2}\sin\theta_{1,2}\cos\theta_{1,2} + \frac{\partial\delta}{\partial\theta_{1,2}}\big)\nu_1 \\
&\qquad - \big(I_{z_2}d_1\sin^2\theta_{1,2} + \frac{\partial\gamma}{\partial\theta_{1,2}}\sin\theta_{1,2} + \gamma(\theta_{1,2})\cos\theta_{1,2}\big)\nu_2\Big]\,\dot{\theta}_{1,2} \\
&\qquad + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} \\
&= \frac{1}{d_2\Delta(\theta_{1,2})}\Big[\big(- I_{z_2}\sin\theta_{1,2}\cos\theta_{1,2} + \frac{\partial\delta}{\partial\theta_{1,2}}\big)\big(r(\theta_{1,2})p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\big) \\
&\qquad - \big(I_{z_2}d_1\sin^2\theta_{1,2} + \frac{\partial\gamma}{\partial\theta_{1,2}}\sin\theta_{1,2} + \gamma(\theta_{1,2})\cos\theta_{1,2}\big)\cdot \\
&\qquad \cdot\big(\sin\theta_{1,2}p - \delta(\theta_{1,2})\dot{\theta}_{1,2}\big)\Big]\,\dot{\theta}_{1,2} \\
&\qquad + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} \\
&= \frac{1}{d_2\Delta(\theta_{1,2})}\big[\big(I_{z_2}\sin\theta_{1,2}\cos\theta_{1,2} - m_1d_1d_2\sin\theta_{1,2}\big)\cdot \\
&\qquad \cdot\big(r(\theta_{1,2})p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\big) \\
&\qquad - \gamma(\theta_{1,2})\cos\theta_{1,2}\big(\sin\theta_{1,2}p - \delta(\theta_{1,2})\dot{\theta}_{1,2}\big)\Big]\,\dot{\theta}_{1,2} \\
&\qquad + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} \\
&= \frac{1}{d_2\Delta(\theta_{1,2})}\big[d_2\beta(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}p + m_1d_2\lambda(\theta_{1,2})\gamma(\theta_{1,2})\dot{\theta}_{1,2}^2\big] \\
&\qquad + r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} \, .
\end{aligned}$$

Finally, using the definitions (5.37), we get (5.41).

■

In the case when viscous friction, of the type considered in the previous section, is present, the momentum equation takes the form below.

Proposition 5.12 (Momentum Equation with Friction)

In the presence of friction and for $d_2 \neq 0$, the nonholonomic momentum evolves according to the equation below:

$$\frac{dp}{dt} = \big[A_1^4(\theta_{1,2})\dot{\theta}_{1,2} - A_1^5(\theta_{1,2})\big]p + \big[A_2^4(\theta_{1,2})\dot{\theta}_{1,2} + A_2^5(\theta_{1,2})\big]\dot{\theta}_{1,2} \, , \tag{5.45}$$

where

$$
\begin{aligned}
A_1^5(\theta_{1,2}) &\stackrel{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})}\big[\eta_1(\theta_{1,2})\sin\theta_{1,2} + \eta_2(\theta_{1,2})r(\theta_{1,2})\big], \\
A_2^5(\theta_{1,2}) &\stackrel{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})}\big[\eta_1(\theta_{1,2})\delta(\theta_{1,2}) + \eta_2(\theta_{1,2})\gamma(\theta_{1,2})\sin\theta_{1,2}\big], \\
\eta_1(\theta_{1,2}) &\stackrel{\text{def}}{=} 2\frac{k_2}{R_2^2}d_1 r(\theta_{1,2})\sin\theta_{1,2}\cos\theta_{1,2} + 2\big(\frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}\frac{L_2^2}{4} + \frac{k_2}{R_2^2}d_1^2\sin^2\theta_{1,2}\big)\sin\theta_{1,2} \\
&= 2\left[\frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}\frac{L_2^2}{4} + \frac{k_2}{R_2^2}d_1\lambda(\theta_{1,2})\right]\sin\theta_{1,2}\ , \\
\eta_2(\theta_{1,2}) &\stackrel{\text{def}}{=} 2\big(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\big)r(\theta_{1,2}) + 2\frac{k_2}{R_2^2}d_1\sin^2\theta_{1,2}\cos\theta_{1,2} \\
&= 2\frac{k_1}{R_1^2}r(\theta_{1,2}) + 2\frac{k_2}{R_2^2}\lambda(\theta_{1,2})\cos\theta_{1,2}\ .
\end{aligned}
\tag{5.46}
$$

If $d_1 > d_2$, then $A_1^5(\theta_{1,2}) > 0$, for all $\theta_{1,2}$.

Proof

In the momentum equation (5.41), we consider an external force 1–form α_e which is due to friction and to the torque $\tau_{1,2}$ at the joint $O_{1,2}$. Thus, α_e has the form (5.28). The following terms from the RHS of (5.41) take, then, the form:

$$
r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} = -\eta_1(\theta_{1,2})\dot\theta_1 - \eta_2(\theta_{1,2})(\dot x_1\cos\theta_1 + \dot y_1\sin\theta_1)\ , \tag{5.47}
$$

where η_1 and η_2 are defined in (5.46).
For $v = (\dot x_1, \dot y_1, \dot\theta_1, \dot\theta_{1,2}) \in \mathcal{D}_q$, we have from (5.13) when $d_2 \neq 0$:

$$
\dot x_1\cos\theta_1 + \dot y_1\sin\theta_1 = \nu_1\ d_2 \quad \text{and} \quad \dot\theta_1 = \nu_2\ d_2\ ,
$$

for $\nu_1, \nu_2 \in I\!R$. Thus,

$$
r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} = -\eta_1\nu_2 d_2 - \eta_2\nu_1 d_2\ .
$$

From (5.44), when $d_2 \neq 0$:

$$
\begin{aligned}
&r(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\sin\theta_{1,2} \\
&\qquad = -\frac{1}{\Delta(\theta_{1,2})}\big[\eta_1(\theta_{1,2})\sin\theta_{1,2} + \eta_2(\theta_{1,2})r(\theta_{1,2})\big]p \\
&\qquad\qquad + \frac{1}{\Delta(\theta_{1,2})}\big[\eta_1(\theta_{1,2})\delta(\theta_{1,2}) + \eta_2(\theta_{1,2})\gamma(\theta_{1,2})\sin\theta_{1,2}\big]\dot\theta_{1,2}\ .
\end{aligned}
$$

From this, using the definitions (5.46), we get (5.45).

It is easy to check from the definitions (5.46) that

$$A_1^5(\theta_{1,2}) = \frac{2}{\Delta(\theta_{1,2})}\left[\left(\frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}\frac{L_2^2}{4}\right)\sin^2\theta_{1,2} + \frac{k_1}{R_1^2}r^2(\theta_{1,2}) + \frac{k_2}{R_2^2}\lambda^2(\theta_{1,2})\right]$$
$$\geq \frac{2}{\Delta(\theta_{1,2})}\frac{k_2}{R_2^2}\lambda^2(\theta_{1,2}) \ .$$

If $d_1 > d_2$, then $\Delta(\theta_{1,2}) > 0$ and also $\lambda(\theta_{1,2}) > 0$ for all $\theta_{1,2}$. Thus, $A_1^5(\theta_{1,2}) > 0$.

∎

Comparing the momentum equations (5.45) and (5.36), we note the extra terms that are due to friction.

When $\dot{\theta}_{1,2} = 0$, i.e. when $\theta_{1,2}(t) = \theta_{1,2}(0) = \text{constant}$, the momentum equation for the Roller Racer model without friction (5.36) takes the form $\dot{p} = 0$, i.e. the momentum is conserved. However, in the same case, the momentum equation for the model with friction (5.45) takes the form:

$$\frac{dp}{dt} = -A_1^5\big(\theta_{1,2}(0)\big)p \ , \tag{5.48}$$

thus

$$p(t) = e^{-A_1^5(\theta_{1,2}(0))t}p(0) \ , \tag{5.49}$$

for a constant $A_1^5(\theta_{1,2}(0)) > 0$, which is the rate at which the momentum decreases exponentially and at which the system will come to rest. This provides a *braking mechanism* for the Roller Racer, which has been noticed in experiments with the ISL's Roller Racer prototypes.

5.3 Reconstruction of Group Motion

Assume that a shape–space trajectory $\theta_{1,2}(.) \subset \mathcal{S}$ has been specified. The corresponding nonholonomic momentum can be determined from the solution of the momentum equation (5.36), in the case of the Roller Racer model without friction, or from the solution of the momentum equation (5.45), in the case of the Roller Racer model with friction. From the definition of the nonholonomic momentum (equation (5.31)) and from the nonholonomic constraints (equations (3.9) and (3.11)), we can reconstruct the group trajectory $g_1(.) = g_1(x_1(.), y_1(.), \theta_1(.)) \subset SE(2)$. This can be done either by first specifying $(\dot{x}_1, \dot{y}_1, \dot{\theta}_1)$ and then integrating to find (x_1, y_1, θ_1) or by first specifying $\xi_1 \in \mathcal{G}$ and then applying the Wei–Norman procedure to find the corresponding $g_1 \in G$.

Proposition 5.13 (Reconstruction of Group Trajectory)

For $g_1 = g_1(x_1, y_1, \theta_1) \in SE(2)$, the corresponding curve in the Lie algebra $\xi_1 = g_1^{-1}\dot{g}_1$ is given by

$$\xi_1 = \xi_1^1(\theta_{1,2}, \dot{\theta}_{1,2})\mathcal{A}_1 + \xi_2^1(\theta_{1,2}, \dot{\theta}_{1,2})\mathcal{A}_2 \ , \tag{5.50}$$

where for $d_1 \neq d_2$, the components of ξ_1 are

$$\begin{aligned} \xi_1^1(\theta_{1,2}, \dot{\theta}_{1,2}) = \dot{\theta}_1 &= \frac{\sin\theta_{1,2}}{\Delta(\theta_{1,2})}p - \frac{I_{z_2}\sin^2\theta_{1,2} + m_1 d_2(d_1\cos\theta_{1,2} + d_2)}{\Delta(\theta_{1,2})}\dot{\theta}_{1,2} \\ &= \frac{1}{\Delta(\theta_{1,2})}[\sin\theta_{1,2}\, p - \delta(\theta_{1,2})\dot{\theta}_{1,2}] \end{aligned} \tag{5.51}$$

and

$$\begin{aligned} \xi_2^1(\theta_{1,2}, \dot{\theta}_{1,2}) &= \dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1 \\ &= \frac{d_1\cos\theta_{1,2} + d_2}{\Delta(\theta_{1,2})}p - \frac{-I_{z_1}d_2 + I_{z_2}d_1\cos\theta_{1,2}}{\Delta(\theta_{1,2})}\sin\theta_{1,2}\dot{\theta}_{1,2} \\ &= \frac{1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}] \ , \end{aligned} \tag{5.52}$$

where

$$\delta(\theta_{1,2}) \stackrel{\text{def}}{=} I_{z_2}\sin^2\theta_{1,2} + m_1 d_2(d_1\cos\theta_{1,2} + d_2)$$

and where $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is the usual basis of $\mathcal{G} = se(2)$. The group trajectory is given by solving

$$\dot{g}_1 = g_1\xi_1 \ , \tag{5.53}$$

where ξ_1 is given by (5.50) or by solving

$$\begin{aligned} \dot{\theta}_1 = \xi_1^1 &= \frac{1}{\Delta(\theta_{1,2})}[\sin\theta_{1,2}\, p - \delta(\theta_{1,2})\dot{\theta}_{1,2}] \ , \\ \dot{x}_1 = \cos\theta_1\, \xi_2^1 &= \frac{\cos\theta_1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}] \ , \\ \dot{y}_1 = \sin\theta_1\, \xi_2^1 &= \frac{\sin\theta_1}{\Delta(\theta_{1,2})}[r(\theta_{1,2})\, p - \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}] \ . \end{aligned} \tag{5.54}$$

The solution of (5.53) or (5.54) can be obtained by quadratures.

Proof

Equation (5.51) is immediate from (5.33). From equations (3.11), (3.12) and (5.33) we get

$$(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1)\sin\theta_{1,2} = \frac{\sin\theta_{1,2}}{\Delta}\left[(d_1\cos\theta_{1,2}+d_2)p + (I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\right] .$$

When $\sin\theta_{1,2} \neq 0$ we get

$$\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1 = \frac{1}{\Delta}\left[(d_1\cos\theta_{1,2}+d_2)p + (I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\right] . \tag{5.55}$$

From (5.31) and (5.33) we get

$$m_1(d_1\cos\theta_{1,2}+d_2)(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) = \frac{m_1}{\Delta}(d_1\cos\theta_{1,2}+d_2)\left[(d_1\cos\theta_{1,2}+d_2)p + (I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\right] .$$

When $d_1\cos\theta_{1,2}+d_2 \neq 0$ we get from this exactly (5.55).
When $d_1 \neq d_2$, either $\sin\theta_{1,2} \neq 0$ or $d_1\cos\theta_{1,2}+d_2 \neq 0$. In either case (5.55) holds. From this we get (5.52).
Finally, from (3.9) we get $\xi^1_3 = 0$. Equations (5.54) are immediate from (5.52) and (3.9).

∎

From (5.52), note that for $d_2 = 0$, the velocity component ξ^1_2 is

$$\xi^1_2 = \dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1 = \frac{d_1\cos\theta_{1,2}}{\Delta}(p - I_{z_2}\sin\theta_{1,2}\dot{\theta}_{1,2}) . \tag{5.56}$$

Now observe that from (5.51) and (5.52) we get

$$\begin{aligned}\begin{pmatrix}\xi^1_1\\ \xi^2_1\end{pmatrix} &= \frac{1}{\Delta}\begin{pmatrix}\sin\theta_{1,2} & -[I_{z_2}\sin^2\theta_{1,2} + m_1d_2(d_1\cos\theta_{1,2}+d_2)]\\ d_1\cos\theta_{1,2}+d_2 & -(-I_{z_1}d_2 + I_{z_2}d_1\cos\theta_{1,2})\sin\theta_{1,2}\end{pmatrix}\begin{pmatrix}p\\ \dot{\theta}_{1,2}\end{pmatrix}\\ &= \frac{1}{\Delta}\begin{pmatrix}\sin\theta_{1,2} & -\delta(\theta_{1,2})\\ r(\theta_{1,2}) & -\gamma(\theta_{1,2})\sin\theta_{1,2}\end{pmatrix}\begin{pmatrix}p\\ \dot{\theta}_{1,2}\end{pmatrix} = B(\theta_{1,2})\begin{pmatrix}p\\ \dot{\theta}_{1,2}\end{pmatrix}\end{aligned} \tag{5.57}$$

and notice that

$$r(\theta_{1,2})\delta(\theta_{1,2}) - \gamma(\theta_{1,2})\sin^2\theta_{1,2} = d_2\Delta(\theta_{1,2}) , \tag{5.58}$$

therefore

$$\det B(\theta_{1,2}) = \frac{d_2}{\Delta(\theta_{1,2})} . \tag{5.59}$$

Thus, in the case $d_2 = 0$, given a group trajectory $\xi_1 \subset \mathcal{G}$, we cannot always solve (5.57) for p and $\dot{\theta}_{1,2}$.

When $d_1 \neq d_2$ and $d_2 \neq 0$, from (5.57) and (5.59), we get

$$\begin{pmatrix} p \\ \dot{\theta}_{1,2} \end{pmatrix} = B^{-1}(\theta_{1,2}) \begin{pmatrix} \xi^1_1 \\ \xi^2_1 \end{pmatrix} = \frac{1}{d_2} \begin{pmatrix} -\gamma(\theta_{1,2}) \sin\theta_{1,2} & \delta(\theta_{1,2}) \\ -r(\theta_{1,2}) & \sin\theta_{1,2} \end{pmatrix} \begin{pmatrix} \xi^1_1 \\ \xi^2_1 \end{pmatrix} . \tag{5.60}$$

5.4 The Nonholonomic Connection

Consider the *kinetic energy inner product* $\ll,\gg$ specified by the Lagrangian (5.1):

$$\ll v, \tilde{v} \gg \stackrel{\text{def}}{=} v^\top \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & I_{z_1} + I_{z_2} & I_{z_2} \\ 0 & 0 & I_{z_2} & I_{z_2} \end{pmatrix} \tilde{v} , \tag{5.61}$$

for $v, \tilde{v} \in T_q Q$.

Proposition 5.14

The orthogonal complement H_q of the subspace $\mathcal{S}_q$ with respect to the constraint subspace $\mathcal{D}_q$, i.e.

$$\mathcal{S}_q \oplus H_q = \mathcal{D}_q , \tag{5.62}$$

where orthogonality is defined with respect to the kinetic energy inner product $\ll,\gg$, is given by

$$H_q = \text{sp}\{\xi^H_Q\} , \tag{5.63}$$

where

$$\xi^H_Q = \gamma(\theta_{1,2}) \sin\theta_{1,2} \big(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1} \big) + \delta(\theta_{1,2}) \frac{\partial}{\partial \theta_1} - \Delta(\theta_{1,2}) \frac{\partial}{\partial \theta_{1,2}} . \tag{5.64}$$

Proof

Since $\dim \mathcal{S}_q = 1$ and $\dim \mathcal{D}_q = 2$, we should have $\dim H_q = 1$. Consider an element $\xi^H_Q \in H_q$. As ξ^H_Q also belongs to $\mathcal{D}_q$, it can be written, as a function of the basis elements of $\mathcal{D}_q$, as

$$\xi^H_Q = \alpha_1 \xi^1_Q + \alpha_2 \xi^2_Q , \tag{5.65}$$

for $\alpha_1, \alpha_2 \in I\!R$.

When $d_2 \neq 0$, we have from (3.16)

$$\begin{aligned} \xi^H_Q = \alpha_1 d_2 (\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) \\ + \alpha_2 d_2 \frac{\partial}{\partial \theta_1} + \big[\alpha_1 \sin\theta_{1,2} - \alpha_2 r(\theta_{1,2})\big] \frac{\partial}{\partial \theta_{1,2}} . \end{aligned}$$

The vector ξ_Q^H should be orthogonal to every $\xi_Q^q \in \mathcal{S}_q$, i.e.

$$\ll \xi_Q^q, \xi_Q^H \gg = 0 \ . \tag{5.66}$$

From (5.61) we get

$$\delta(\theta_{1,2})\alpha_1 - \gamma(\theta_{1,2}) \sin\theta_{1,2}\alpha_2 = 0 \ .$$

Choose $\alpha_1 = \gamma(\theta_{1,2}) \sin\theta_{1,2}$ and $\alpha_2 = \delta(\theta_{1,2})$. Using (5.58) and dividing by d_2, we get (5.64).

In the case $d_2 = 0$ we have from (3.17):

$$\xi_Q^H = \alpha_1 d_1 \cos\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \alpha_1 \sin\theta_{1,2} \frac{\partial}{\partial \theta_1} + \alpha_2 \frac{\partial}{\partial \theta_{1,2}} \ .$$

From orthogonality to ξ_Q^q, we obtain $\alpha_1 = I_{z_2} \sin\theta_{1,2}$ and $\alpha_2 = \big[(I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1 d_1^2 \cos^2\theta_{1,2}\big]$, thus

$$\begin{aligned} \xi_Q^H = I_{z_2} d_1 \sin\theta_{1,2} \cos\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + I_{z_2} \sin^2\theta_{1,2} \frac{\partial}{\partial \theta_1} \\ - \big[(I_{z_1} + I_{z_2}) \sin^2\theta_{1,2} + m_1 d_1^2 \cos^2\theta_{1,2}\big] \frac{\partial}{\partial \theta_{1,2}} \ . \end{aligned}$$

Note that, when $d_2 = 0$, this expression is the same as (5.64).

∎

Proposition 5.15

The orthogonal complement $\mathcal{U}_q$ of the subspace $\mathcal{S}_q$ with respect to the subspace $T_q Orb(q)$, i.e.

$$\mathcal{S}_q \oplus \mathcal{U}_q = T_q Orb(q) \ , \tag{5.67}$$

where orthogonality is defined with respect to the kinetic energy inner product $\ll, \gg$, is given by

$$\mathcal{U}_q = \mathrm{sp}\{\xi_Q^{\mathcal{U}_1},\ \xi_Q^{\mathcal{U}_2}\} \ , \tag{5.68}$$

where

$$\begin{aligned} \xi_Q^{\mathcal{U}_1} &= -\sin\theta_1 \frac{\partial}{\partial x_1} + \cos\theta_1 \frac{\partial}{\partial y_1} \ , \\ \xi_Q^{\mathcal{U}_2} &= (I_{z_1} + I_{z_2}) \sin\theta_{1,2}\big(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}\big) - m_1(d_1 \cos\theta_{1,2} + d_2) \frac{\partial}{\partial \theta_1} \ . \end{aligned} \tag{5.69}$$

Proof

Since dim $\mathcal{S}_q = 1$ and dim $T_qOrb(q) = 3$, we have dim $\mathcal{U}_q = 2$. Let $\xi_Q^{\mathcal{U}_1}$ and $\xi_Q^{\mathcal{U}_2}$ be two basis elements of $\mathcal{U}_q$. Since the $\xi_Q^{\mathcal{U}_i}$, $i = 1,2$ also belong to $T_qOrb(q)$, they can expressed as a function of its basis elements as

$$\xi_Q^{\mathcal{U}_i} = u_1^i \frac{\partial}{\partial x_1} + u_2^i \frac{\partial}{\partial y_1} + u_3^i \frac{\partial}{\partial \theta_1} \ ,$$

for $u_j^i \in I\!R$. The $\xi_Q^{\mathcal{U}_i}$ need to be mutually linearly independent and orthogonal to $\xi_Q^q \in \mathcal{S}_q$. This last requirement gives

$$m_1 r(\theta_{1,2})(u_1^i \cos\theta_1 + u_2^i \sin\theta_1) + (I_{z_1} + I_{z_2})\sin\theta_{1,2} u_3^i = 0, \ i = 1,2 \ .$$

Two linearly independent vectors that fulfill this condition are the ones given in (5.69).

∎

From equation (3.3) we see that the configuration space for the Roller Racer system is $Q = SE(2) \times S^1$. From left invariance of the system's kinematics, the tangent space T_qQ to the configuration space is

$$T_qQ = \left\{(\dot{g}_1, \dot{\theta}_{1,2}) \;\middle|\; g_1 \in SE(2), \theta_{1,2} \in S^1\right\} = \left\{(g_1\xi_1, \dot{\theta}_{1,2}) \;\middle|\; \xi_1 \in se(2), \dot{\theta}_{1,2} \in I\!R\right\} . \tag{5.70}$$

Consider the space $\tilde{\mathcal{S}} = S^1 \times S^1$ and consider the projection $\tilde{\pi} : Q \to \tilde{\mathcal{S}}$. It can be easily seen that $T\tilde{\pi}$ is onto, thus $\tilde{\pi}$ is a submersion. Then, $\tilde{\pi} : Q \to \tilde{\mathcal{S}}$ is a bundle with associated group $G = SE(2)$ and with vertical space

$$\tilde{V}_q = \mathrm{Ker}(T_q\tilde{\pi}) = \left\{v_q \in T_qQ \;\middle|\; \dot{\theta}_1 = 0, \dot{\theta}_{1,2} = 0\right\} .$$

The nonholonomic kinematic constraints determine an *Ehresmann connection* (Bloch, Krishnaprasad, Marsden & Murray [1994]), (Marsden, Montgomery & Ratiu [1990]) on this bundle, with vertical subspace $\tilde{V}_q$ and with horizontal subspace $\tilde{H}_q = \mathcal{D}_q = \mathrm{sp}\{\xi_Q^1, \xi_Q^2\}$. It can be easily checked that $\tilde{V}_q \oplus \tilde{H}_q = T_qQ$. The corresponding matrix form $\tilde{A}_q$ of the connection can be written locally (when $\sin\theta_{1,2} \neq 0$) as:

$$\tilde{A}_q = \begin{pmatrix} 1 & 0 & -\frac{\cos\theta_1}{\sin\theta_{1,2}} r(\theta_{1,2}) & -\frac{\cos\theta_1}{\sin\theta_{1,2}} d_2 \\ 0 & 1 & -\frac{\sin\theta_1}{\sin\theta_{1,2}} r(\theta_{1,2}) & -\frac{\sin\theta_1}{\sin\theta_{1,2}} d_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Notice that $\tilde{H}_q = \mathrm{Ker}\tilde{A}_q$. Also, for $v \in \tilde{V}_q$, we have $\tilde{A}_q v = v$. The horizontal projection (projection of a vector in T_qQ into a vector in $\tilde{H}_q$) corresponds to the matrix $I\!I - \tilde{A}_q$, where $I\!I$ is the 4×4 identity matrix.

Such a formulation suggests expressing the configuration variables and the corresponding velocities as functions of the variables in $\tilde{S}$ (i.e. θ_1 and $\theta_{1,2}$) and of the corresponding velocities. However, since the shape $\theta_{1,2}$ is the only directly controllable configuration variable in our system, the formulation of an Ehresmann connection on the bundle $(Q, \tilde{S}, \tilde{\pi}, G)$ is not particularly helpful in this instance.

Therefore, we consider a different bundle, where the corresponding shape space is the quotient space of Q with the group G, which expresses better the ultimate dependence of all configuration variables on the shape $\theta_{1,2}$. We can show that this is a principal fiber bundle. By considering the Lagrangian dynamics, in addition to the kinematic constraints, we can synthesize a principal fiber bundle connection for this system, which reflects the dependence of all configuration velocities on the shape variation $\dot{\theta}_{1,2}$.

Consider, then, the configuration space $Q = SE(2) \times S^1$ of the Roller Racer, its shape space $\mathcal{S} = S^1$, the group $G = SE(2)$ and the canonical projection

$$\pi : Q \longrightarrow \mathcal{S} : \; (g_{_1}, \theta_{1,2}) \mapsto \theta_{1,2} \; . \tag{5.71}$$

Proposition 5.16 (Principal Fiber Bundle)

The quadruple $(Q, \mathcal{S}, \pi, G)$, together with the left action Φ of G on Q defined by equation (4.1) is a (trivial) *principal fiber bundle.*

Proof

The configuration space Q is trivial, since $Q = \mathcal{S} \times G$. The shape space $\mathcal{S}$ is the quotient space of Q with G. The projection π is differentiable and its differential is

$$T_q\pi : T_qQ \longrightarrow T_{\pi(q)}\mathcal{S} : \; (g_{_1}\xi_{_1}, \dot{\theta}_{1,2}) \mapsto \dot{\theta}_{1,2} \; . \tag{5.72}$$

Thus, $(Q, \mathcal{S}, \pi, G)$ meets the requirements of Definition 2.6.

∎

Proposition 5.17 (Nonholonomic Connection)

The nonholonomic kinematic constraints and the system dynamics determine a *connection* on the principal fiber bundle $(Q, \mathcal{S}, \pi, G)$. The horizontal subspace of the connection is the subspace H_q defined in (5.63) and (5.64), i.e. the orthogonal complement of the subspace $\mathcal{S}_q$ with respect to the constraint distribution $\mathcal{D}_q$, with orthogonality defined with respect to the kinetic energy inner product. When $d_1 \neq d_2$, the horizontal subspace is

$$
\begin{aligned}
H_q &\stackrel{\text{def}}{=} \{v \in T_qQ \mid \ v \in \mathrm{sp}\{\xi_Q^H\}\} \\
&= \{v = (\dot{x}_1,\ \dot{y}_1,\ \dot{\theta}_1,\ \dot{\theta}_{1,2}) \in T_qQ \mid \\
&\qquad \dot{x}_1 = -\frac{\gamma(\theta_{1,2})\sin\theta_{1,2}}{\Delta(\theta_{1,2})}\cos\theta_1\ \dot{\theta}_{1,2},\ \dot{y}_1 = -\frac{\gamma(\theta_{1,2})\sin\theta_{1,2}}{\Delta(\theta_{1,2})}\sin\theta_1\ \dot{\theta}_{1,2}, \\
&\qquad \dot{\theta}_1 = -\frac{\delta(\theta_{1,2})}{\Delta(\theta_{1,2})}\ \dot{\theta}_{1,2}\ \}\ .
\end{aligned}
\tag{5.73}
$$

The vertical subspace of the connection is

$$
V_q \stackrel{\text{def}}{=} \{v \in T_qQ \mid \ T_q\pi = 0\} = \{v = (\dot{x}_1,\ \dot{y}_1,\ \dot{\theta}_1,\ \dot{\theta}_{1,2}) \in T_qQ \mid \ \dot{\theta}_{1,2} = 0\}\ . \tag{5.74}
$$

Proof

It is easy to see that the horizontal subspace H_q defined in (5.73) is

$$
\begin{aligned}
H_q &\stackrel{\text{def}}{=} \{(g_1\xi_1,\ \dot{\theta}_{1,2}) \mid \ g_1 \in SE(2), \\
&\qquad \xi_1 = -A_{loc}(\theta_{1,2})\dot{\theta}_{1,2},\ A_{loc}(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{\delta(\theta_{1,2})}{\Delta(\theta_{1,2})}\mathcal{A}_1 + \frac{\gamma(\theta_{1,2})\sin\theta_{1,2}}{\Delta(\theta_{1,2})}\mathcal{A}_2\ \in se(2)\}\ .
\end{aligned}
\tag{5.75}
$$

To show property (1) of Definition 2.7, consider a non–zero $v \in V_q \cap H_q$. Since v is non–zero and belongs to H_q, we have from (5.75) that $\dot{\theta}_{1,2} \neq 0$. But then, because of (5.74), v cannot belong also to V_q, as we supposed. Thus, $V_q \cap H_q = \{0\}$. Moreover, $\dim V_q + \dim H_q = 3 + 1 = 4 = \dim T_qQ$. Thus, $V_q \oplus H_q = T_qQ$.
To show property (2) of Definition 2.7, consider a $g \in G$. From left–invariance

$$
\begin{aligned}
T_q\Phi_g \cdot H_q = g \cdot H_q &= g \cdot \{(g_1\xi_1, \dot{\theta}_{1,2}) \mid \ \xi_1 = -A_{loc}(\theta_{1,2})\dot{\theta}_{1,2}\} \\
&\stackrel{\text{def}}{=} \{(gg_1\xi_1, \dot{\theta}_{1,2}) \mid \ \xi_1 = -A_{loc}(\theta_{1,2})\dot{\theta}_{1,2}\}
\end{aligned}
$$

and

$$
H_{g\cdot q} = \{v \in T_{g\cdot q}Q \mid \ \xi_1 = -A_{loc}(\theta_{1,2})\dot{\theta}_{1,2}\} = \{(gg_1\xi_1, \dot{\theta}_{1,2}) \mid \ \xi_1 = -A_{loc}(\theta_{1,2})\dot{\theta}_{1,2}\}\ .
$$

Then, obviously, $T_q\Phi_g \cdot H_q = H_{g\cdot q}$.
The differentiability of H_q with respect to $q \in Q$, follows from the smooth dependence of A_{loc} on the shape $\theta_{1,2}$ and from the left–invariance of our system.

∎

Physically, V_q is the set of all possible rigid motions of the system on the plane that keep shape constant; these "frozen–shape" motions do not need to satisfy the nonholonomic constraints. On the other hand, H_q is the set of all possible motions of the system on the plane that comply with the nonholonomic constraints and with the dynamics. Observe that all such motions are due to shape variations.

With the above definition of A_{loc} in equation (5.73), the reconstructed group trajectory equations (5.50), (5.51) and (5.52) take the form:

$$\xi_1 = g_1^{-1}\dot{g}_1 = -A_{loc}(\theta_{1,2})\dot{\theta}_{1,2} + I\!I^{-1}(\theta_{1,2})\ p\ , \tag{5.76}$$

where

$$I\!I^{-1}(\theta_{1,2}) = \frac{\sin\theta_{1,2}}{\Delta(\theta_{1,2})}\mathcal{A}_1 + \frac{r(\theta_{1,2})}{\Delta(\theta_{1,2})}\mathcal{A}_2 \tag{5.77}$$

is the local form of the inverse of the locked inertia tensor of the Roller Racer.

Let the set of Lie algebra elements, whose infinitessimal generators belong to $\mathcal{S}_q$, be denoted as $\mathcal{G}^q$. From (4.10): $\mathcal{G}^q = \mathrm{sp}\{\xi^q\}$. The *locked inertia tensor* $I\!I(q)$ relative to $\mathcal{G}^q$ is defined in (Bloch, Krishnaprasad, Marsden & Murray [1994])as

$$I\!I(q):\ \mathcal{G}^q \longrightarrow (\mathcal{G}^q)^\star:\ \xi^q \longmapsto < I\!I(q)\,\xi^q,\ .\ >\ , \tag{5.78}$$

where, for $\eta^q \in \mathcal{G}^q$, with corresponding infinitessimal generator $\eta^q_Q \in \mathcal{S}_q$, we define

$$< I\!I(q)\,\xi^q,\ \eta^q > \stackrel{\text{def}}{=} \ll \xi^q_Q,\ \eta^q_Q \gg\ , \tag{5.79}$$

and where $\ll\ .,\ .\gg$ is the kinetic energy inner product defined earlier in (5.61).

It is easy to verify from (4.5) and (5.32), that for the Roller Racer

$$< I\!I(q)\,\xi^q,\ \xi^q > = \ll \xi^q_Q,\ \xi^q_Q \gg = \Delta(\theta_{1,2})\ . \tag{5.80}$$

Since $\eta^q_Q = \beta\,\xi^q_Q$, for some $\beta \in I\!R$, we have

$$< I\!I(q)\,\xi^q,\ \eta^q > = \ll \xi^q_Q,\ \eta^q_Q \gg = \beta\ \ll \xi^q_Q,\ \xi^q_Q \gg = \beta\,\Delta(\theta_{1,2})\ . \tag{5.81}$$

5.5 The Reduced Dynamics

Proposition 5.18 (Reduced Dynamics with External Forces)

In the case $d_2 \neq 0$, the reduced dynamics of the Roller Racer take the form:

$$\begin{aligned}\ddot{\theta}_{1,2} = {} & B^4_1(\theta_{1,2})\dot{\theta}_{1,2}p + B^4_2(\theta_{1,2})\dot{\theta}^2_{1,2} \\ & + B^4_3(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + B^4_4(\theta_{1,2})F_{\theta_1} + B^4_5(\theta_{1,2})F_{\theta_{1,2}}\ ,\end{aligned} \tag{5.82}$$

where $\alpha_e = (F_{x_1},\ F_{y_1},\ F_{\theta_1},\ F_{\theta_{1,2}})$ is the external force 1–form of the total forces or torques applied to the system, with

$$\Delta_1(\theta_{1,2}) \stackrel{\text{def}}{=} I_{z_1} I_{z_2} \sin^2\theta_{1,2} + m_1(I_{z_1} d_2^2 + I_{z_2} d_1^2 \cos^2\theta_{1,2}) \ ,$$

and with

$$\begin{aligned}
B_1^4(\theta_{1,2}) &\stackrel{\text{def}}{=} -\frac{A_2^4(\theta_{1,2})}{\Delta_1(\theta_{1,2})} \ , \\
B_2^4(\theta_{1,2}) &\stackrel{\text{def}}{=} \frac{m_1\gamma(\theta_{1,2})\sin\theta_{1,2}}{\Delta(\theta_{1,2})\Delta_1(\theta_{1,2})}\big[\gamma(\theta_{1,2})\cos\theta_{1,2} + d_1\delta(\theta_{1,2})\big] \ , \\
B_3^4(\theta_{1,2}) &\stackrel{\text{def}}{=} -\frac{\gamma(\theta_{1,2})\sin\theta_{1,2}}{\Delta_1(\theta_{1,2})} \ , \\
B_4^4(\theta_{1,2}) &\stackrel{\text{def}}{=} -\frac{\delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})} \ , \\
B_5^4(\theta_{1,2}) &\stackrel{\text{def}}{=} \frac{\Delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})} \ .
\end{aligned} \tag{5.83}$$

Proof

Consider the Lagrange–d'Alembert principle (5.10) with a test vector $u \in \mathcal{D}_q$, which we restrict to the orthogonal complement $H_q \subset \mathcal{D}_q$ of $\mathcal{S}_q$ with respect to $\mathcal{D}_q$. Orthogonality is defined using the kinetic energy inner product $\ll,\gg$ defined in (5.61). Without loss of generality, choose u as

$$u = \xi_Q^H = \gamma(\theta_{1,2})\sin\theta_{1,2}\big(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}\big) + \delta(\theta_{1,2})\frac{\partial}{\partial\theta_1} - \Delta(\theta_{1,2})\frac{\partial}{\partial\theta_{1,2}} \ . \tag{5.84}$$

Assume $d_2 \neq 0$. From (5.10), with u given by (5.84) and $\dot{v}$ given by (5.15), and since $\dot{v}_4 \equiv \ddot{\theta}_{1,2}$, we have:

$$\begin{aligned}
m_1&\big(\dot{\nu}_1 d_2\cos\theta_1 - \nu_1 d_2\sin\theta_1\dot{\theta}_1\big)\gamma(\theta_{1,2})\sin\theta_{1,2}\cos\theta_1 \\
&+ m_1\big(\dot{\nu}_1 d_2\sin\theta_1 + \nu_1 d_2\cos\theta_1\dot{\theta}_1\big)\gamma(\theta_{1,2})\sin\theta_{1,2}\sin\theta_1 \\
&+ \big[(I_{z_1} + I_{z_2})\dot{\nu}_2 d_2 + I_{z_2}\ddot{\theta}_{1,2}\big]\delta(\theta_{1,2}) - I_{z_2}\big(\dot{\nu}_2 d_2 + \ddot{\theta}_{1,2}\big)\Delta \\
&\qquad = \gamma(\theta_{1,2})\sin\theta_{1,2}(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\delta(\theta_{1,2}) - F_{\theta_{1,2}}\Delta(\theta_{1,2}) \ ,
\end{aligned} \tag{5.85}$$

for $\nu_1, \nu_2 \in I\!R$. Then

$$\begin{aligned}
m_1 &d_2\gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\nu}_1 \\
&+ \big[I_{z_1}\delta(\theta_{1,2}) + I_{z_2}(\delta(\theta_{1,2}) - \Delta(\theta_{1,2}))\big] d_2\dot{\nu}_2 + I_{z_2}(\delta(\theta_{1,2}) - \Delta(\theta_{1,2}))\ddot{\theta}_{1,2} \\
&\qquad = \gamma(\theta_{1,2})\sin\theta_{1,2}(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + F_{\theta_1}\delta(\theta_{1,2}) - F_{\theta_{1,2}}\Delta(\theta_{1,2}) \ ,
\end{aligned} \tag{5.86}$$

Observe that $\Delta(\theta_{1,2}) - \delta(\theta_{1,2}) = I_{z_1}\sin^2\theta_{1,2} + m_1 d_1 \cos\theta_{1,2} r(\theta_{1,2})$, then $I_{z_1}\delta(\theta_{1,2}) + I_{z_2}\big(\delta(\theta_{1,2}) - \Delta(\theta_{1,2})\big) = -m_1\gamma(\theta_{1,2})r(\theta_{1,2})$. Then, the LHS of (5.86) becomes

$$m_1 d_2 \gamma(\theta_{1,2})\sin\theta_{1,2}\dot{\nu}_1 - m_1 d_2\gamma(\theta_{1,2})r(\theta_{1,2})\dot{\nu}_2 + I_{z_2}\big(\delta(\theta_{1,2}) - \Delta(\theta_{1,2})\big)\ddot{\theta}_{1,2}$$

or

$$m_1 d_2 \gamma(\theta_{1,2})\big(\sin\theta_{1,2}\dot{\nu}_1 - r(\theta_{1,2})\dot{\nu}_2\big) + I_{z_2}\big(\delta(\theta_{1,2}) - \Delta(\theta_{1,2})\big)\ddot{\theta}_{1,2}\,. \tag{5.87}$$

By differentiating (5.44), the terms $\dot{\nu}_1$ and $\dot{\nu}_2$ above can be expressed as functions of $\theta_{1,2},\ \dot{\theta}_{1,2},\ \ddot{\theta}_{1,2},\ p,\ \dot{p}$:

$$\begin{aligned}
\dot{\nu}_1 = \frac{1}{d_2\Delta^2}\Bigg[& r(\theta_{1,2})\Delta(\theta_{1,2})\dot{p} + \big(\frac{\partial r}{\partial\theta_{1,2}}\Delta(\theta_{1,2}) - r(\theta_{1,2})\frac{\partial\Delta}{\partial\theta_{1,2}}\big)\dot{\theta}_{1,2}\, p \\
& - \gamma(\theta_{1,2})\Delta(\theta_{1,2})\sin\theta_{1,2}\ddot{\theta}_{1,2} \\
& + \left(\gamma(\theta_{1,2})\sin\theta_{1,2}\frac{\partial\Delta}{\partial\theta_{1,2}} - \frac{\partial\gamma}{\partial\theta_{1,2}}\Delta(\theta_{1,2})\sin\theta_{1,2} - \gamma(\theta_{1,2})\Delta(\theta_{1,2})\cos\theta_{1,2}\right)\dot{\theta}_{1,2}^2\Bigg], \\
\dot{\nu}_2 = \frac{1}{d_2\Delta^2}\Bigg[& \Delta(\theta_{1,2})\sin\theta_{1,2}\dot{p} + \big(\Delta(\theta_{1,2})\cos\theta_{1,2} - \frac{\partial\Delta}{\partial\theta_{1,2}}\sin\theta_{1,2}\big)\dot{\theta}_{1,2}\, p \\
& - \Delta(\theta_{1,2})\delta(\theta_{1,2})\ddot{\theta}_{1,2} + \big(\frac{\partial\Delta}{\partial\theta_{1,2}}\delta(\theta_{1,2}) - \frac{\partial\delta}{\partial\theta_{1,2}}\Delta(\theta_{1,2})\big)\dot{\theta}_{1,2}^2\Bigg]\,.
\end{aligned} \tag{5.88}$$

Thus, the LHS of (5.86) becomes, after some calculations using (5.87) and (5.88):

$$\begin{aligned}
& \big[m_1 d_2\gamma(\theta_{1,2}) + I_{z_2}(\delta(\theta_{1,2}) - \Delta(\theta_{1,2}))\big]\ddot{\theta}_{1,2} \\
& + \frac{m_1\gamma(\theta_{1,2})}{\Delta(\theta_{1,2})}\left(\frac{\partial r}{\partial\theta_{1,2}}\sin\theta_{1,2} - r(\theta_{1,2})\cos\theta_{1,2}\right)\dot{\theta}_{1,2}\, p \\
& - \frac{m_1\gamma(\theta_{1,2})}{\Delta(\theta_{1,2})}\left(d_2\frac{\partial\Delta}{\partial\theta_{1,2}} + \frac{\partial\gamma}{\partial\theta_{1,2}}\sin^2\theta_{1,2} + \gamma(\theta_{1,2})\sin\theta_{1,2}\cos\theta_{1,2} - r(\theta_{1,2})\frac{\partial\delta}{\partial\theta_{1,2}}\right)\dot{\theta}_{1,2}^2\,.
\end{aligned} \tag{5.89}$$

The parenthesis in the second term above can be shown to be equal to $-\lambda(\theta_{1,2})$, while the parenthesis of the third term can be shown to be equal to $-\big(\gamma(\theta_{1,2})\cos\theta_{1,2} + d_1\delta(\theta_{1,2})\big)\sin\theta_{1,2}$. Thus, equation (5.82) follows.

∎

Corollary 5.19 (Reduced Dynamics without External Forces)

In the case $d_2 \neq 0$, the reduced dynamics of the Roller Racer, in the absence of external forces or torques, take the following form:

$$\ddot{\theta}_{1,2} = B_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + B_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 + B_5^4(\theta_{1,2})\tau_{1,2}\,, \tag{5.90}$$

where $\tau_{1,2}$ is the torque applied to the joint $O_{1,2}$.

Proof

Immediate from the previous Proposition, when $\alpha_e = (F_{x_1},\ F_{y_1},\ F_{\theta_1},\ F_{\theta_{1,2}}) = (0,\ 0,\ 0,\ \tau_{1,2})$.

∎

Corollary 5.20 (Reduced Dynamics with Friction)

In the presence of friction and for $d_2 \neq 0$, the reduced dynamics take the form:

$$\ddot{\theta}_{1,2} = \left[B_1^4(\theta_{1,2})\dot{\theta}_{1,2} + B_6^4(\theta_{1,2})\right] p + \left[B_2^4(\theta_{1,2})\dot{\theta}_{1,2} - B_7^4(\theta_{1,2})\right] \dot{\theta}_{1,2} + B_5^4(\theta_{1,2})\ \tau_{1,2}\ , \tag{5.91}$$

where B_1^4, B_2^4 and B_5^4 were defined previously in (5.83) and where

$$\begin{aligned} B_6^4(\theta_{1,2}) &\stackrel{\text{def}}{=} \frac{A_2^5(\theta_{1,2})}{\Delta_1(\theta_{1,2})}\ , \\ B_7^4(\theta_{1,2}) &\stackrel{\text{def}}{=} \frac{A_3^5(\theta_{1,2})}{\Delta_1(\theta_{1,2})}\ , \end{aligned} \tag{5.92}$$

with A_2^5 as defined in (5.46), with Δ_1 as defined in (5.83) and with

$$\begin{aligned} A_3^5(\theta_{1,2}) &\stackrel{\text{def}}{=} \frac{1}{\Delta(\theta_{1,2})}\left[\eta_3(\theta_{1,2})\gamma(\theta_{1,2})\sin\theta_{1,2} + \eta_4(\theta_{1,2})\delta(\theta_{1,2}) + \eta_5(\theta_{1,2})\Delta(\theta_{1,2})\right] \\ &= \frac{2}{\Delta(\theta_{1,2})}\left[\frac{k_1}{R_1^2}\left(\frac{L_1^2}{4}\delta^2(\theta_{1,2}) + \gamma^2(\theta_{1,2})\sin^2\theta_{1,2}\right)\right. \\ &\qquad + \frac{k_2}{R_2^2}\left(\frac{L_2^2}{4}\delta^2(\theta_{1,2}) + \frac{L_2^2}{4}\Delta^2(\theta_{1,2}) + \right. \\ &\qquad\qquad \left.\left. + \big(\gamma(\theta_{1,2})\cos\theta_{1,2} + d_1\delta(\theta_{1,2})\big)^2 \sin^2\theta_{1,2}\right)\right]\ , \end{aligned} \tag{5.93}$$

where

$$\begin{aligned} \eta_3(\theta_{1,2}) &\stackrel{\text{def}}{=} 2\left[\big(\frac{k_1}{R_1^2} + \frac{k_2}{R_2^2}\cos^2\theta_{1,2}\big)\gamma(\theta_{1,2}) + \frac{k_2}{R_2^2}d_1\delta(\theta_{1,2})\cos\theta_{1,2}\right]\sin\theta_{1,2}, \\ \eta_4(\theta_{1,2}) &\stackrel{\text{def}}{=} 2\frac{k_2}{R_2^2}d_1\gamma(\theta_{1,2})\sin^2\theta_{1,2}\cos\theta_{1,2} + 2\left(\frac{k_1}{R_1^2}\frac{L_1^2}{4} + \frac{k_2}{R_2^2}\frac{L_2^2}{4} + \frac{k_2}{R_2^2}d_1^2\sin^2\theta_{1,2}\right)\delta(\theta_{1,2}), \\ \eta_5(\theta_{1,2}) &\stackrel{\text{def}}{=} 2\frac{k_2}{R_2^2}\frac{L_2^2}{4}\Delta(\theta_{1,2}). \end{aligned} \tag{5.94}$$

If $d_1 \neq d_2$, then $A_3^5(\theta_{1,2}) > 0$, for all $\theta_{1,2}$.

Proof

From the reduced dynamics of the Roller Racer with external forces given by equation (5.82) and from the external force 1–form due to friction, given by equation (5.46), we have for the last 3 terms of (5.82), using the definitions of (5.83):

$$
\begin{aligned}
B_3^4(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + B_4^4(\theta_{1,2})F_{\theta_1} + B_5^4(\theta_{1,2})F_{\theta_{1,2}} = \\
= \frac{1}{\Delta_1(\theta_{1,2})}\eta_3(\theta_{1,2})(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) + \frac{1}{\Delta_1(\theta_{1,2})}\eta_4(\theta_{1,2})\dot{\theta}_1 \\
- \frac{1}{\Delta_1(\theta_{1,2})}\eta_5(\theta_{1,2})\dot{\theta}_{1,2} + \frac{\Delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})}\tau_{1,2}\ ,
\end{aligned}
\tag{5.95}
$$

where η_3, η_4 and η_5 are defined in (5.93).
In the case $d_2 \neq 0$, we have for $v = (\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2}) \in \mathcal{D}_q$, from (5.13):

$$
\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1 = \nu_1 d_2 \quad \text{and} \quad \dot{\theta}_1 = \nu_2 d_2\ ,
$$

for $\nu_1, \nu_2 \in I\!R$. From this and from (5.44) we get:

$$
\begin{aligned}
B_3^4(\theta_{1,2})(F_{x_1}\cos\theta_1 + F_{y_1}\sin\theta_1) + B_4^4(\theta_{1,2})F_{\theta_1} + B_5^4(\theta_{1,2})F_{\theta_{1,2}} = \\
= \frac{d_2}{\Delta_1(\theta_{1,2})}\big[\eta_3(\theta_{1,2})\nu_1 + \eta_4(\theta_{1,2})\nu_2\big] - \frac{1}{\Delta_1(\theta_{1,2})}\eta_5(\theta_{1,2})\dot{\theta}_{1,2} + \frac{\Delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})}\tau_{1,2} \\
= \frac{1}{\Delta\Delta_1}\big[\eta_3(\theta_{1,2})r(\theta_{1,2}) + \eta_4(\theta_{1,2})\sin\theta_{1,2}\big]p \\
+ \frac{1}{\Delta\Delta_1}\big[\eta_3(\theta_{1,2})\gamma(\theta_{1,2})\sin\theta_{1,2} + \eta_4(\theta_{1,2})\delta(\theta_{1,2}) + \eta_5(\theta_{1,2})\Delta(\theta_{1,2})\big]\dot{\theta}_{1,2} \\
+ \frac{\Delta(\theta_{1,2})}{\Delta_1(\theta_{1,2})}\tau_{1,2} \\
= B_6^4(\theta_{1,2})\ p + B_7^4(\theta_{1,2})\ \dot{\theta}_{1,2} + B_5^4(\theta_{1,2})\ \tau_{1,2}\ .
\end{aligned}
\tag{5.96}
$$

It is an easy calculation to show that

$$
B_6^4(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{1}{\Delta_1(\theta_{1,2})}\frac{1}{\Delta(\theta_{1,2})}\big[\eta_3(\theta_{1,2})r(\theta_{1,2}) + \eta_4(\theta_{1,2})\sin\theta_{1,2}\big] = \frac{1}{\Delta_1(\theta_{1,2})}A_2^5(\theta_{1,2})\ ,
$$

with $A_2^5(\theta_{1,2})$ as defined in (5.46).

∎

6 Motion Control of the Roller Racer

6.1 Preliminaries

Consider the smooth affine nonlinear control system:

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x)u_j \ , \tag{6.1}$$

where $x = (x_1, \dots, x_n)$ are local coordinates for the smooth manifold M with dim $M = n$ and $u = (u_1, \dots, u_m) \in U \subset I\!R^m$, the set of admissible controls. The unique solution of (6.1) at time $t \geq 0$ with initial condition $x(t_0) = x_0$ and input function $u(.)$ is denoted $x(t, t_0, x_0, u)$ or simply $x(t)$.

The discussion in this section follows (Nijmeijer & vanderSchaft [1990]), unless otherwise noted. We are interested in systems like the unicycle, where the number of controls is less than the dimension of the system and whose tangent linearization is uncontrollable. Therefore, tools from nonlinear control theory are necessary to analyze these systems. We define below various notions of accessibility, controllability and feedback linearization, along with some related results, that will be useful in our subsequent analysis.

Definition 6.1 (Controllability)

The system (6.1) is *controllable*, if, for every $x_1, x_2 \in M$, there exists a finite time $T > 0$ and an admissible control $u : [0, T] \to U$ such that $x(T, 0, x_1, u) = x_2$.

∎

Definition 6.2 (Reachable Set)

The reachable set $R^V(x_0, T)$ is the set of points in M which are reachable from $x_0 \in M$ at exactly time $T > 0$, following system trajectories which, for $t \leq T$, remain in the neighborhood V of x_0.

Consider also $R^V_T(x_0) \overset{\text{def}}{=} \bigcup_{t \leq T} R^V(x_0, t)$, the set of points in M reachable from x_0 at time less or equal to T.

∎

Definition 6.3 (Local Accessibility)

The system (6.1) is *locally accessible* from $x_0 \in M$, if, for any neighborhood V of x_0 and all $T > 0$, the set $R^V_T(x_0)$ contains a non–empty open set. If the system is locally accessible from any $x_0 \in M$, then it is *locally accessible*.

∎

Proposition 6.4 (Accessibility Algebra and Distribution)

The *accessibility algebra* $\mathcal{C}$ is the smallest subalgebra of the Lie algebra $V^{\infty}(M)$ of vector fields on M, which contains the vector fields $f, g_1, \ldots, g_m$. The *accessibility distribution* C is the involutive distribution

$$C(x) = \mathrm{sp}\{X(x) \,|\, X \in \mathcal{C}\} \; .$$

Every element of $\mathcal{C}$ is a linear combination of repeated Lie brackets of the form

$$[X_k, [X_{k-1}, [\ldots, [X_1, X_0] \ldots]]] \; ,$$

where $X_i,\; i \in \{0, \ldots k\},\; k = 0, 1, \ldots$ belongs to the set $\{f, g_1, \ldots, g_m\}$.

∎

Proposition 6.5

If the Accessibility Rank Condition at x_0 is satisfied, i.e. if

$$\dim C(x_0) = n \; , \tag{6.2}$$

then the system (6.1) is locally accessible from x_0. If the Accessibility Rank Condition is satisfied at every $x \in M$, then the system is locally accessible. If the system (6.1) is locally accessible, then $\dim C(x) = n$, for x in an open and dense subset of M.

∎

In (Isidori [1989]), the subalgebra $\mathcal{C}$ is called the Control Lie Algebra and the condition (6.2) is called the Controllability Rank Condition. Another common term for condition (6.2) is the Lie Algebra Rank Condition (LARC) (Sussmann [1987]).

Definition 6.6 (Local Strong Accessibility)

The system (6.1) is locally strongly accessible from $x_0 \in M$, if, for any neighborhood V of x_0 and for any $T > 0$ sufficiently small, the set $R^V(x_0, T)$ contains a non–empty open set.

∎

Proposition 6.7 (Strong Accessibility Algebra and Distribution)

The *strong accessibility algebra* $\mathcal{C}_0$ is the smallest subalgebra of the Lie algebra $V^{\infty}(M)$ containing the control vector fields $g_1, \ldots, g_m$, which is invariant under the drift vector field f, i.e. $[f, X] \in \mathcal{C}_0,\; \forall X \in \mathcal{C}_0$. The *strong accessibility distribution* C_0 is the corresponding involutive distribution

$$C_0(x) = \mathrm{sp}\{X(x) \,|\, X \in \mathcal{C}_0\} \; .$$

Every element of the algebra $\mathcal{C}_0$ is a linear combination of repeated Lie brackets of the form

$$[X_k, [X_{k-1}, [\ldots, [X_1, g_j] \ldots]]] \ ,$$

for $j \in \{1, \ldots, m\}$ and where $X_i, \ i \in \{1, \ldots k\}, \ k = 0, 1, \ldots$ belongs to $\{f, g_1, \ldots, g_m\}$. Observe that the drift vector field f is not contained explicitly in these expressions.

■

Proposition 6.8

If the Strong Accessibility Rank Condition at $x_0 \in M$ is satisfied, i.e. if

$$\dim C_0(x_0) = n \ , \tag{6.3}$$

then the system (6.1) is locally strongly accessible from x_0. If the Strong Accessibility Rank Condition is satisfied at every $x \in M$, then the system is locally strongly accessible. If the system (6.1) is locally strongly accessible, then $\dim C_0(x) = n$, for x in an open and dense subset of M.

■

For systems without drift (i.e. where $f = 0$ in (6.1)), accessibility is equivalent to controllability. However, this is no longer true for systems with drift and various notions of controllability have been developped. Below we consider only the notion of small–time local controllability for which relatively simple verification tests have been established, as well as links to the closed–loop control of nonholonomic systems.

Definition 6.9 (Small–Time Local Controllability STLC (Sussmann [1987]))

The system (6.1) is *small–time locally controllable* from $x_0 \in M$, if, for any neighborhood V of x_0 and any $T > 0$, x_0 is an interior point of the set $R_T^V(x_0)$, i.e. a whole neighborhood of x_0 is reachable from x_0 at arbitrarily small time.

■

Sussmann (Sussmann [1987]) showed that a sufficient condition for STLC requires that the system (6.1) be accessible and that "bad" brackets, i.e. brackets where the degree of the drift vector field (the number of times the drift vector field appears in the bracket) is odd, while the sum of the degrees of the control vector fields is even, be identically zero or be "neutralized" by brackets of lesser degree (i.e. be a linear combination of such brackets).

In (Sussmann [1983]) a condition for *lack* of STLC is given for single–input systems (Proposition 6.3 of (Sussmann [1983])). An equivalent form of this condition, derived in the proof of this Proposition is presented below (see also the discussion on single–input systems following the proof of Theorem 7.3 in (Sussmann [1987])).

Proposition 6.10 (Lack of STLC (Sussmann [1983]))

Consider an analytic affine nonlinear system with a single input, of the form

$$\dot{x} = f(x) + u\ g(x)\ , \tag{6.4}$$

with $|u| \leq 1$, $f(x_0) = 0$ and $g(x_0) \neq 0$, for some $x_0 \in M$. Assume that the bracket $\big[g,[g,f]\big](x_0)$ does not belong to the linear span of the vector fields $\{ad_f^j\ g\ (x_0),\ \ j = 0,1,\ldots\}$. Then, the system is *not* STLC from x_0.

∎

Certain nonlinear systems can be transformed, at least locally, into a linear controllable system, via a state coordinate transformation and a static state feedback. This process is called *static feedback linearization.* Other nonlinear systems can be transformed into a linear controllable system via a dynamic state feedback and a coordinate transformation involving the extended state of the system. This process is called *dynamic feedback linearization.*

Definition 6.11 (Static Feedback Linearization)

Consider the nonlinear control system (6.1):

$$\dot{x} = f(x) + G(x)\ u\ , \tag{6.5}$$

where $G(x)$ is the matrix whose j–th column is $g_j(x)$. Let x_0 be an equilibrium of f, i.e. $f(x_0) = 0$. This systems is *static feedback linearizable* around x_0, if there exists a state transformation $z = \phi(x)$ and regular static state feedback $u = \alpha(x) + \beta(x)\ v$, with ϕ, α and β defined in a neighborhood of x_0, with α and β smooth mappings, and with $\phi(x_0) = 0$, $\alpha(x_0) = 0$ and β an $m \times m$ nonsingular matrix, so that the feedback transformed system

$$\dot{x} = \tilde{f}(x) + \tilde{G}(x)\ v \stackrel{\text{def}}{=} \big[f(x) + G(x)\ \alpha(x)\big] + \big[G(x)\ \beta(x)\big]\ v\ , \tag{6.6}$$

transforms under $z = \phi(x)$ into the linear controllable system

$$\dot{z} = A\ z + B\ v\ , \tag{6.7}$$

where $A\ z \stackrel{\text{def}}{=} (\phi_\star \tilde{f})(z)$ and $B \stackrel{\text{def}}{=} (\phi_\star \tilde{g}_j)(z)$.

∎

Proposition 6.12 ((Nijmeijer & vanderSchaft [1990]), Theorem 6.3)

Consider system (6.5) with $f(x_0) = 0$. Assume that the strong accessibility rank condition holds at x_0. This system is static feedback linearizable if and only if the distributions $D_1, \ldots, D_n$ defined by

$$D_k(x) = sp\{ad_f^r g_1(x), \ldots, ad_f^r g_m(x) \mid r = 0, 1, \ldots, k-1\},\ k = 1, 2, \ldots \tag{6.8}$$

are all involutive and constant dimensional in a neighborhood of x_0.

∎

Corollary 6.13 ((Nijmeijer & vanderSchaft [1990]), Corollary 6.17)

Consider a single–input system (6.1) with $f(x_0) = 0$. This system is static feedback linearizable around x_0 if and only if $\dim D_n(x_0) = n$ and D_{n-1} is involutive around x_0.

∎

Definition 6.14 (Dynamic Feedback Linearization) (Pomet [1995])

Consider a control system $\dot{x} = f(x, u)$. If there exists a *dynamic feedback* with state z, input (x, v) and output u of the form $\dot{z} = g(z, x, v),\ u = \gamma(z, x, v)$, which, when applied to the original system, gives an extended one with state (x, z) and input v, of the form $\dot{x} = f(x, \gamma(x, z, v)),\ \dot{z} = g(x, z, v)$ and which, by a change of coordinates $X = \phi(x, z)$ of the extended state of the system, transforms into a linear controllable system, then we call the original system *dynamic feedback linearizable.*

∎

Proposition 6.15 (Charlet, Levine & Marino [1989]; Pomet [1995])

A single–input nonlinear system is dynamic feedback linearizable if and only if it is static feedback linearizable.

∎

Remark 6.16 (Differential Flatness)

As (Pomet [1995]) remarks, dynamic feedback linearization, as defined above, is equivalent to the concept of *differential flatness* introduced by (Fliess et al. [1995]). From the previous proposition then, differential flatness is equivalent, for single–input systems, to static feedback linearization.

6.2 The Base–Momentum Subsystem of the Dynamics

We refer to the momentum equation and the reduced dynamics as the *Base–Momentum (B–M) subsystem* of the dynamics of the Roller Racer. In this section we only consider the momentum equation without external forces or friction.

The reduced dynamics can often be transformed by nonlinear static state feedback into the form

$$\ddot{\theta}_{1,2} = u \ . \tag{6.9}$$

For control purposes, we assume that $\theta_{1,2}$ and $\dot{\theta}_{1,2}$ are available from proprioceptive sensors. When it is possible to estimate p, the reduced dynamics of the system without external forces (5.90) can be transformed into the form of equation (6.9) by the following nonlinear static state feedback

$$\tau_{1,2} = \frac{1}{B_5^4(\theta_{1,2})}\left[u - B_1^4(\theta_{1,2})\dot{\theta}_{1,2}p - B_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2\right] \ . \tag{6.10}$$

Assume then that, after feedback linearization of the reduced dynamics, the *B–M* subsystem of the dynamics takes the form:

$$\begin{aligned} \frac{dp}{dt} &= A_1^4(\theta_{1,2})\dot{\theta}_{1,2}\, p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \ , \\ \frac{d\theta_{1,2}}{dt} &= \dot{\theta}_{1,2} \ , \\ \frac{d\dot{\theta}_{1,2}}{dt} &= u \ . \end{aligned} \tag{6.11}$$

Defining the state vector $z \stackrel{\text{def}}{=} (p,\ \theta_{1,2},\ \dot{\theta}_{1,2})^\top \in M$, where $M \stackrel{\text{def}}{=} I\!R^2 \times S^1$, the *B–M* subsystem (6.11) takes the form of an affine nonlinear system with a single control $u \in I\!R$:

$$\dot{z} = f(z) + g(z)u \ , \tag{6.12}$$

and with

$$f(z) \stackrel{\text{def}}{=} \begin{pmatrix} A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \\ \dot{\theta}_{1,2} \\ 0 \end{pmatrix} \quad \text{and} \quad g(z) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ . \tag{6.13}$$

The equilibria of the *B–M* subsystem are states $z_e \in M$ where $f(z_e) = 0$. It can be easily seen that these are of the form $z_e = (p_e,\ \theta_{1,2_e},\ 0)^\top \in M$, with $p_e \in I\!R$ and $\theta_{1,2_e} \in S^1$. In particular, the origin $z_0 = (0,\ 0,\ 0)^\top \in M$ is an equilibrium.

The tangent linearization of the B–M subsystem (6.12) is *not* controllable at equilibria, since the matrix

$$\left[g \;\middle|\; \frac{\partial f}{\partial z} g \;\middle|\; \left(\frac{\partial f}{\partial z}\right)^2 g \right]\Bigg|_{z_e} = \begin{pmatrix} 0 & A_1^4(\theta_{1,2_e}) & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is singular.

Define

$$A_3^4(\theta_{1,2}) \stackrel{\text{def}}{=} A_1^4(\theta_{1,2}) A_2^4(\theta_{1,2}) - \frac{\partial A_2^4(\theta_{1,2})}{\partial \theta_{1,2}} \tag{6.14}$$

and, iteratively,

$$A_{i+1}^4(\theta_{1,2}) \stackrel{\text{def}}{=} -A_1^4(\theta_{1,2}) A_i^4(\theta_{1,2}) + \frac{\partial A_i^4(\theta_{1,2})}{\partial \theta_{1,2}} \;, \quad \text{for } i = 3, 4, \ldots \tag{6.15}$$

When $d_1 > d_2$, the roots $\theta_{1,2}^\star$ of $A_2^4(\theta_{1,2})$ correspond to the solutions of $\gamma(\theta_{1,2}) = 0$, i.e. to $\theta_{1,2}^\star = \cos^{-1} \frac{I_{z_1} d_2}{I_{z_2} d_1}$. Notice that at roots of $A_2^4(\theta_{1,2})$ such that $I_{z_1} d_2 \neq I_{z_2} d_1$, we have $\theta_{1,2}^\star \neq 0, \pi$ and $\frac{\partial A_2^4(\theta_{1,2}^\star)}{\partial \theta_{1,2}} = -\frac{m_1}{\Delta^2} I_{z_2} d_1 \lambda \sin \theta_{1,2}^\star \neq 0$. Thus, when $A_2^4(\theta_{1,2}) = 0$ and $I_{z_1} d_2 \neq I_{z_2} d_1$, we have from (6.14) that $A_3^4(\theta_{1,2}) \neq 0$.

Proposition 6.17 (Strong Accessibility of the B–M subsystem)

Assume $d_1 > d_2$ and $I_{z_1} d_2 \neq I_{z_2} d_1$. The B–M subsystem (6.12) is locally strongly accessible from equilibria $z_e = (p_e,\ \theta_{1,2_e},\ 0)^\top$.

Proof

Consider the following brackets, which are of the type required in Proposition 6.7:

$$[f, g]\Big|_{z_e} = -\begin{pmatrix} A_1^4(\theta_{1,2}) p + 2 A_2^4(\theta_{1,2}) \dot{\theta}_{1,2} \\ 1 \\ 0 \end{pmatrix}\Bigg|_{z_e} = -\begin{pmatrix} A_1^4(\theta_{1,2_e}) p_e \\ 1 \\ 0 \end{pmatrix} \tag{6.16}$$

and

$$[[f, g], g]\Big|_{z_e} = \begin{pmatrix} 2 A_2^4(\theta_{1,2}) \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} 2 A_2^4(\theta_{1,2_e}) \\ 0 \\ 0 \end{pmatrix} . \tag{6.17}$$

If $\theta_{1,2_e}$ is such that $A_2^4(\theta_{1,2_e}) \neq 0$, observe that

$$\mathrm{sp}\Big\{ g,\ [f, g],\ [[f, g], g] \Big\}(z_e) = I\!R^3 = T_{z_e} M \;,$$

i.e. the system satisfies the strong accessibility rank condition at z_e.

If $\theta_{1,2_e}$ is such that $A_2^4(\theta_{1,2_e}) = 0$ and $I_{z_1} d_2 \neq I_{z_2} d_1$, we saw above that $A_3^4(\theta_{1,2_e}) \neq 0$. Consider the bracket

$$\begin{aligned} ad^2_{[f,g]} g\Big|_{z_e} &= \Big[[f,g],\ [[f,g],g]\Big]\Big|_{z_e} = \Big[g, \Big[f, [g,[f,g]]\Big]\Big]\Big|_{z_e} \\ &= \begin{pmatrix} 2A_3^4(\theta_{1,2}) \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} 2A_3^4(\theta_{1,2_e}) \\ 0 \\ 0 \end{pmatrix} . \end{aligned} \tag{6.18}$$

Then

$$\mathrm{sp}\Big\{g,\ [f,g],\ \Big[[f,g],\ [[f,g],g]\Big]\ \Big\}(z_e) = \mathbb{R}^3 = T_{z_e} M\ ,$$

i.e. the system satisfies again the strong accessibility rank condition at z_e.

∎

However, since (6.12) is a system with drift, its accessibility does not imply its controllability. In particular, it is possible to show the following:

Theorem 6.18 (Small–Time Local Controllability of the B–M subsystem)

The B–M subsystem is *not* STLC from equilibria z_e where $A_2^4(\theta_{1,2_e}) \neq 0$.

Proof

From (6.17), we have

$$[g,[g,f]]\Big|_{z_e} = \begin{pmatrix} 2A_2^4(\theta_{1,2_e}) \\ 0 \\ 0 \end{pmatrix} . \tag{6.19}$$

Consider the vector fields $\mathrm{ad}_f^j g,\ j = 0, 1, \ldots$ evaluated at z_e. The vector fields $\mathrm{ad}_f^0\ g(z_e) = g(z_e)$ and $\mathrm{ad}_f^1\ g(z_e) = [f,g]\ (z_e)$ are given by (6.13) and (6.16) respectively. The vector field $\mathrm{ad}_f^2\ g(z_e)$ can be expressed, using (6.14), as:

$$\begin{aligned} \mathrm{ad}_f^2\ g\ (z_e) &= [f,[f,g]]\Big|_{z_e} \\ &= \begin{pmatrix} [A_1^4(\theta_{1,2})A_2^4(\theta_{1,2}) - \frac{\partial A_2^4(\theta_{1,2})}{\partial \theta_{1,2}}]\dot{\theta}_{1,2}^2 \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} A_3^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \end{aligned}$$

Similarly, it can be established using (6.15) that

$$\mathrm{ad}_f^j\ g\ (z_e) = \begin{pmatrix} [-A_1^4(\theta_{1,2})A_j^4(\theta_{1,2}) + \frac{\partial A_j^4(\theta_{1,2})}{\partial \theta_{1,2}}]\dot{\theta}_{1,2}^j \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} A_{j+1}^4(\theta_{1,2})\dot{\theta}_{1,2}^j \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{6.20}$$

for $j \geq 3$. Thus,

$$\begin{aligned} \mathrm{sp}\{\mathrm{ad}_f^j\ g\ (z_e),\ j=0,1,\dots\} &= \mathrm{sp}\{g\ (z_e),\ [f,g]\ (z_e)\} \\ &= \mathrm{sp}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} A_1^4(\theta_{1,2_e})p_e \\ 1 \\ 0 \end{pmatrix} \right\}. \end{aligned} \tag{6.21}$$

Obviously, from (6.19) and (6.21), the bracket $\big[g,[g,f]\big]\ (z_e)$ does not belong to $\mathrm{sp}\{\mathrm{ad}_f^j\ g(z_e),\ j=0,1,\dots\}$ when $A_2^4(\theta_{1,2_e}) \neq 0$. Thus, from Proposition 6.10, the *B–M* subsystem is not STLC from such equilibria.

■

Proposition 6.19 (Feedback Linearizability of the *B–M* subsystem)

The *B–M* subsystem is *not* static feedback linearizable around equilibria $z_e = (p_e, \theta_{1,2_e}, 0)^\top$.

Proof

The dimension of the state space is $n=3$. The family of distributions defined in equation (6.8) is:

$$\begin{aligned} D_1(z) &= \mathrm{sp}\{g\ (z)\} \\ D_{n-1}(z) = D_2(z) &= \mathrm{sp}\{g\ (z),\ [f,g]\ (z)\} \\ D_n(z) = D_3(z) &= \mathrm{sp}\{g\ (z),\ [f,g]\ (z), [f,[f,g]]\ (z)\} \end{aligned} \tag{6.22}$$

At the equilibrium z_e the distribution D_n is

$$D_n(z_e) = D_3(z_e) = \mathrm{sp}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} A_1^4(\theta_{1,2_e})p_e \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Obviously, $\dim D_n(z_e) = 2 < n = 3$. The result follows from Proposition 6.12.

■

Remark 6.20 (Differential Flatness)

In view of Remark 6.16, the *B–M* subsystem is neither dynamic feedback linearizable, nor differentially flat.

6.3 The Full System

The full dynamics of the Roller Racer are given by equations (5.36) and $\dot{g}_1 = g_1 \xi_1$, where ξ_1 is given by (5.50). In order to put these in the form of an affine nonlinear control system, consider the shape acceleration $\ddot{\theta}_{1,2}$ as the control u of the system. Moreover, consider local coordinates (x_1, y_1, θ_1) for $g_1 \in SE(2)$. Letting $z \stackrel{\text{def}}{=} (\theta_1,\ x_1,\ y_1,\ p,\ \theta_{1,2},\ \dot{\theta}_{1,2})^\top \in M = I\!R^6$, and under the assumption of feedback linearizability of the reduced dynamics, the dynamics take the form:

$$\dot{z} = f(z) + g(z)u\ , \tag{6.23}$$

with $u \in I\!R$ and

$$f(z) \stackrel{\text{def}}{=} \begin{pmatrix} \frac{1}{\Delta(\theta_{1,2})}(\sin\theta_{1,2}\ p - \delta(\theta_{1,2})\ \dot{\theta}_{1,2}) \\ \frac{\cos\theta_1}{\Delta(\theta_{1,2})}(r(\theta_{1,2})\ p - \gamma(\theta_{1,2})\sin\theta_{1,2}\ \dot{\theta}_{1,2}) \\ \frac{\sin\theta_1}{\Delta(\theta_{1,2})}(r(\theta_{1,2})\ p - \gamma(\theta_{1,2})\sin\theta_{1,2}\ \dot{\theta}_{1,2}) \\ A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \\ \dot{\theta}_{1,2} \\ 0 \end{pmatrix} \quad \text{and} \quad g(z) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The *equilibria* of the system are states $z_e \in M$ of the form

$$z_e = (\theta_{1_e},\ x_{1_e},\ y_{1_e},\ 0,\ \theta_{1,2_e},\ 0)^\top \in M,$$

i.e. states where, not only the shape $\theta_{1,2}$ is constant, but also the nonholonomic momentum p is zero.

Theorem 6.21 (Accessibility of the Roller Racer)

The dynamics of the Roller Racer (equation (6.23)) with $d_1 > d_2$ are locally accessible at equilibria z_e.

Proof

Consider the following brackets:

$$[f,g]\Big|_{z_e} = \begin{pmatrix} \frac{\delta}{\Delta} \\ \frac{\cos\theta_1}{\Delta}\gamma\ \sin\theta_{1,2} \\ \frac{\sin\theta_1}{\Delta}\gamma\ \sin\theta_{1,2} \\ -A_1^4(\theta_{1,2})p - 2A_2^4(\theta_{1,2})\dot{\theta}_{1,2} \\ -1 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} \frac{\delta(\theta_{1,2_e})}{\Delta(\theta_{1,2_e})} \\ \frac{\cos\theta_{1_e}}{\Delta(\theta_{1,2_e})}\gamma(\theta_{1,2_e})\sin\theta_{1,2_e} \\ \frac{\sin\theta_{1_e}}{\Delta(\theta_{1,2_e})}\gamma(\theta_{1,2_e})\sin\theta_{1,2_e} \\ 0 \\ -1 \\ 0 \end{pmatrix}, \tag{6.24}$$

$$[[f,g],g]\Big|_{z_e} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2A_2^4(\theta_{1,2}) \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2A_2^4(\theta_{1,2_e}) \\ 0 \\ 0 \end{pmatrix},$$

$$\left[\left[[f,g],g\right],f\right]\Big|_{z_e} = \begin{pmatrix} h_1(\theta_{1,2}) \\ \cos\theta_1 h_2(\theta_{1,2}) \\ \sin\theta_1 h_2(\theta_{1,2}) \\ 2A_3^4(\theta_{1,2})\dot{\theta}_{1,2} \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} h_1(\theta_{1,2_e}) \\ \cos\theta_{1_e} h_2(\theta_{1,2_e}) \\ \sin\theta_{1_e} h_2(\theta_{1,2_e}) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $h_1(\theta_{1,2}) \stackrel{\text{def}}{=} 2A_2^4 \frac{\sin\theta_{1,2}}{\Delta}$ and $h_2(\theta_{1,2}) \stackrel{\text{def}}{=} 2A_2^4 \frac{r}{\Delta}$,

$$\begin{aligned}\left[[f,g],\ \left[\left[[f,g],g\right],f\right]\right]\Big|_{z_e} &= \begin{pmatrix} h_3(\theta_{1,2}) \\ \cos\theta_1 h_4(\theta_{1,2}) - \sin\theta_1 h_5(\theta_{1,2}) \\ \sin\theta_1 h_4(\theta_{1,2}) + \cos\theta_1 h_5(\theta_{1,2}) \\ 2A_4^4(\theta_{1,2})\dot{\theta}_{1,2} \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} \\ &= \begin{pmatrix} h_3(\theta_{1,2_e}) \\ \cos\theta_{1_e} h_4(\theta_{1,2_e}) - \sin\theta_{1_e} h_5(\theta_{1,2_e}) \\ \sin\theta_{1_e} h_4(\theta_{1,2_e}) + \cos\theta_{1_e} h_5(\theta_{1,2_e}) \\ 0 \\ 0 \\ 0 \end{pmatrix},\end{aligned}$$

where $h_3(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{\partial h_1}{\partial\theta_{1,2}}$, $h_4(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{\partial h_2}{\partial\theta_{1,2}}$ and $h_5(\theta_{1,2}) \stackrel{\text{def}}{=} 2A_2^4 \frac{d_2}{\Delta}$,

$$\begin{aligned}&\left[\left[\left[[f,g],g\right],f\right],\ \left[[f,g],\ \left[\left[[f,g],g\right],f\right]\right]\right]\Big|_{z_e} \\ &\qquad = \begin{pmatrix} 0 \\ \cos\theta_1 h_6(\theta_{1,2}) - \sin\theta_1 h_7(\theta_{1,2}) \\ \sin\theta_1 h_6(\theta_{1,2}) + \cos\theta_1 h_7(\theta_{1,2}) \\ 0 \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} \\ &\qquad = \begin{pmatrix} 0 \\ \cos\theta_{1_e} h_6(\theta_{1,2_e}) - \sin\theta_{1_e} h_7(\theta_{1,2_e}) \\ \sin\theta_{1_e} h_6(\theta_{1,2_e}) + \cos\theta_{1_e} h_7(\theta_{1,2_e}) \\ 0 \\ 0 \\ 0 \end{pmatrix},\end{aligned}$$

where

$$h_6(\theta_{1,2}) \stackrel{\text{def}}{=} -h_1(\theta_{1,2})h_5(\theta_{1,2}) = -4\left(A_2^4\right)^2 \frac{d_2 \sin\theta_{1,2}}{\Delta^2}$$

and

$$h_7(\theta_{1,2}) \stackrel{\text{def}}{=} h_1(\theta_{1,2})h_4(\theta_{1,2}) - h_2(\theta_{1,2})h_3(\theta_{1,2}) = 4\left(A_2^4\right)^2 \frac{\lambda}{\Delta^2}.$$

The last three brackets span $I\!R^3$ whenever they are linearly independent. This happens whenever

$$\det \begin{pmatrix} h_1 & h_2 & 0 \\ h_3 & h_4 & h_5 \\ 0 & h_6 & h_7 \end{pmatrix} = h_6^2 + h_7^2 = 16 \left(A_2^4\right)^2 \frac{\lambda^2 + d_2^2 \sin^2\theta_{1,2}}{\Delta^4} \neq 0 \ .$$

If $d_1 > d_2$, we know that $\lambda > 0$. If $A_2^4(\theta_{1,2_e}) \neq 0$, this determinant is nonzero. Thus, if $\theta_{1,2_e}$ is such that $A_2^4(\theta_{1,2_e}) \neq 0$, we have at z_e :

$$\begin{aligned} \mathrm{sp}\Bigg\{ g,\ [f,g],\ [[f,g],g], & \\ & \Big[[[f,g],g]\,,f\Big], \\ & \Big[[f,g],\ \Big[[[f,g],g]\,,f\Big]\Big], \\ & \Big[\Big[[[f,g],g]\,,f\Big],\ \Big[[f,g],\ \Big[[[f,g],g]\,,f\Big]\Big]\Big] \Bigg\} = I\!R^6 = T_{z_e}M \ , \end{aligned}$$

i.e. the system satisfies the accessibility rank condition.

If $A_2^4(\theta_{1,2_e}) = 0$, we know that $A_3^4(\theta_{1,2_e}) \neq 0$. The accessibility rank condition can be satisfied by higher–order brackets, which involve A_3^4 instead of A_2^4. Consider then the brackets

$$\Big[[f,g],\ [[f,g],g]\Big]\Big|_{z_e} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2A_3^4(\theta_{1,2}) \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2A_3^4(\theta_{1,2_e}) \\ 0 \\ 0 \end{pmatrix} ,$$

$$\Big[\Big[[f,g],\ [[f,g],g]\Big],\ f\Big]\Big|_{z_e} = \begin{pmatrix} h_8(\theta_{1,2}) \\ \cos\theta_1 h_9(\theta_{1,2}) \\ \sin\theta_1 h_9(\theta_{1,2}) \\ 2A_4^4(\theta_{1,2})\dot{\theta}_{1,2} \\ 0 \\ 0 \end{pmatrix}\Bigg|_{z_e} = \begin{pmatrix} h_8(\theta_{1,2_e}) \\ \cos\theta_{1_e} h_9(\theta_{1,2_e}) \\ \sin\theta_{1_e} h_9(\theta_{1,2_e}) \\ 0 \\ 0 \\ 0 \end{pmatrix} ,$$

where $h_8(\theta_{1,2}) \stackrel{\text{def}}{=} 2A_3^4\frac{\sin\theta_{1,2}}{\Delta}$ and $h_9(\theta_{1,2}) \stackrel{\text{def}}{=} 2A_3^4\frac{r}{\Delta}$,

$$\left[[f,g],\ \left[\left[[f,g],\ [[f,g],g]\right]\right]\right|_{z_e} = \left.\begin{pmatrix} h_{10}(\theta_{1,2}) \\ \cos\theta_1 h_{11}(\theta_{1,2}) - \sin\theta_1 h_{12}(\theta_{1,2}) \\ \sin\theta_1 h_{11}(\theta_{1,2}) + \cos\theta_1 h_{12}(\theta_{1,2}) \\ 2A_5^4(\theta_{1,2})\dot{\theta}_{1,2} \\ 0 \\ 0 \end{pmatrix}\right|_{z_e}$$

$$= \begin{pmatrix} h_{10}(\theta_{1,2_e}) \\ \cos\theta_{1_e} h_{11}(\theta_{1,2_e}) - \sin\theta_{1_e} h_{12}(\theta_{1,2_e}) \\ \sin\theta_{1_e} h_{11}(\theta_{1,2_e}) + \cos\theta_{1_e} h_{12}(\theta_{1,2_e}) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $h_{10}(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{\partial h_8}{\partial\theta_{1,2}}$, $h_{11}(\theta_{1,2}) \stackrel{\text{def}}{=} \frac{\partial h_9}{\partial\theta_{1,2}}$ and $h_{12}(\theta_{1,2}) \stackrel{\text{def}}{=} 2A_3^4\frac{d_2}{\Delta}$,

$$\left[\left[\left[[f,g],\ [[f,g],g]\right],\ f\right],\ \left[[f,g],\ \left[\left[[f,g],\ [[f,g],g]\right],\ f\right]\right]\right]\Big|_{z_e}$$

$$= \left.\begin{pmatrix} 0 \\ \cos\theta_1 h_{13}(\theta_{1,2}) - \sin\theta_1 h_{14}(\theta_{1,2}) \\ \sin\theta_1 h_{13}(\theta_{1,2}) + \cos\theta_1 h_{14}(\theta_{1,2}) \\ 0 \\ 0 \\ 0 \end{pmatrix}\right|_{z_e}$$

$$= \begin{pmatrix} 0 \\ \cos\theta_{1_e} h_{13}(\theta_{1,2_e}) - \sin\theta_{1_e} h_{14}(\theta_{1,2_e}) \\ \sin\theta_{1_e} h_{13}(\theta_{1,2_e}) + \cos\theta_{1_e} h_{14}(\theta_{1,2_e}) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$h_{13}(\theta_{1,2}) \stackrel{\text{def}}{=} -h_8(\theta_{1,2})h_{12}(\theta_{1,2}) = -4\left(A_3^4\right)^2\frac{d_2\sin\theta_{1,2}}{\Delta^2}$$

and

$$h_{14}(\theta_{1,2}) \stackrel{\text{def}}{=} h_8(\theta_{1,2})h_{11}(\theta_{1,2}) - h_9(\theta_{1,2})h_{10}(\theta_{1,2}) = 4\left(A_3^4\right)^2\frac{\lambda}{\Delta^2}.$$

The last three brackets span $I\!R^3$ whenever they are linearly independent. This happens whenever

$$\det\begin{pmatrix} h_8 & h_9 & 0 \\ h_{10} & h_{11} & h_{12} \\ 0 & h_{13} & h_{14} \end{pmatrix} = h_{13}^2 + h_{14}^2 = 16\left(A_3^4\right)^2\frac{\lambda^2 + d_2^2\sin^2\theta_{1,2}}{\Delta^4} \neq 0\,.$$

If $d_1 > d_2$, we know that $\lambda > 0$. If $A_3^4(\theta_{1,2_e}) \neq 0$, this determinant is nonzero.

Thus at equilibria z_e where $\theta_{1,2_e}$ is such that $A_2^4(\theta_{1,2_e}) = 0$, the system satisfies again the accessibility rank condition, since

$$\begin{aligned}
\mathrm{sp}\Big\{ g,\ [f,g],\ \Big[[f,g],\ [[f,g],g]\Big],\\
&\Big[\Big[[f,g],\ [[f,g],g]\Big],\ f\Big],\\
&\Big[[f,g],\ \Big[\Big[[f,g],\ [[f,g],g]\Big],\ f\Big]\Big],\\
&\Big[\Big[\Big[[f,g],\ [[f,g],g]\Big],\ f\Big],\ \Big[[f,g],\ \Big[\Big[[f,g],\ [[f,g],g]\Big],\ f\Big]\Big]\Big]\Big\}\\
&= I\!R^6 = T_{z_e}M\ .
\end{aligned}$$

∎

Corollary 6.22 (Small–Time Local Controllability of the Roller Racer)

The dynamics of the Roller Racer (equation (6.23)) *are not* STLC at equilibria z_e where $A_2^4(\theta_{1,2_e}) \neq 0$.

Proof

Immediate from Theorem 6.18.

∎

Other undulatory locomotors, like the snakeboard, were shown to be STLC (Ostrowski & Burdick [1995]). The nontrivial second term in the momentum equation (5.36) of the Roller Racer ($A_2^4(\theta_{1,2})\ \dot\theta_{1,2}^2$), plays a crucial role in its property of being accessible, but not being STLC.

Proposition 6.23 (Differential Flatness of the Roller Racer)

The dynamics of the Roller Racer (equation (6.23)) *are not* differentially flat around equilibria z_e.

Proof

Our system has $n = 6$. Consider the family of distributions defined in (6.8):

$$D_k(x) = sp\{g(x),\ ad_f g(x),\ \dots, ad_f^{k-1} g(x)\},\ k = 1, \dots, n. \tag{6.25}$$

Consider in particular $D_3(x) = sp\{g(x),\ ad_f g(x),\ ad_f^2 g(x)\}$. It is easy to see that $ad_f^2 g(z_e) = \big[f,\ [f,\ g]\big]\big|_{z_e} = \frac{\partial [f,\ g]}{\partial x}(z_e) f(z_e)\ - \frac{\partial f}{\partial x}(z_e)\ [f,\ g](z_e) = 0$, since $f(z_e) = 0$

and since from (6.24): $[f,\ g](z_e) = (\star,\ \star,\ \star,\ 0,\ 1,\ 0)^\top$, where $\star$ are non–zero elements. Also $\frac{\partial f}{\partial \theta_{1,2}} = \alpha_1(\theta_1,\ \theta_{1,2},\ \dot{\theta}_{1,2})\, p + \alpha_2(\theta_1,\ \theta_{1,2},\ \dot{\theta}_{1,2})\, \dot{\theta}_{1,2}$, for appropriate column vectors α_i, and, thus, $\frac{\partial f}{\partial \theta_{1,2}}(z_e) = 0$. Finally

$$\frac{\partial f}{\partial(x_1,\ y_1,\ \theta_1)} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\sin\theta_1}{\Delta}(r\, p - \gamma\ \sin\theta_{1,2}\, \dot{\theta}_{1,2}) & 0 & 0 \\ \frac{\cos\theta_1}{\Delta}(r\, p - \gamma\ \sin\theta_{1,2}\, \dot{\theta}_{1,2}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

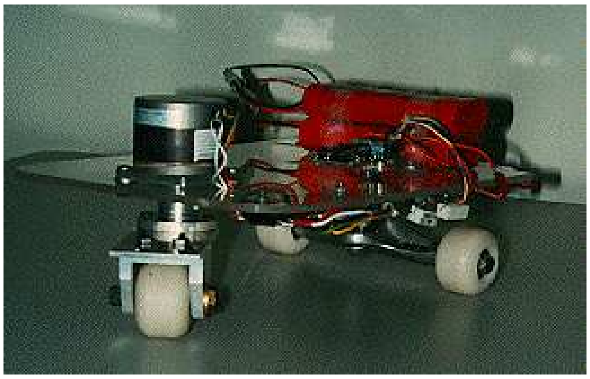
thus $\frac{\partial f}{\partial(x_1,\, y_1,\, \theta_1)}(z_e) = 0$.
Then $\dim\, D_3(z_e) = 2 < 3$ and thus $\dim\, D_n(z_e) < n$. From Corollary 6.13, Proposition 6.15 and Remark 6.16, the system is not differentially flat around equilibria z_e.

∎

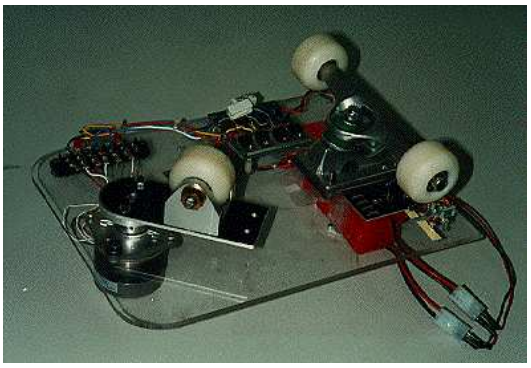
It is interesting to observe that, even though the Roller Racer resembles a unicycle with one trailer, which is hitched not to the center of the unicycle's wheel axis, but to a point displaced from the center (also referred to as kingpin hitch), and this last system has been shown to be differentially flat (Rouchon et al. [1993]), the peculiar actuation scheme of the Roller Racer makes it non–flat.

7 Simulation and Experimental Results

A computer–controlled prototype of the Roller Racer system was built at the Intelligent Servosystems Laboratory (fig. 7.1). The assumption of our model that the only feature of the body motion of a Roller Racer rider which is crucial to the propulsion of this mechanism is the swinging of the steering arm around the pivot axis, was verified using this and other similar prototypes.



(a)



(b)

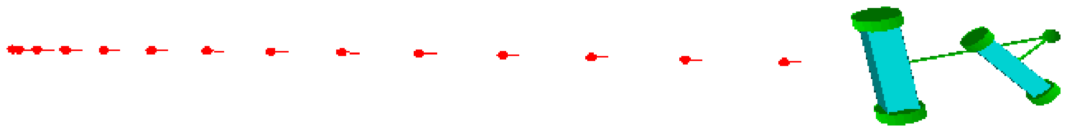
Fig. 7.1: Roller Racer Prototype

The model of the dynamics of the Roller Racer system, which was developed in the previous sections, was used in computer simulations of the system on Silicon Graphics workstations and in Mathematica and Simparc (Astraudo & Borrelly [1992]) simulations on SUN SPARCstations.

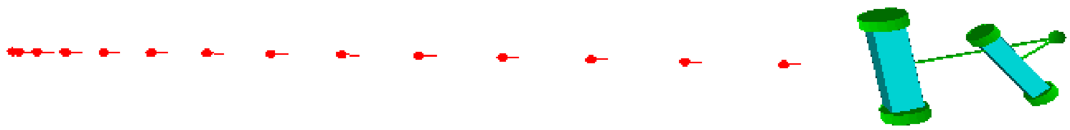
A periodic shape trajectory of period $T_{1,2}$ of the form

$$\theta_{1,2}(t) = \theta_{1,2}(0) + \alpha_{1,2}\sin(\omega_{1,2}t + \phi_{1,2}) \ , \tag{7.1}$$

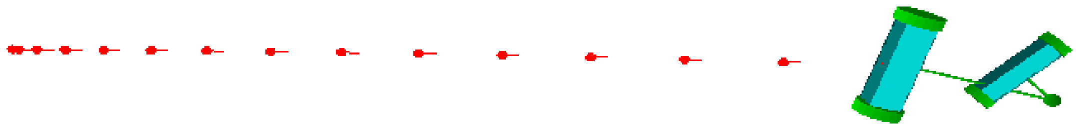
with $\omega_{1,2} = \frac{2\pi}{T_{1,2}}$ is used in the simulations. The average value of $\theta_{1,2}$ is $\theta_{1,2}(0)$. Setting this average to π, as in fig. 7.2, generates a "straight–line" motion. In fig. 7.2 three successive snapshots of the system's motion are shown from the Silicon Graphics 3–d animation. The trajectory of the system is shown by needle–like markers, so that the position and orientation of the system at the end of a period of the shape control becomes evident. As the system moves to the right, the spacing between these needles becomes larger, since, as momentum builds up, the system moves faster. Setting $\theta_{1,2}(0)$ to a value other than π or zero, as in fig. 7.3 (where $\theta_{1,2}(0) = 3.31$ rad), generates a rotation around the point where the axes of the platforms intersect, when the system is in the configuration corresponding to this average value. Once momentum has built up through periodic shape variations, we can stop varying the shape periodically and use $\theta_{1,2}$ just to steer the system.



(a)

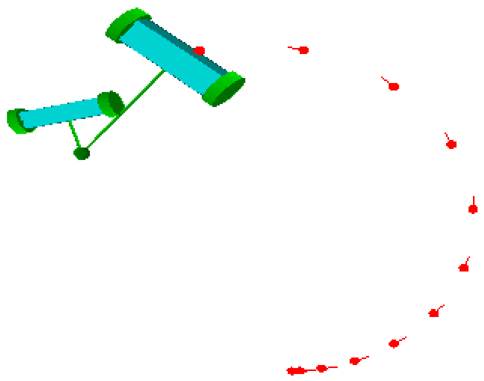


(b)

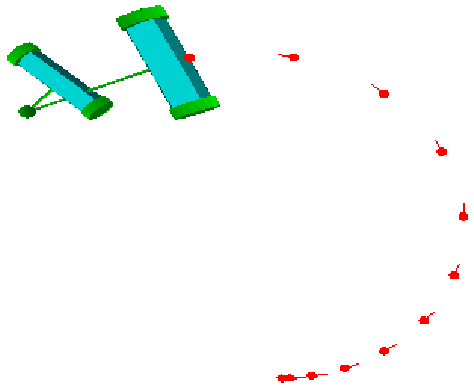


(c)

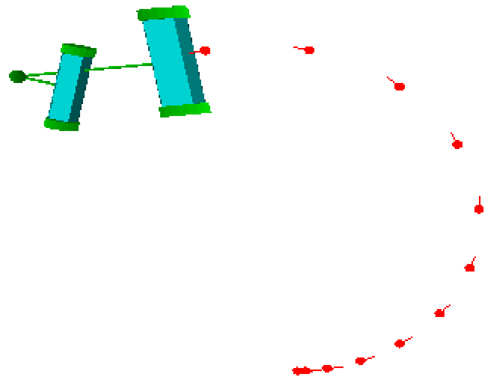
Fig. 7.2: “Straight–line” motion of the Roller Racer



(a)



(b)



(c)

Fig. 7.3: “Turning” motion of the Roller Racer

7.1 Gaits

More detailed simulations of the system were done using Mathematica and Simparc.

Initially we consider the model without friction or external forces. The Roller Racer parameters used in these simulations are $m_1 = 1$, $d_1 = 5$, $d_2 = 1$, $I_{z_1} = 10$, $I_{z_2} = 1$ $(I_{z_1} d_2 > I_{z_2} d_1)$.

In all the (x_1, y_1) plots that follow, the system starts at $(0, 0)$ and is initially oriented towards the positive x–axis ($\theta_1 = 0$).

a. Forward Translation

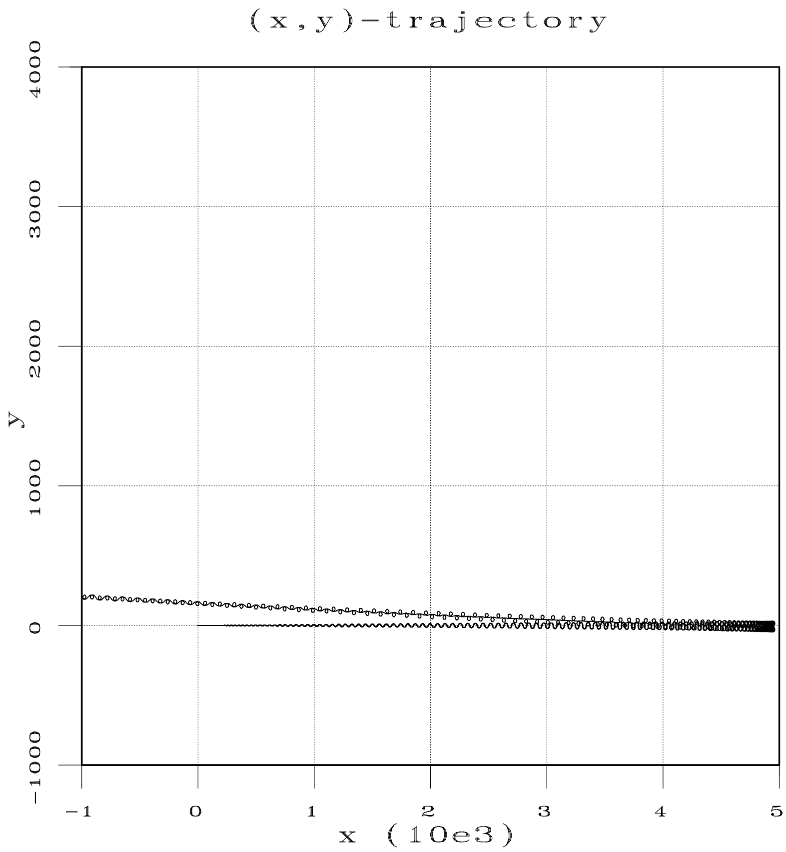
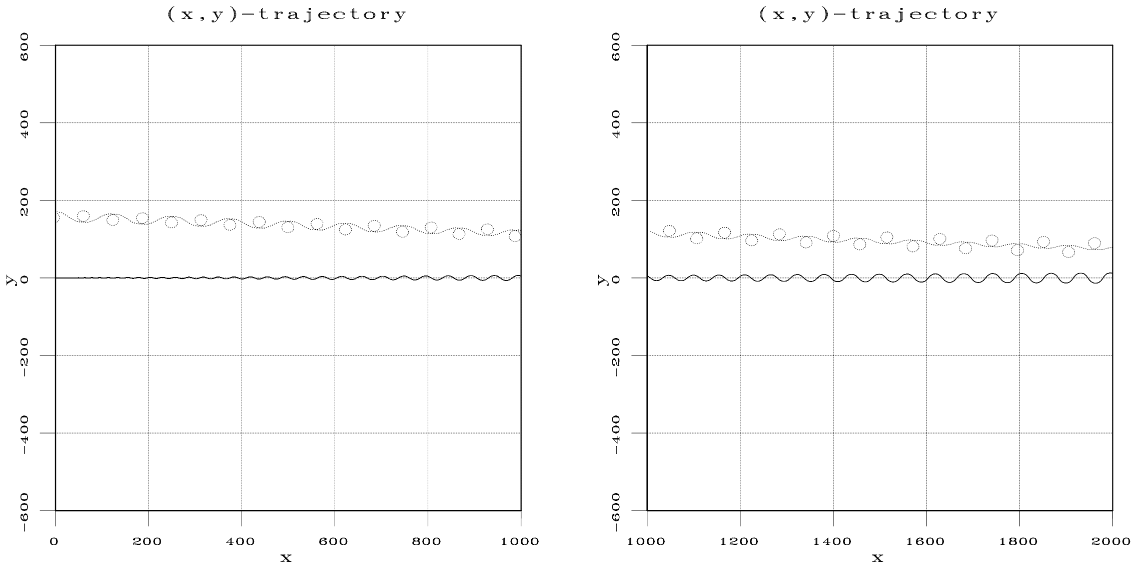


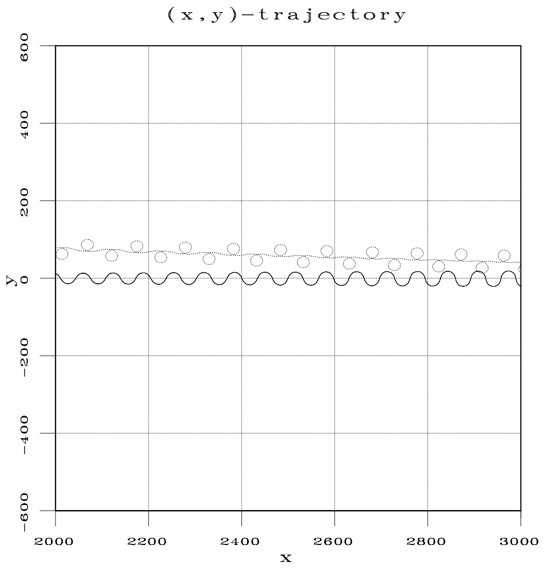
Fig. 7.4: Roller Racer Forward Translation

When $\theta_{1,2}(0) = \pi$, the system translates forward. In the present simulations, the system starts at $(x_1, y_1) = (0, 0)$ with amplitude of oscillation $\alpha_{1,2} = 0.3$ and frequency $\omega_{1,2} = 1$.

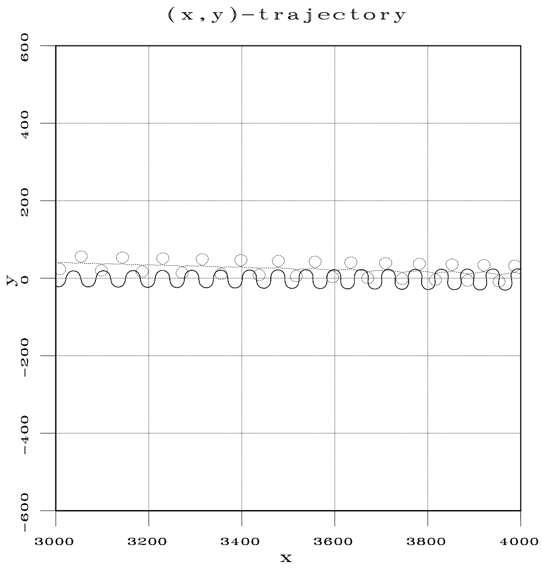
In fig. 7.4, the (x_1, y_1)–trajectory is shown. The system initially translates to the right, while oscillating about the x–axis. These oscillations become more pronounced as the momentum increases, giving rise to elastica–like trajectory segments (c.f. §3.2.6 of (Tsakiris [1995])), which at some point reverse direction and the system starts moving to the left, creating the upper branch of the trajectory of fig. 7.4. In fig. 7.5, this trajectory is magnified (as we move from fig. 7.5.a to 7.5.e, we are moving to the left of the x–axis), in order to show better the elastica–like trajectory segments. The lower branch of the trajectory, the one that corresponds to a translation of the Roller Racer to the left, is shown as a solid line. The upper branch of the trajectory, the one that corresponds to a translation to the right, is shown as a dotted line. Notice that the scales of the x and y axes are different. A further magnification of the trajectory close to the direction reversal point is shown in fig. 7.6.



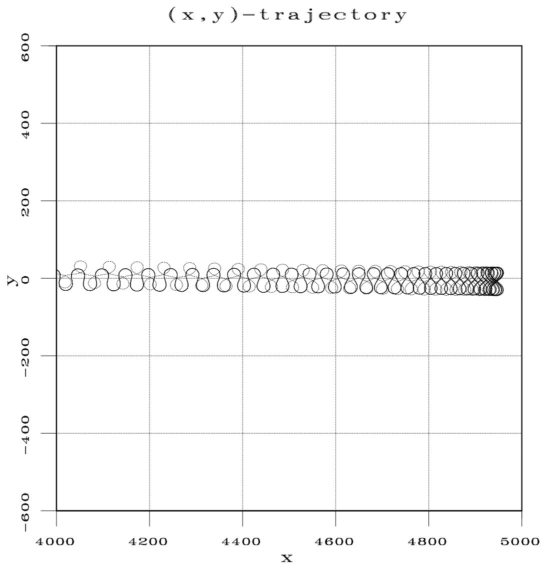
(a) (b)



(c)



(d)



(e)

Fig. 7.5: Forward Translation: Magnification of Trajectory

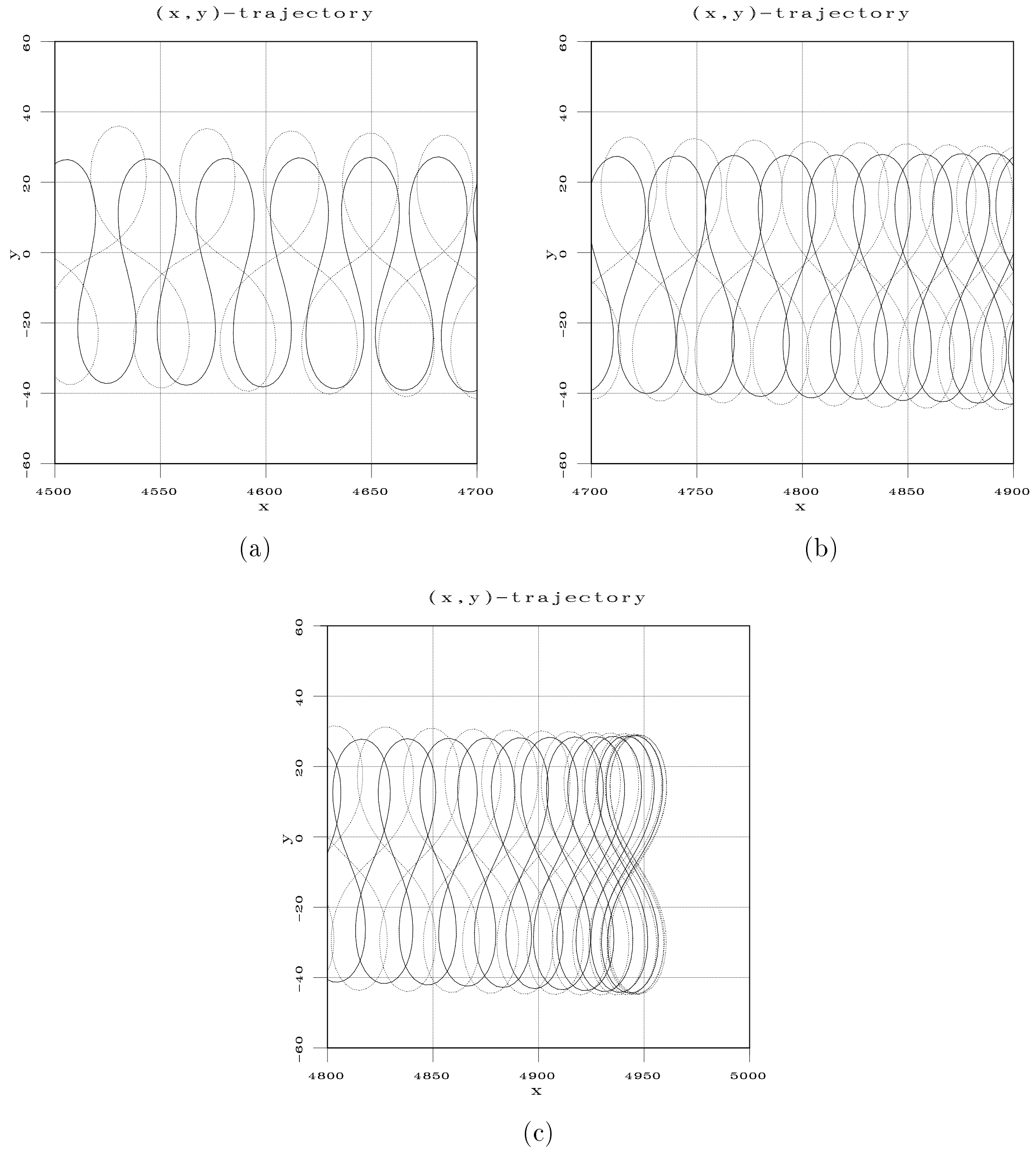
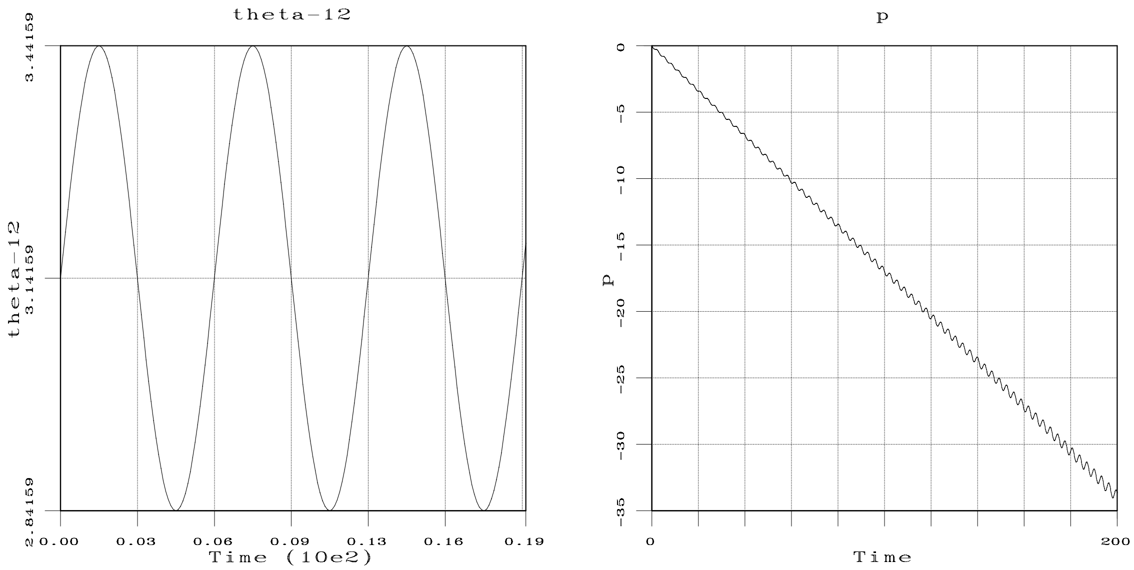


Fig. 7.6: Forward Translation: Magnification of Trajectory near Reversal Point

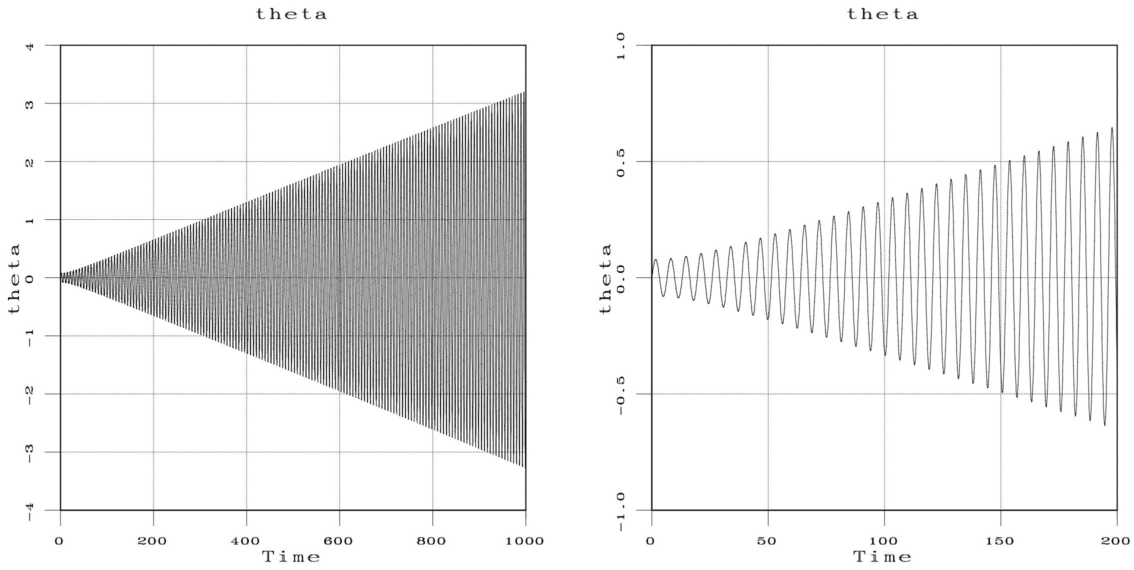
The corresponding sinusoidal shape angle $\theta_{1,2}(t)$ with average π and frequency $\omega_{1,2} = 1$ (thus period $T_{1,2} = 2\pi$ sec) is shown in fig. 7.7.a, while the nonholonomic momentum $p(t)$ is shown in fig. 7.7.b. The group variable $\theta_1(t)$ is shown in fig. 7.8, showing that the system oscillates with increasing amplitude (as the nonholonomic momentum increases), but the average of this oscillation is zero. Thus, the system translates on a more or less straight–line trajectory (c.f. figs. 7.1 and 7.4).



(a) Shape Angle $\theta_{1,2}$

(b) Nonholonomic Momentum p

Fig. 7.7: Forward Translation



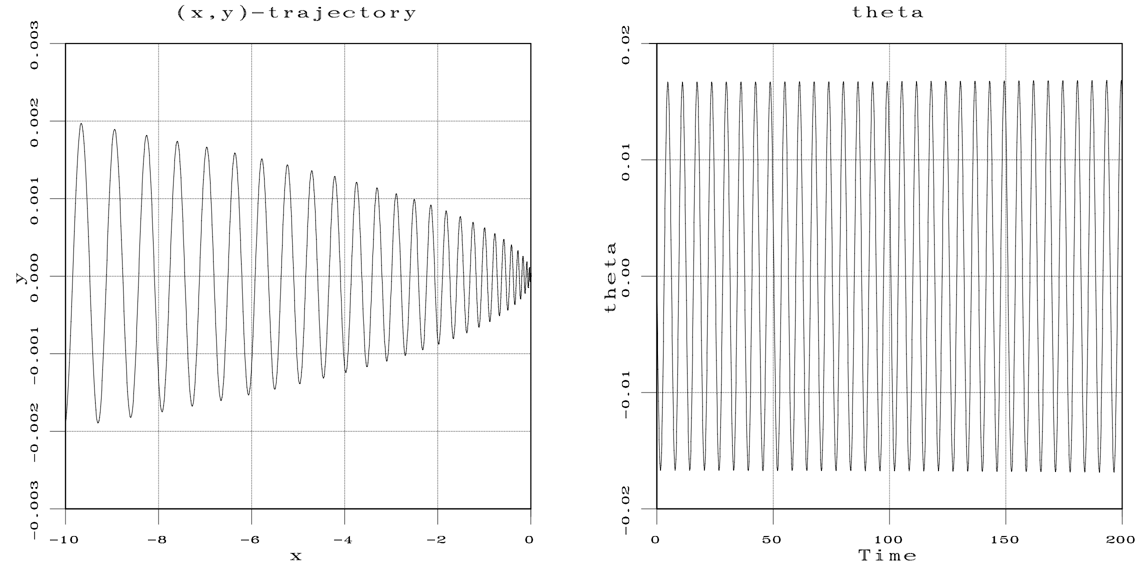
(a) Angle θ_1

(b) Magnification of initial part

Fig. 7.8: Forward Translation: Angle θ_1

b. Backward Translation

When $\theta_{1,2}(0) = 0$, the system translates backwards. In the present simulations, the system starts at $(x_1, y_1) = (0, 0)$ pointing towards the positive x–axis and the shape oscillation has amplitude $\alpha_{1,2} = 0.1$ and frequency $\omega_{1,2} = 1$. The corresponding group variables $(x_1,\ y_1,\ \theta_1)$ are shown in fig. 7.9. Observe that y_1 and θ_1 merely oscillate around zero, while the magnitude of x_1 increases.



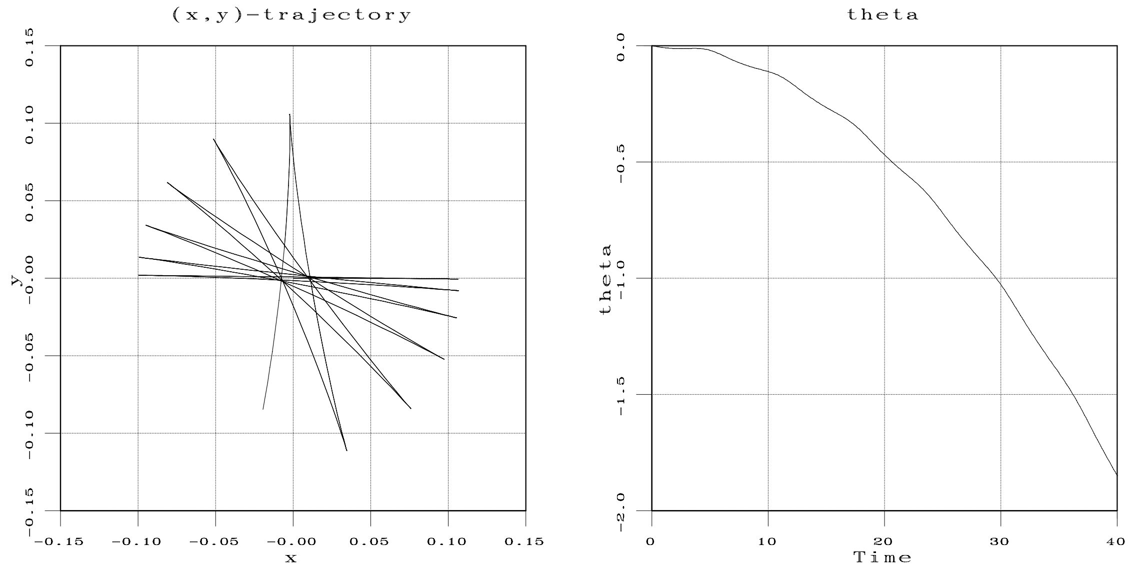
(a) (x_1, y_1)–trajectory (b) θ_1–trajectory

Fig. 7.9: Backward Translation

c. Pure Rotation

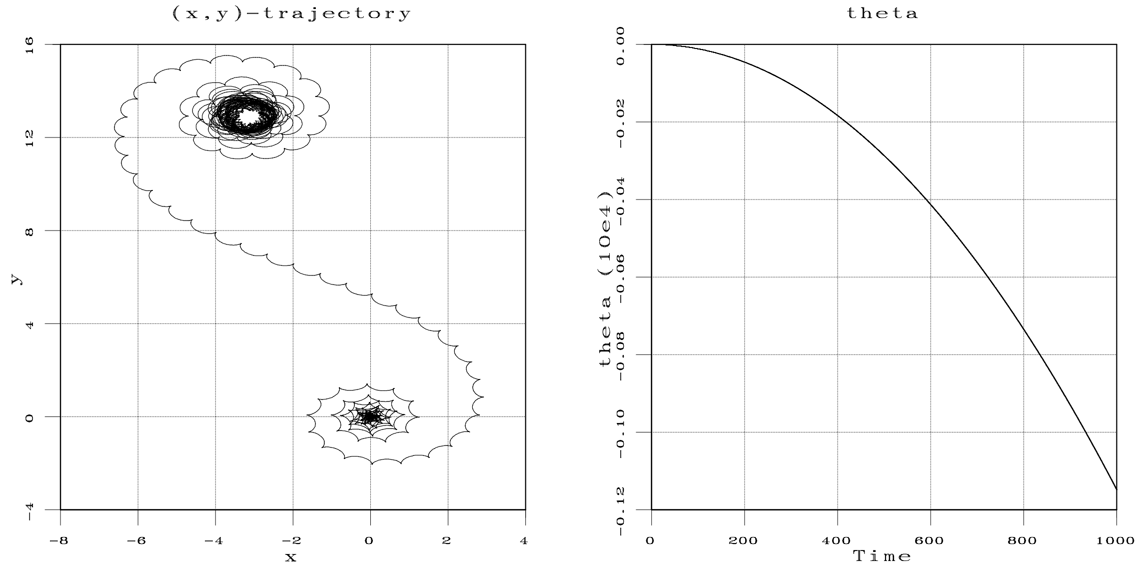
When the instantaneous center of rotation of the system is, on the average, at the middle of the rear wheel axis, i.e. when $\theta_{1,2}(0)$ is a root of $r(\theta_{1,2}) = 0$, which we denote as $\theta_{1,2}^{r=0}$, the Roller Racer rotates without translating (on the average). This can be seen in fig. 7.10 for a clockwise rotation with $\theta_{1,2}(0) = \theta_{1,2}^{r=0} = 1.7721542$ rad and in fig. 7.12 for a counter–clockwise rotation with $\theta_{1,2}(0) = -\theta_{1,2}^{r=0} = -1.7721542$ rad. However, as momentum increases, the system trajectories become more complicated (figs. 7.11 and 7.13).

When the average of the shape oscillation ($\theta_{1,2}(0)$) is not set to $0, \pi$ or $\pm\, \theta_{1,2}^{r=0}$, the system rotates around the average position of the instantaneous center of rotation (fig. 7.3).



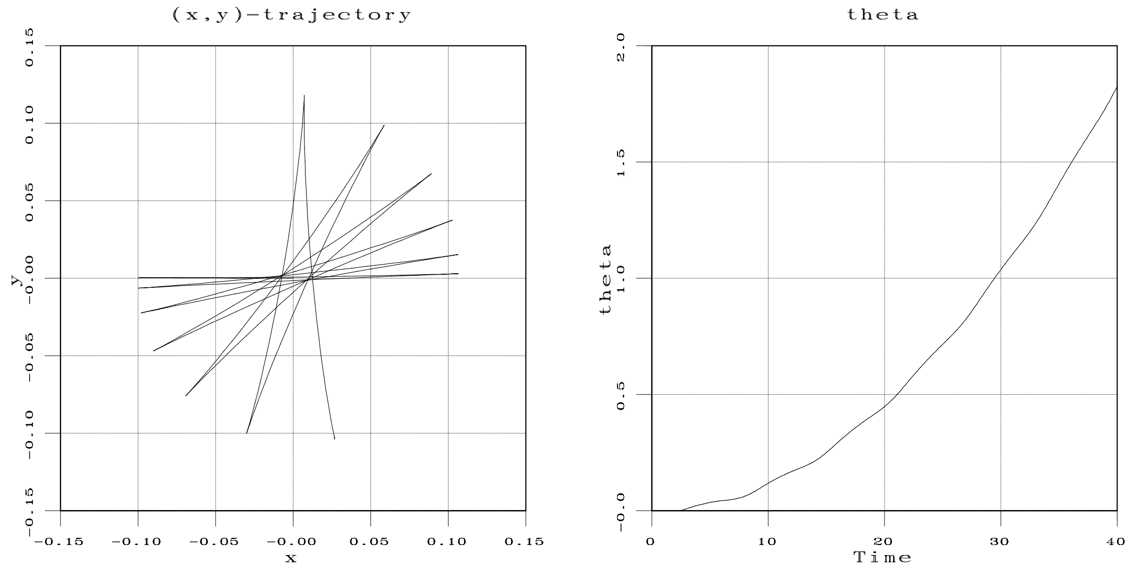
(a) (x_1, y_1) (b) θ_1

Fig. 7.10: Clockwise Rotation by $\frac{\pi}{2}$



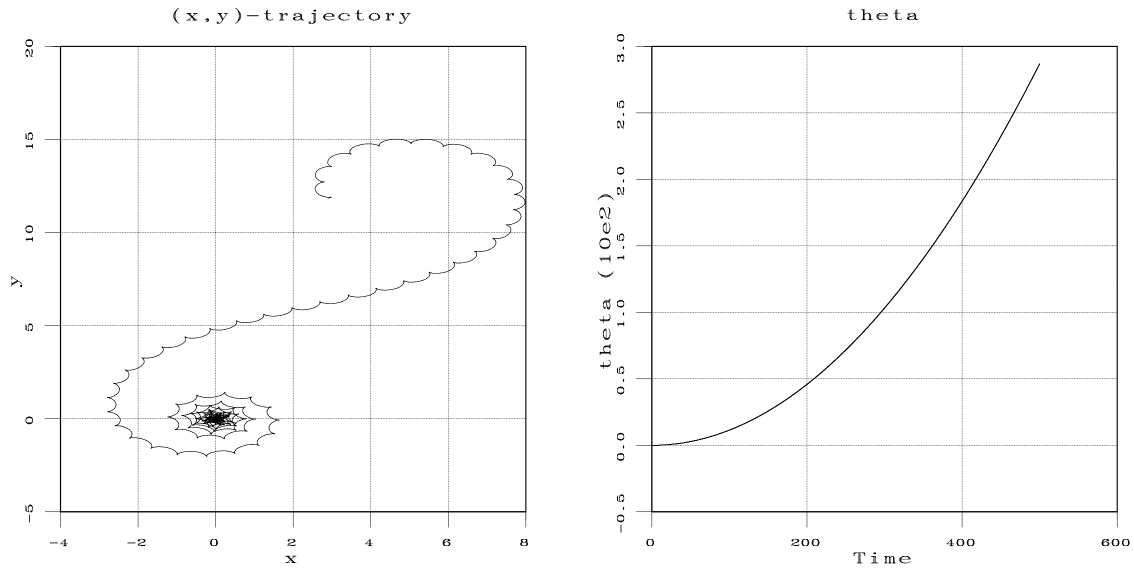
(a) (x_1, y_1) (b) θ_1

Fig. 7.11: Large Clockwise Rotation



(a) (x_1, y_1) (b) θ_1

Fig. 7.12: Counter–Clockwise Rotation by $\frac{\pi}{2}$



(a) (x_1, y_1) (b) θ_1

Fig. 7.13: Large Counter–Clockwise Rotation

7.2 Geometric and Dynamic Phase

Consider a periodic shape variation of the type of equation (7.1) corresponding to forward translation of the system with $\theta_{1,2}(0) = \pi$, $\alpha_{1,2} = 0.1$ and $\omega_{1,2} = 1$. We consider the momentum equation without friction (5.36).

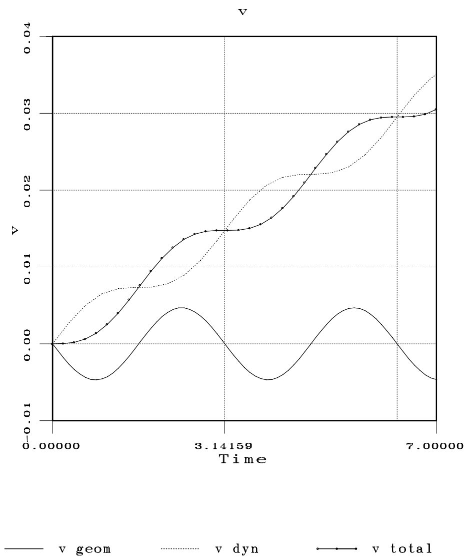
The notions of *geometric* and *dynamic phase* (Bloch, Krishnaprasad, Marsden & Murray [1994]), (Marsden, Montgomery & Ratiu [1990]) describe how much the system moved after one period of the oscillatory controls. The group velocity, given by the reconstructed group motion equations (5.76)

$$\xi_{1} = g_{1}^{-1}\dot{g}_{1} = A_{loc}(\theta_{1,2})\dot{\theta}_{1,2} + \mathbb{I}^{-1}(\theta_{1,2})\, p$$

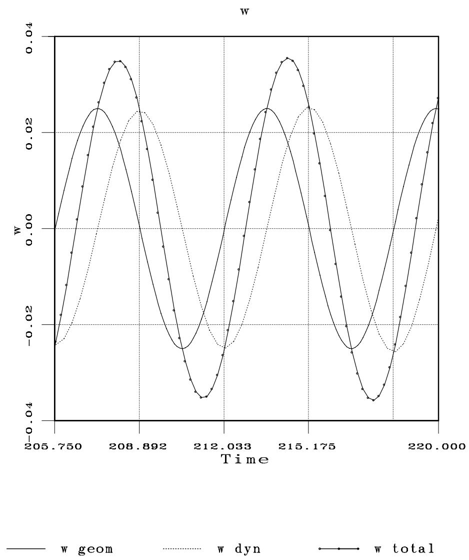
is composed of two parts: the system motion due to the first term $A_{loc}(\theta_{1,2})\,\dot{\theta}_{1,2}$ (where the nonholonomic momentum plays no role) is called the *geometric phase*, while the system motion due to the second term $\mathbb{I}^{-1}(\theta_{1,2})\, p$ is called the *dynamic phase*. Thus, the geometric phase is $\int_0^{T_{1,2}} \dot{g}_{1}(t)dt$, with $\dot{g}_{1}(t) = g_{1}(t)A_{loc}(\theta_{1,2}(t))\dot{\theta}_{1,2}(t)$ and the dynamic phase is $\int_0^{T_{1,2}} \dot{g}_{1}(t)dt$, with $\dot{g}_{1}(t) = g_{1}(t)\mathbb{I}^{-1}(\theta_{1,2}(t))\, p(t)$. The dynamic phase obviously depends on the initial value of the momentum $p(0)$. In the simulation results presented below, we suppose that the system starts at rest, i.e. that $p(0) = 0$.

The contributions of the geometric and dynamic phases to the total motion of the system, for a few periods of the shape controls, are shown below. First the components of the velocities $v = \mathcal{A}_2^\flat(\xi_{1}) = \xi_{2}^1$ and $\omega = \mathcal{A}_1^\flat(\xi_{1}) = \xi_{1}^1$ that are due to each of the two terms above are shown in fig. 7.14. It is obvious that the total v and ω of the system is the sum of these two terms. Since we consider a shape oscillation that gives rise to forward translation of the system, we expect v to increase on the average, while ω remains zero on the average. Indeed, in fig. 7.14.a showing the translational velocities v, the curve marked "v geom" is $\mathcal{A}_2^\flat\big(A_{loc}(\theta_{1,2})\dot{\theta}_{1,2}\big)$, and it varies periodically with zero average value, the curve marked "v dyn" is $\mathcal{A}_2^\flat\big(\mathbb{I}^{-1}(\theta_{1,2})\; p\big)$, and it also varies periodically, but with increasing average value, while the curve marked "v total" is $v = \mathcal{A}_2^\flat(\xi_{1})$, and is obviously the sum of the above. In fig. 7.14.b showing the rotational velocities ω, the curve marked "w geom" is $\mathcal{A}_1^\flat\big(A_{loc}(\theta_{1,2})\dot{\theta}_{1,2}\big)$, and it varies periodically with zero average value, the curve marked "w dyn" is $\mathcal{A}_1^\flat\big(\mathbb{I}^{-1}(\theta_{1,2})\; p\big)$, and it also varies periodically, but with zero average value, while the curve marked "w total" is $\omega = \mathcal{A}_1^\flat(\xi_{1})$, and is obviously the sum of the above.

The components of x_1, y_1 and θ_1, that are due to each of the above two terms, are shown in fig. 7.15. It is easy to see from this figure that the geometric phase, over one period of the periodic shape controls, is zero (this is shown by the curves marked "(x,y) geom" and "th geom" in these figures). In fig. 7.15.a, the contribution of "(x,y)



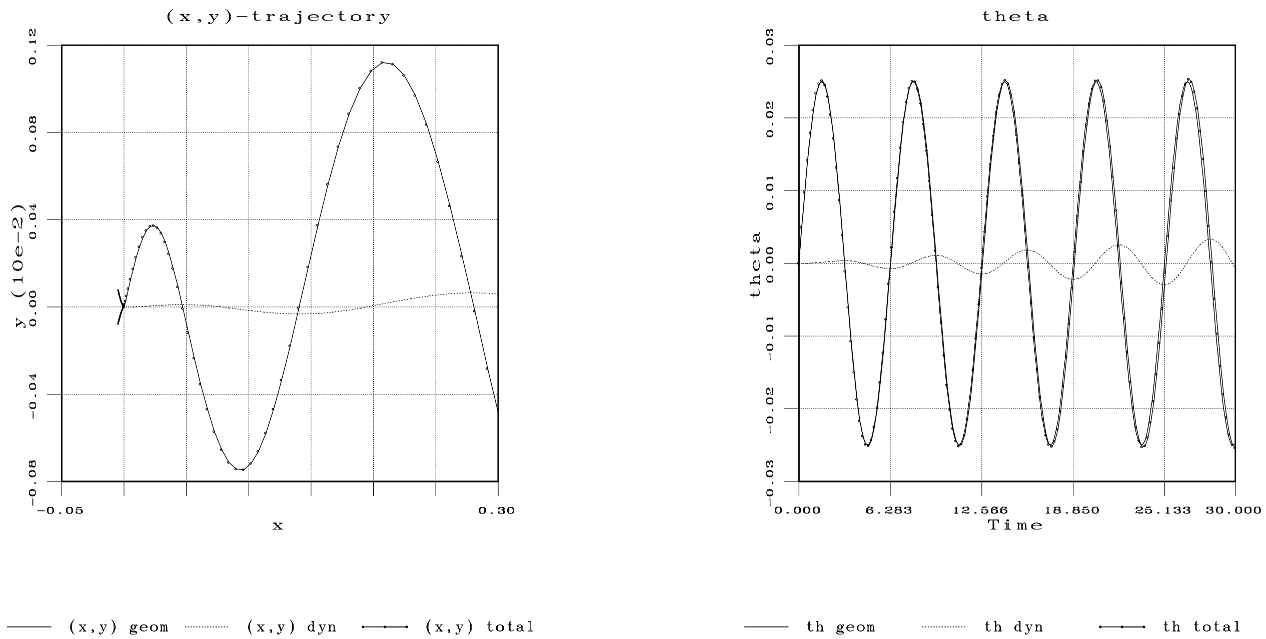
(a) Evolution of v



(b) Evolution of ω

Fig. 7.14: Geometric and Dynamic Phase: Velocities $\xi_1 = g_1^{-1}\dot{g}_1$

geom" is the swallowtail to the left of point $(0, 0)$. In fig. 7.15.b, the curve "th geom", an oscillation around zero, initially overlaps the curve "th total". Obviously, the geometric phase in this case is zero. However, the dynamic phase is not zero. In fig. 7.15.a, the curve "(x,y) dyn" has an evident non–zero component in the x–direction, while the motions in the y and θ directions appear to be again oscillations around zero.



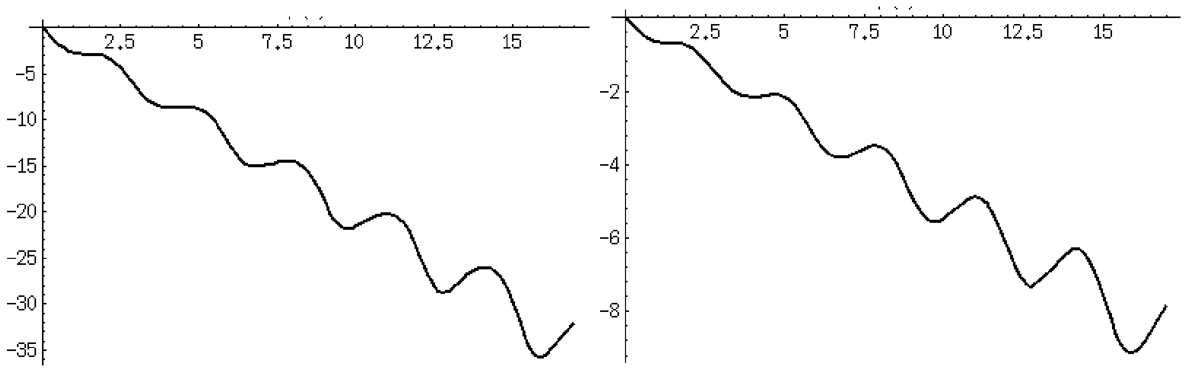
(a) (x_1, y_1)–trajectory (b) θ_1–trajectory

Fig. 7.15: Geometric and Dynamic Phase: State $(x_1,\ y_1,\ \theta_1)$

7.3 Parametric Study of the System

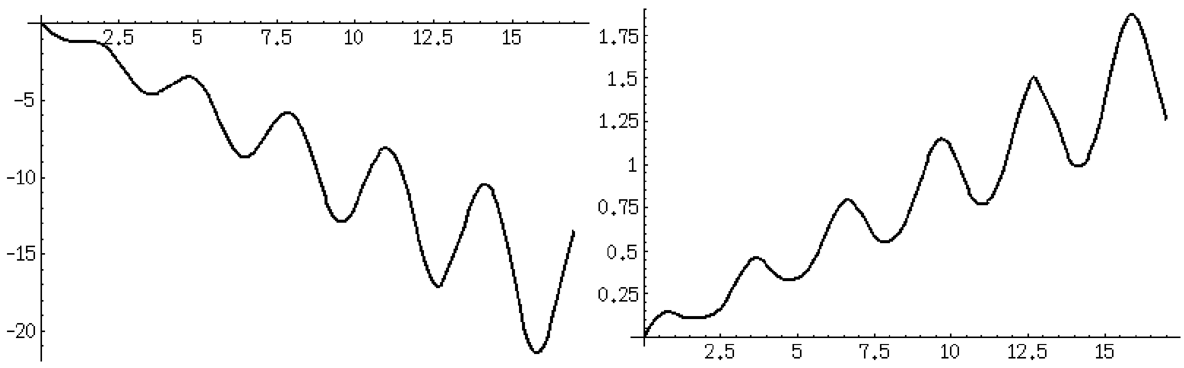
Subsequently, we use Mathematica and Simparc simulations of the dynamics of the Roller Racer, for the system without external forces or friction, in order to study the dependence of its motion on the various size and inertia parameters, as well as its dependence on the amplitude and the frequency of the sinusoidal controls (equation (7.1)).

We consider first the nonholonomic momentum p, given by equation (5.38) and its behavior under various size–inertia relationships. We set $\alpha_{1,2} = 1$, $\omega_{1,2} = 1$ and assume that the system is initially at rest. From fig. 7.16, we observe that, when $I_{z_1} d_2 > I_{z_2} d_1$, the momentum p is always negative when we set $\theta_{1,2}(0)$ to π or 0, as predicted by Proposition 5.10. When $I_{z_1} d_2 < I_{z_2} d_1$, we get positive p when $\theta_{1,2}(0) = 0$ and negative p when $\theta_{1,2}(0) = \pi$ (fig. 7.17).



(a) $\theta_{1,2}(0) = \pi$ (b) $\theta_{1,2}(0) = 0$

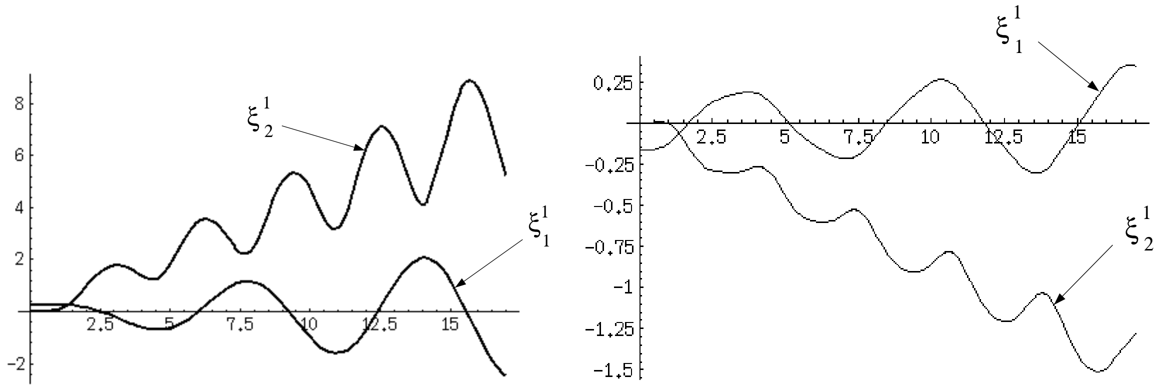
Fig. 7.16: Nonholonomic momentum p : Case $I_{z_1} d_2 > I_{z_2} d_1$



(a) $\theta_{1,2}(0) = \pi$ (b) $\theta_{1,2}(0) = 0$

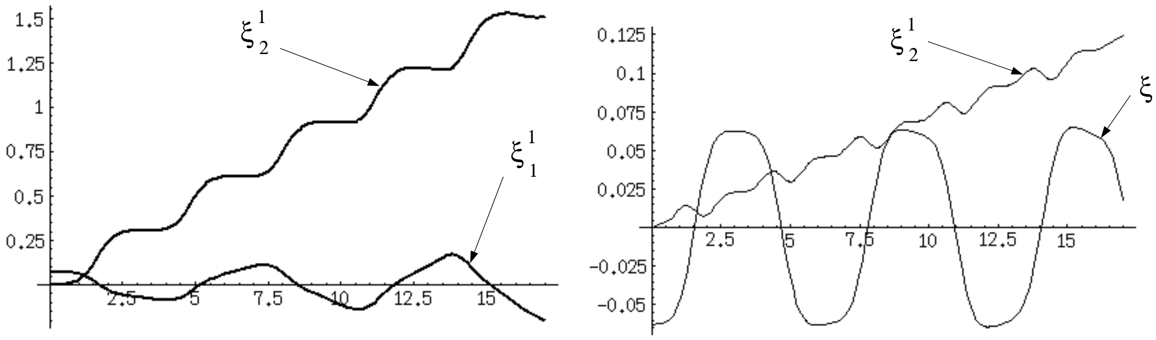
Fig. 7.17: Nonholonomic momentum p : Case $I_{z_1} d_2 < I_{z_2} d_1$

Next, we consider the components ξ_1^1 and ξ_2^1 of the global velocity ξ_1, which are given by equations (5.51) and (5.52) and we study their behavior under various size–inertia relationships. We set $\alpha_{1,2} = 1$ and $\omega_{1,2} = 1$. From fig. 7.18, we observe that, when $I_{z_1} d_2 > I_{z_2} d_1$, we can generate both positive and negative values of ξ^1_2, by setting $\theta_{1,2}(0)$ to π or 0 respectively. Since ξ^1_1 oscillates around zero, the corresponding motion of the system is, respectively, a forward or backward translation, as expected. The corresponding group variable trajectories (x_1, y_1, θ_1) are similar to the ones in figs. 7.4-7.9. However, when $I_{z_1} d_2 < I_{z_2} d_1$, no backward translation is possible, since in both cases of $\theta_{1,2}(0)$ being 0 or π, the heading speed ξ^1_2 is positive (fig. 7.19).



(a) $\theta_{1,2}(0) = \pi$ (b) $\theta_{1,2}(0) = 0$

Fig. 7.18: System Translation: Case $I_{z_1} d_2 > I_{z_2} d_1$



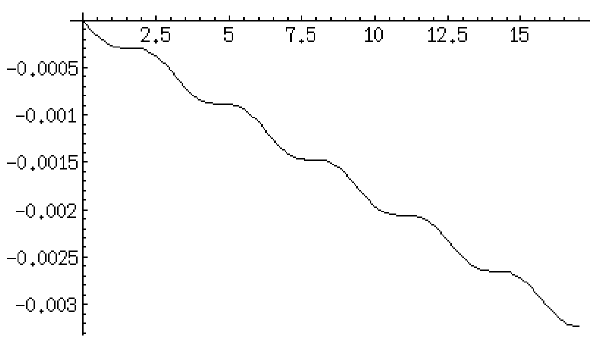
(a) $\theta_{1,2}(0) = \pi$ (b) $\theta_{1,2}(0) = 0$

Fig. 7.19: System Translation: Case $I_{z_1} d_2 < I_{z_2} d_1$

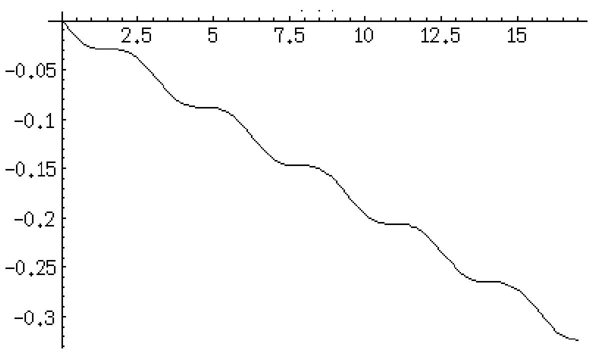
Consider now the case $I_{z_1} d_2 > I_{z_2} d_1$. We explore the effect of the amplitude $\alpha_{1,2}$ and frequency $\omega_{1,2}$ of the sinusoidal control on the nonholonomic momentum p, on the global velocity ξ_1 and on the group variables x_1, y_1, θ_1.

Let $\theta_{1,2}(0) = \pi$ and $\omega_{1,2} = 1$ and consider the effect of the amplitude $\alpha_{1,2}$ on the nonholonomic momentum p. We vary the amplitude from 0.01 rad to 2 rad.

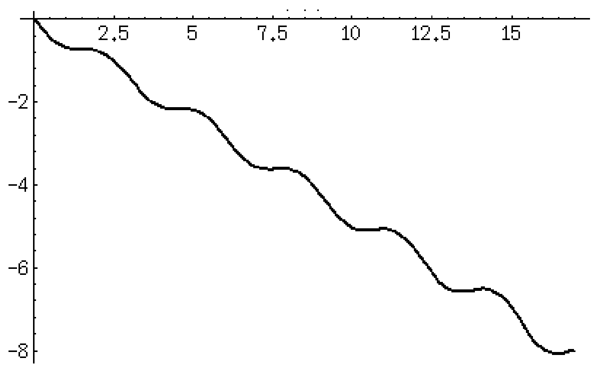
From fig. 7.20, we observe that for $\alpha_{1,2}$ up to 1, an increase of the amplitude produces up to a 10–fold increase in p.



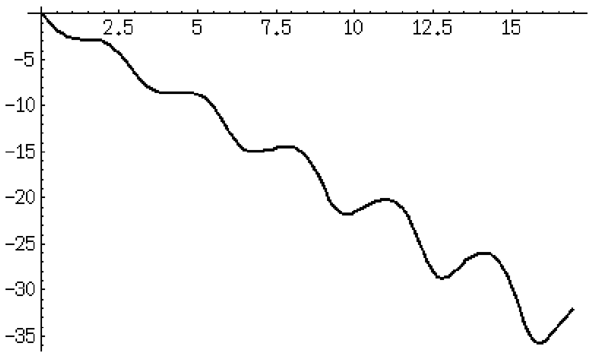
(a) $\alpha_{1,2} = 0.01$



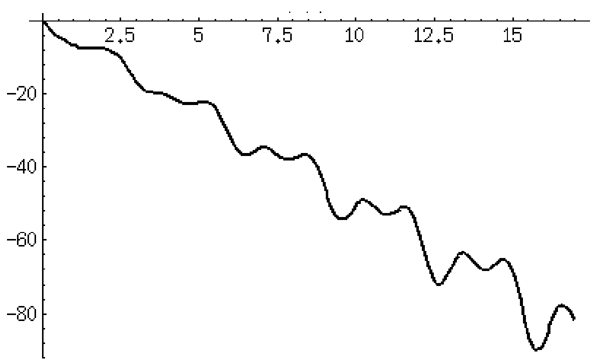
(b) $\alpha_{1,2} = 0.1$



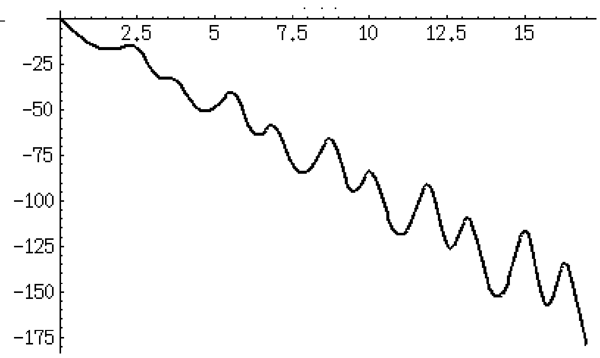
(c) $\alpha_{1,2} = 0.5$



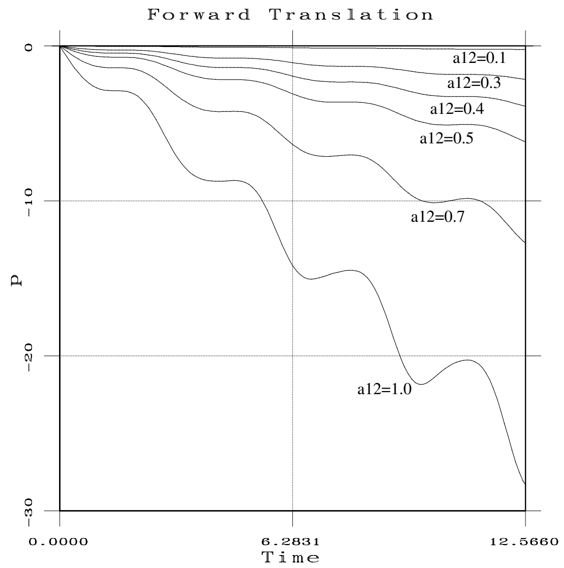
(d) $\alpha_{1,2} = 1.0$



(e) $\alpha_{1,2} = 1.5$



(f) $\alpha_{1,2} = 2.0$

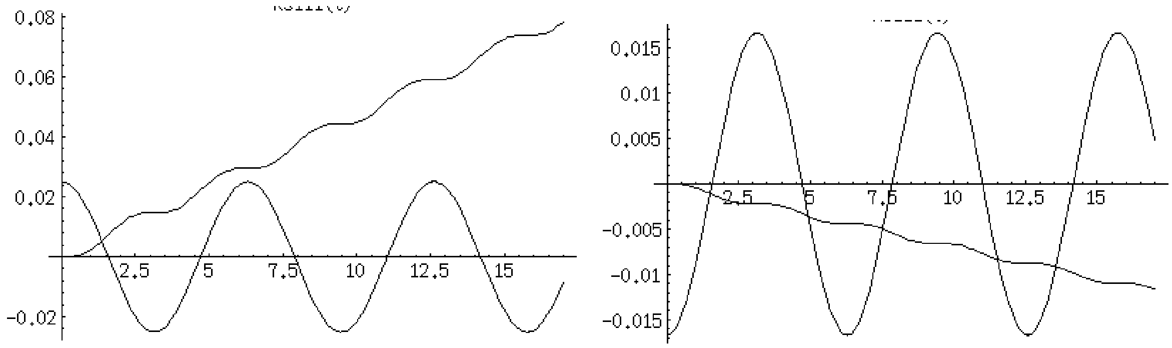


(g) $\alpha_{1,2} = 0.1$ to 1.0

Fig. 7.20: Effect of amplitude $\alpha_{1,2}$ on the nonholonomic momentum p

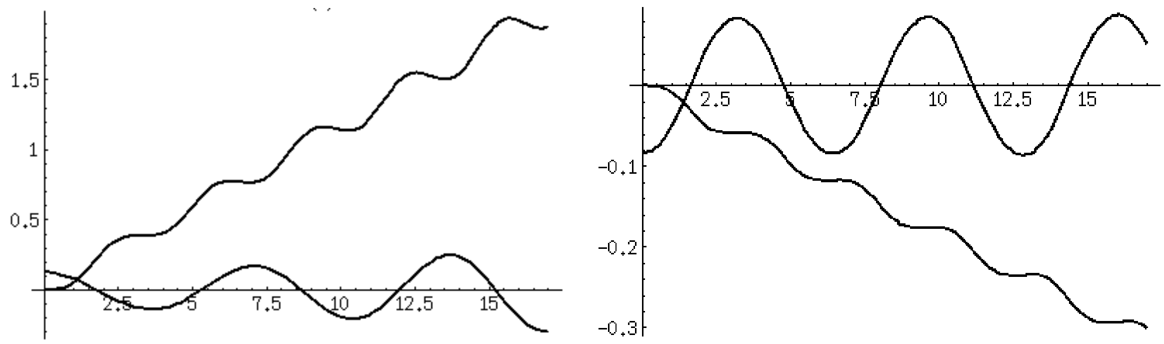
Let $\theta_{1,2}(0) = 0$ or π and $\omega_{1,2} = 1$ and consider the effect of the amplitude $\alpha_{1,2}$ on the components ξ^1_1 and ξ^1_2 of the global velocity ξ_1. This corresponds to the case when the system translates forward ($\theta_{1,2}(0) = \pi$) or backwards ($\theta_{1,2}(0) = 0$). We vary the amplitude from 0.1 rad to 2.0 rad.

Observe from fig. 7.21 that for values of $\alpha_{1,2}$ up to 1, an increase of the amplitude produces up to a 10–fold increase in ξ^1_2. Thus, shape oscillations of larger amplitude, produce faster motion of the system. However, further increases in $\alpha_{1,2}$ produce a significant oscillation of ξ^1_2 without a significant increase in its average value. During this oscillation, ξ^1_2 can become negative in the case $\theta_{1,2}(0) = \pi$ or positive in the case $\theta_{1,2}(0) = 0$.



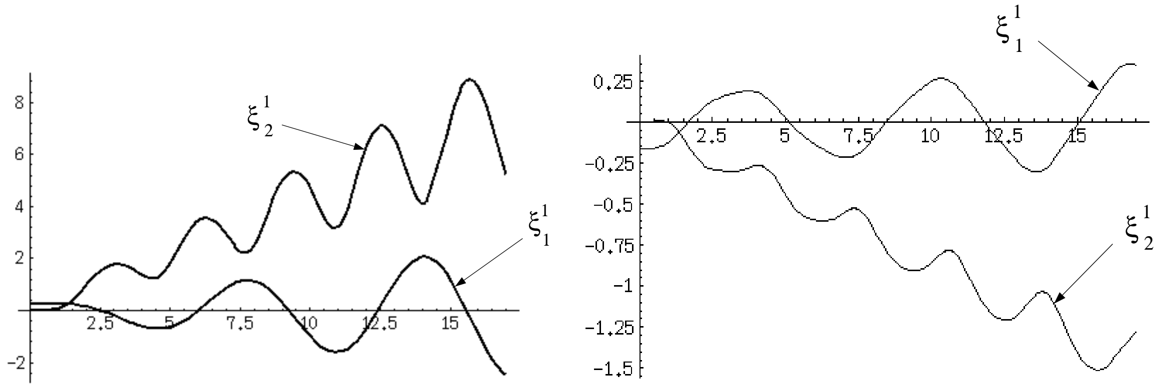
(a1) $\theta_{1,2}(0) = \pi$ (a2) $\theta_{1,2}(0) = 0$

(a) $\alpha_{1,2} = 0.1$



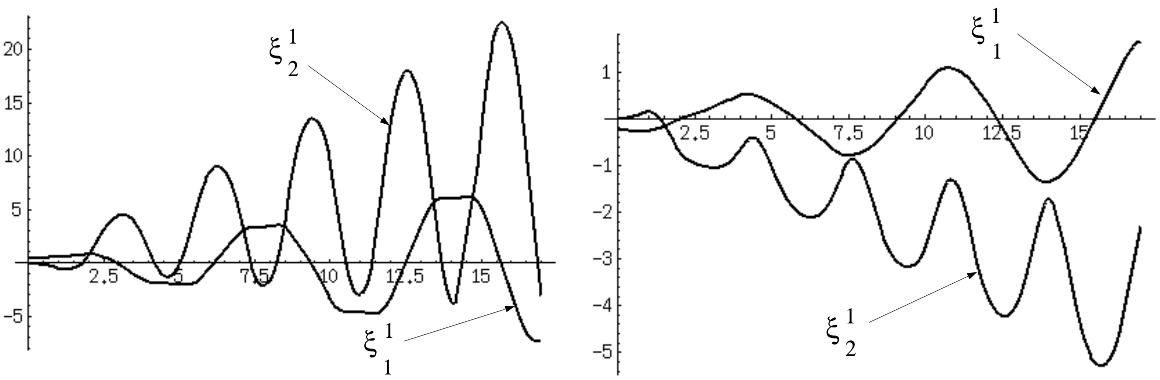
(b1) $\theta_{1,2}(0) = \pi$ (b2) $\theta_{1,2}(0) = 0$

(b) $\alpha_{1,2} = 0.5$



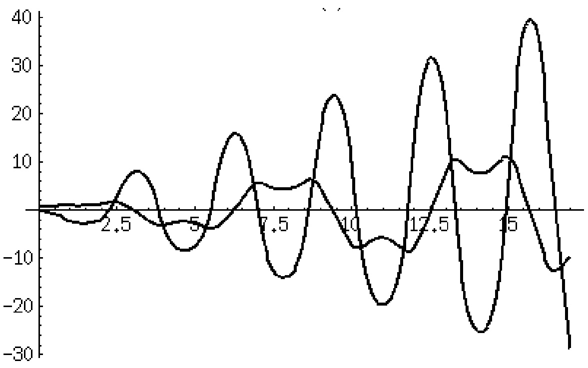
(c1) $\theta_{1,2}(0) = \pi$ (c2) $\theta_{1,2}(0) = 0$

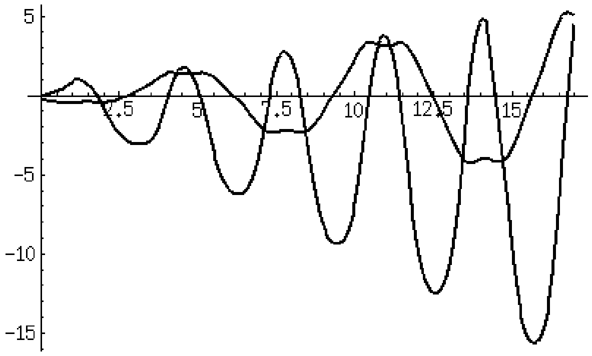
(c) $\alpha_{1,2} = 1.0$



(d1) $\theta_{1,2}(0) = \pi$ (d2) $\theta_{1,2}(0) = 0$

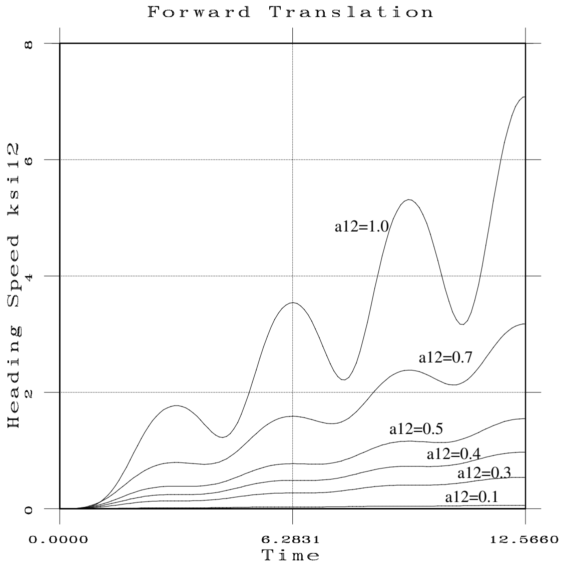
(d) $\alpha_{1,2} = 1.5$





(e1) $\theta_{1,2}(0) = \pi$ (e2) $\theta_{1,2}(0) = 0$

(e) $\alpha_{1,2} = 2.0$



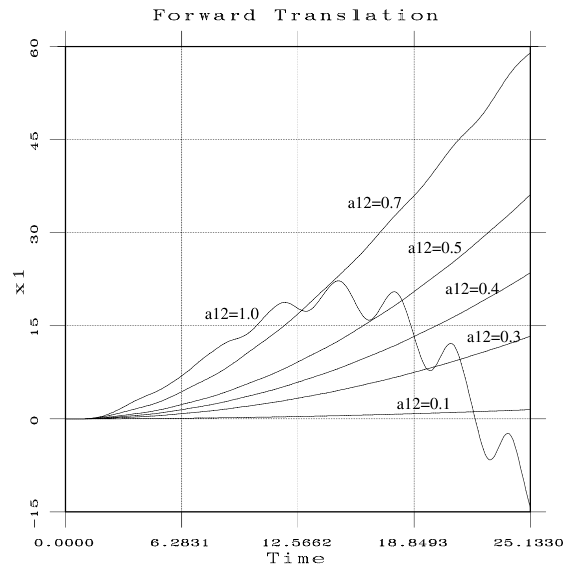
(f) $\alpha_{1,2} = 0.1$ to 1.0 for $\theta_{1,2}(0) = \pi$

Fig. 7.21: Effect of amplitude $\alpha_{1,2}$ on velocity ξ_1

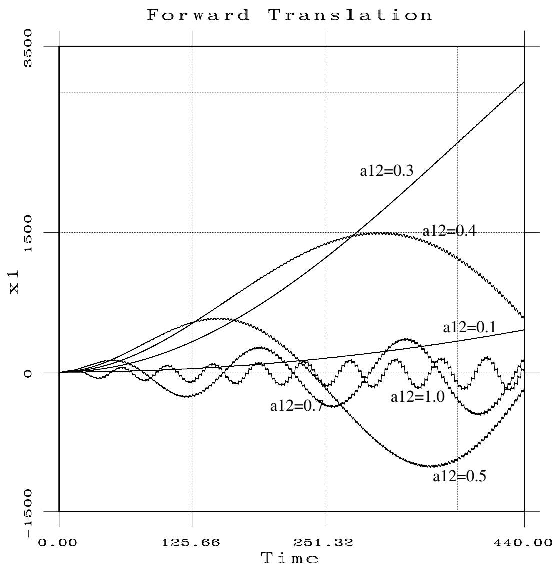
Fig. 7.22 shows the evolution of the group variables x_1, y_1, θ_1 for a forward translation of the system ($\theta_{1,2} = \pi$, $\omega_{1,2} = 1.0$) and for control oscillation amplitude $\alpha_{1,2}$ varying from 0.1 to 1.0.

Fig. 7.22.a shows the evolution of x_1, fig. 7.22.c shows this of y_1 and fig. 7.22.d shows this of θ_1 for a time duration of four time periods of the controls, while fig. 7.22.b shows the evolution of x_1 for a time duration of twenty time periods of the controls. It is obvious that y_1 and θ_1 merely oscillate around zero. This is not the case for x_1. Fig. 7.22.a shows that for short times (of 1–2 periods of the controls), the bigger $\alpha_{1,2}$ is, the bigger the system's forward motion. However, as can be seen in fig. 7.22.b, this is no longer true for longer time periods.

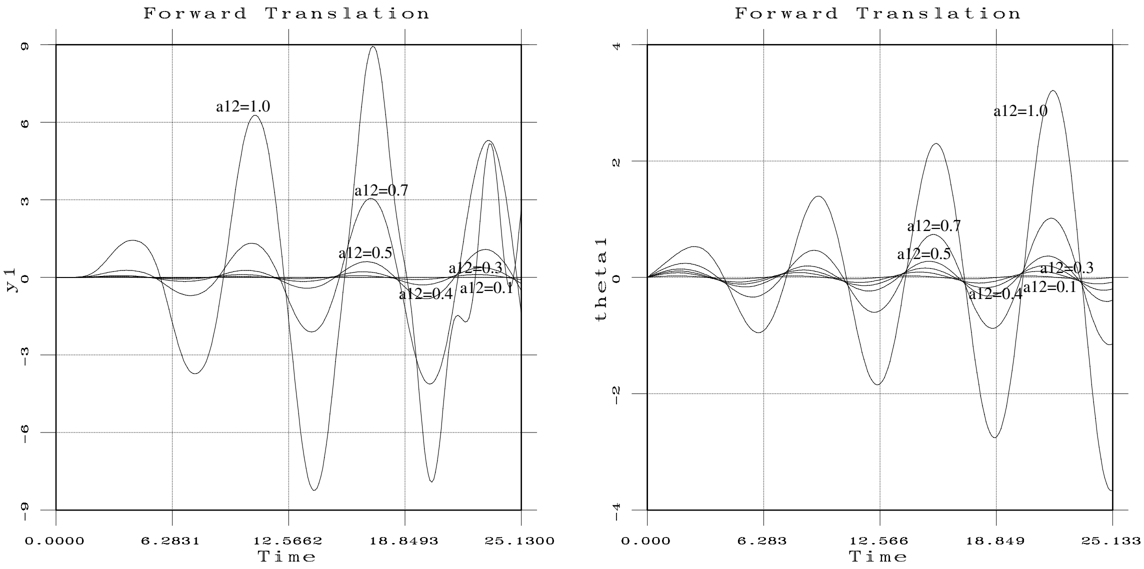
From the above it appears that small–amplitude motion gives forward translation without too much oscillation in the group variables, which closely approximates a straight–line motion.



(a) x_1 for 4 $T_{1,2}$



(b) x_1 for 20 $T_{1,2}$

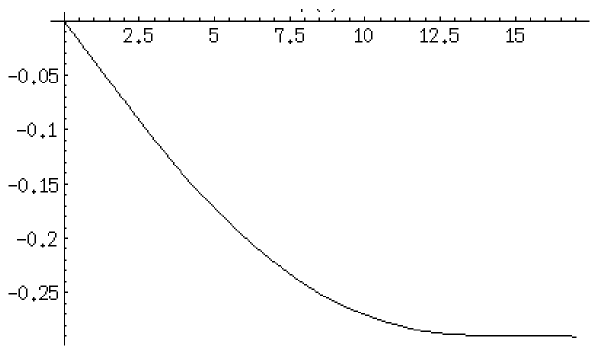


(c) y_1 for 4 $T_{1,2}$ (d) θ_1 for 4 $T_{1,2}$

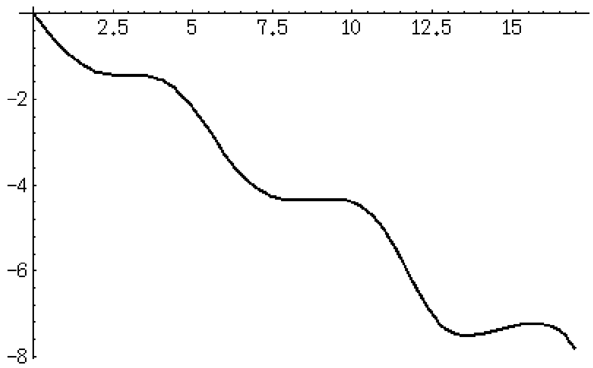
Fig. 7.22: Effect of amplitude $\alpha_{1,2}$ on x_1, y_1, θ_1

Let $\theta_{1,2}(0) = \pi$ and $\alpha_{1,2} = 1$ and consider the effect of the frequency $\omega_{1,2}$ on the nonholonomic momentum p (fig. 7.23). We vary the frequency from 0.1 Hz to 2 Hz.

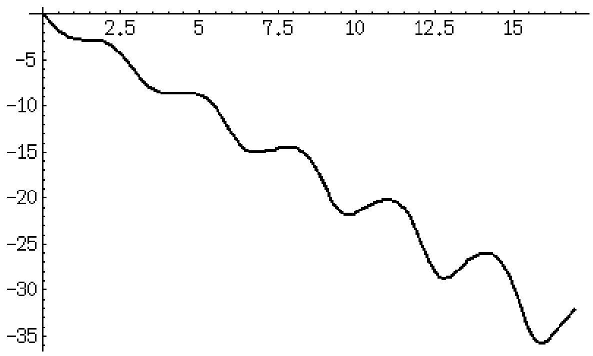
Figs. 7.23.e and 7.23.f show p for $\omega_{1,2} = 0.1$ and 1.0 for a time duration of two periods of the oscillatory controls ($2\ T_{1,2} = 40\ \pi$ and $4\ \pi$ *sec* respectively). Notice that this increase in $\omega_{1,2}$ produces a 10–fold increase in p.



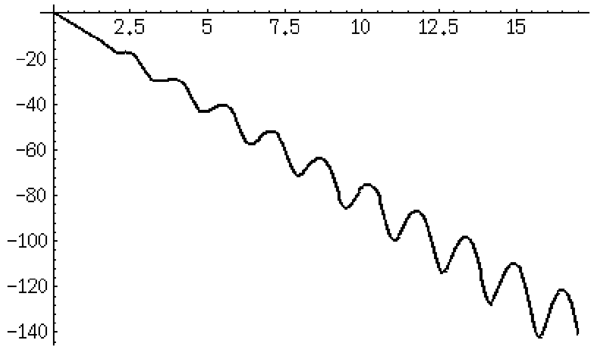
(a) $\omega_{1,2} = 0.1$



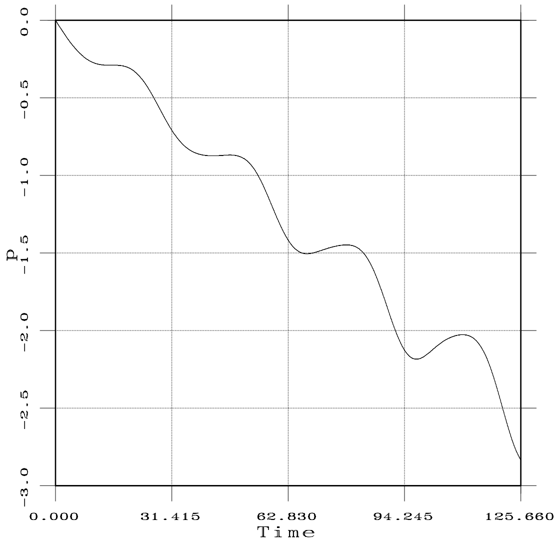
(b) $\omega_{1,2} = 0.5$



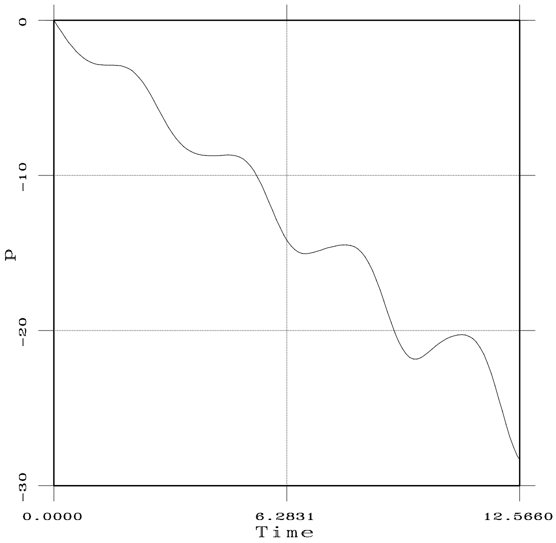
(c) $\omega_{1,2} = 1.0$



(d) $\omega_{1,2} = 2.0$



(e) $\omega_{1,2} = 0.1$

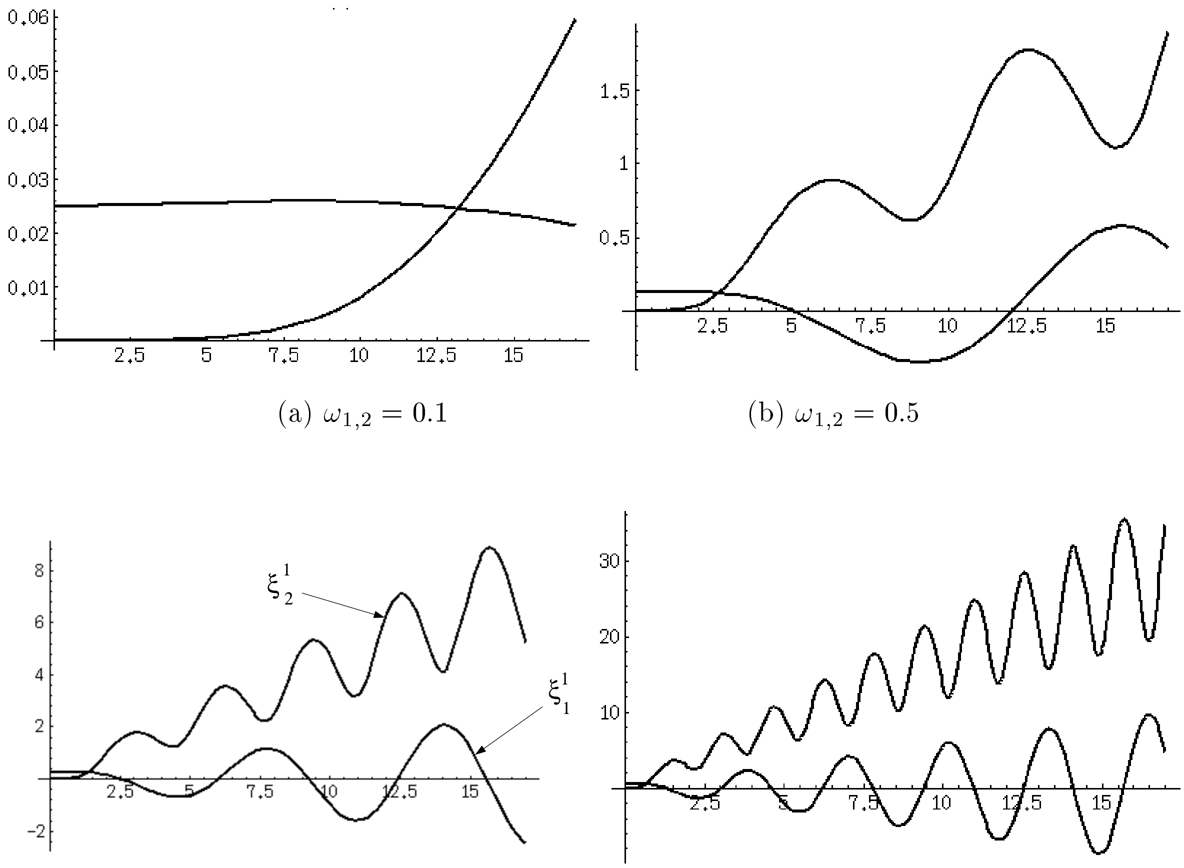


(f) $\omega_{1,2} = 1.0$

Fig. 7.23: Effect of frequency $\omega_{1,2}$ on p

Let $\theta_{1,2}(0) = \pi$ and $\alpha_{1,2} = 1$ and consider the effect of the frequency $\omega_{1,2}$ on the components ξ^1_1 and ξ^1_2 of the global velocity ξ_1. We vary the frequency from 0.1 to 2.0.

From fig. 7.24.a we observe that slow oscillations produce almost insignificant motion of the system. Faster oscillations increase substantially the average translational velocity of the system, but also its oscillation around zero.



(a) $\omega_{1,2} = 0.1$ (b) $\omega_{1,2} = 0.5$

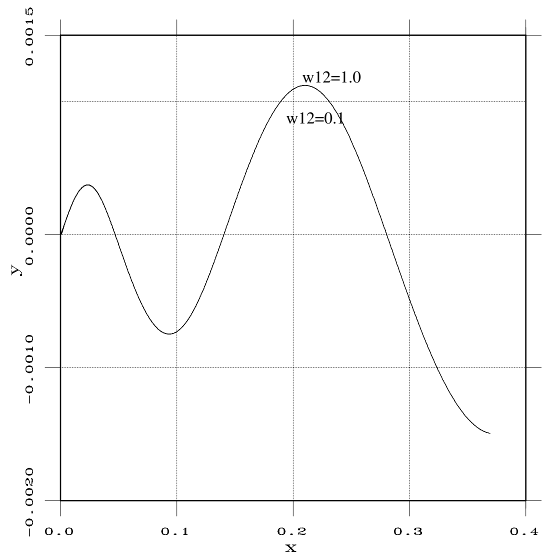
(c) $\omega_{1,2} = 1.0$ (d) $\omega_{1,2} = 2.0$

Fig. 7.24: Effect of frequency $\omega_{1,2}$ on ξ_1

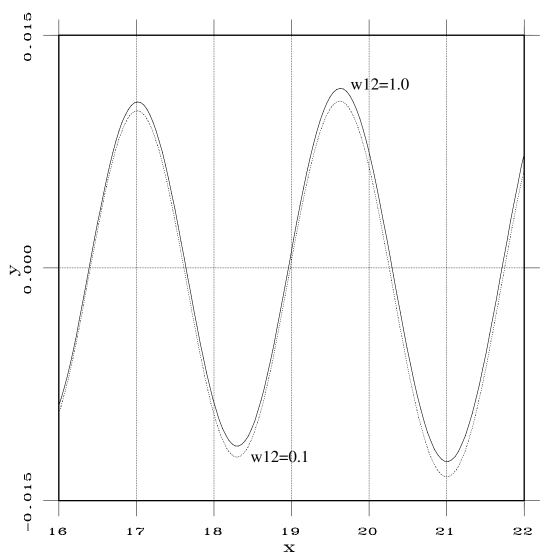
Let $\theta_{1,2}(0) = \pi$ and $\alpha_{1,2} = 0.1$ and consider the effect of the frequency $\omega_{1,2}$ on the group variables x_1, y_1, θ_1. We vary the frequency from 0.1 to 1.0 Hz.

Fig. 7.25 shows the (x, y)–trajectory of the system for $\omega_{1,2} = 0.1$, superimposed to the corresponding trajectory for $\omega_{1,2} = 1.0$, for a time duration of 2 $T_{1,2}$ (40 π and 4 π *sec* respectively). Fig. 25.a shows the initial part of the trajectory, where nonholonomic momentum is low, and fig. 25.b shows a later part of it, where nonholonomic momentum is higher. The trajectory corresponding to $\omega_{1,2} = 1.0$ appears as a solid line, while the one for $\omega_{1,2} = 0.1$ appears as a dotted line.

When nonholonomic momentum is low, the trajectories for $\omega_{1,2} = 1.0$ and $\omega_{1,2} = 0.1$ are geometrically almost the same (fig. 25.a); it is the time traversal of the trajectory that becomes faster as $\omega_{1,2}$ increases. However, as nonholonomic momentum increases, both the geometry of the trajectory and its time traversal become different (fig. 25.b).



(a) Low p



(b) High p

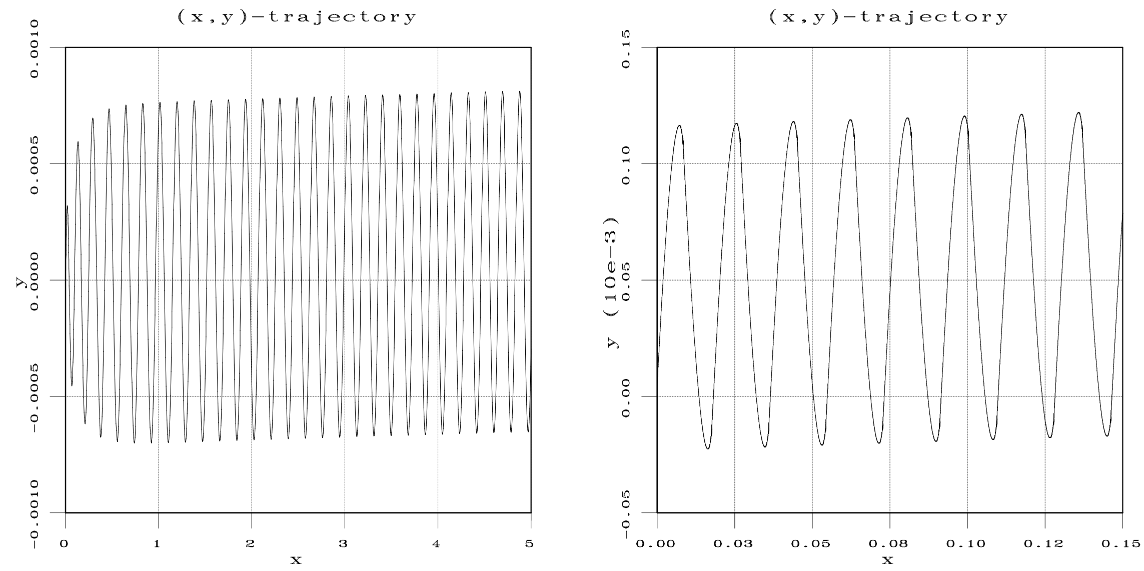
Fig. 7.25: Effect of frequency $\omega_{1,2}$ on (x, y)–trajectory

7.4 Model with Friction

We consider the Roller Racer model with friction (momentum equation (5.45)) with the following parameters (in addition to the ones mentioned earlier): $k_1 = k_2 = 0.01,\ R_1 = R_2 = 0.5,\ L_1 = 1, L_2 = 0.25$.

a. Forward Translation:

The control input (7.1) is considered with $\theta_{1,2}(0) = \pi$ and $\alpha_{1,2} = 0.1$. We use $\omega_{1,2} = 1.0$ in fig. 7.26.a and $\omega_{1,2} = 0.1$ in fig. 7.26.b.



(a) $\omega_{1,2} = 1$ (b) $\omega_{1,2} = 0.1$

Fig. 7.26: Model with Friction: Forward Translation

The nonholonomic momentum corresponding to the trajectory of fig. 7.26.a is shown in fig. 7.27. Comparing this with fig. 7.7.b, we observe that, contrary to the continuously increasing, on the average, p of fig. 7.7.b, here, each shape oscillation pumps just enough energy into the system to overcome friction. This is similar to the real system's behavior observed by the ISL prototypes.

b. Clockwise Rotation:

The control input (7.1) is considered with $\theta_{1,2}(0) = \theta_{1,2}^{r=0}$ and $\omega_{1,2} = 1$. We use $\alpha_{1,2} = 0.1$ in fig. 7.28 and $\alpha_{1,2} = 1.0$ in fig. 7.30. The nonholonomic momentum corresponding to the trajectory of fig. 7.28, is shown in fig. 7.29.

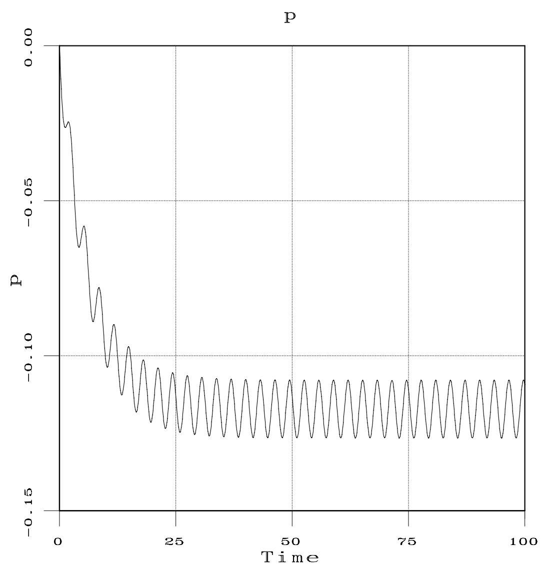
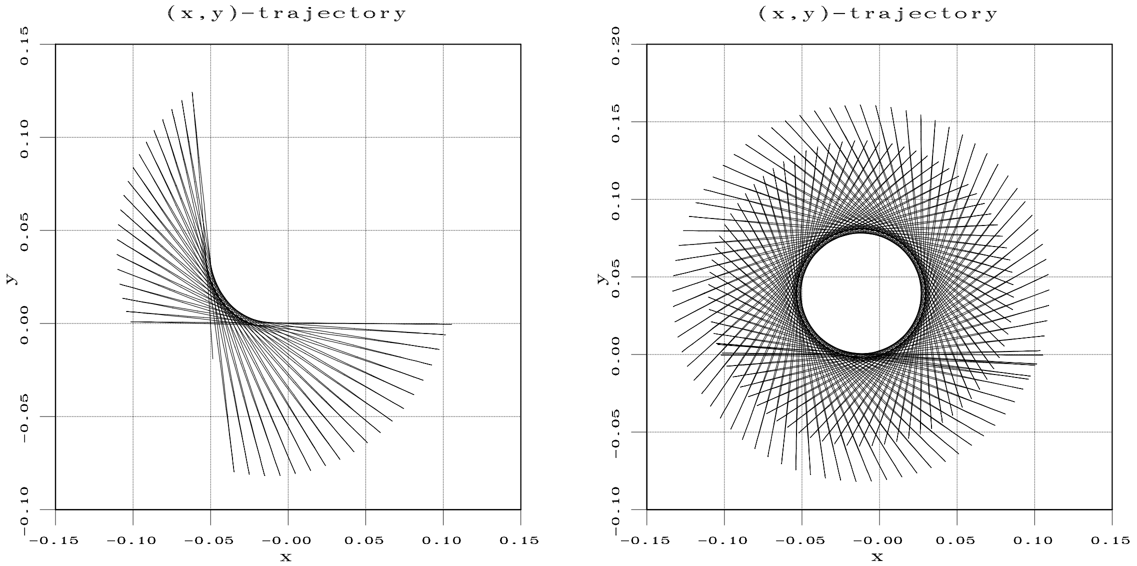


Fig. 7.27: Model with Friction: Forward Translation: Nonholonomic Momentum p



(a) Rotation by $\frac{\pi}{2}$ (b) Large Rotation

Fig. 7.28: Model with Friction: Rotation with $\alpha_{1,2} = 0.1$

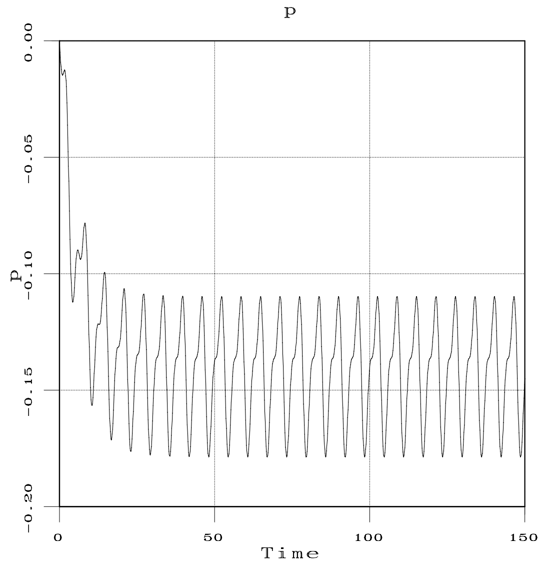


Fig. 7.29: Model with Friction: Clockwise Rotation: Nonholonomic Momentum p

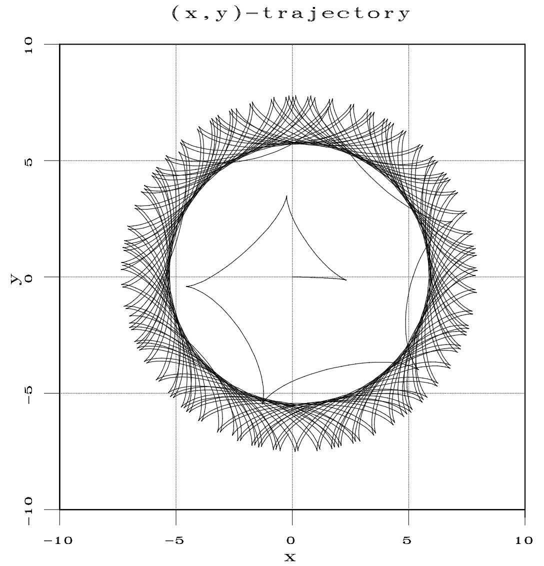


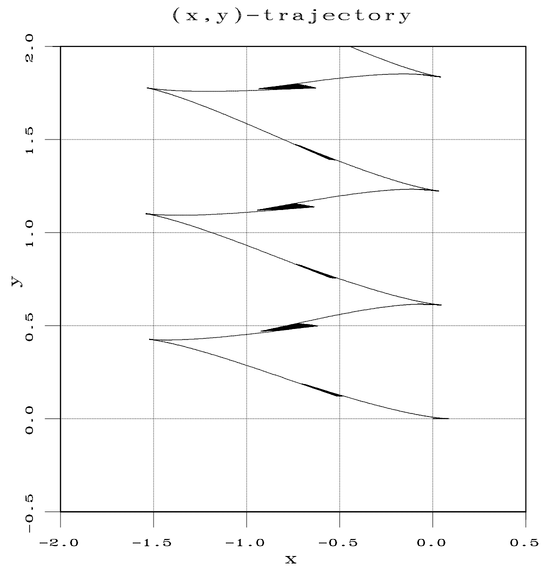
Fig. 7.30: Model with Friction: Large Rotation with $\alpha_{1,2} = 1$

c. Parallel Parking:

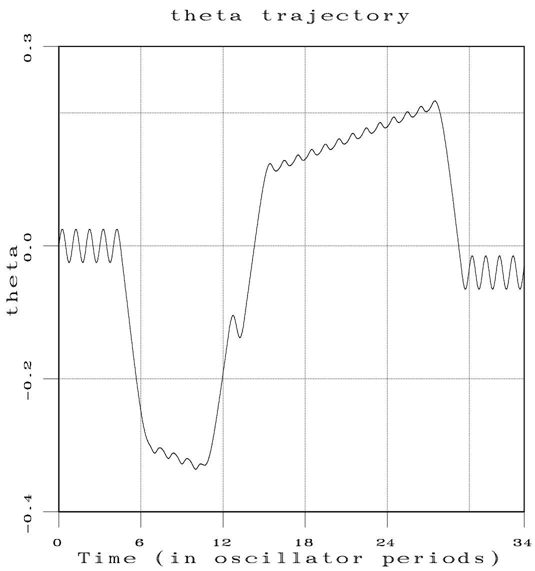
We set the friction coefficients to $k_1 = k_2 = 0.1$, leaving the rest of the parameters as before.

In order to create a "parallel parking" behavior, the idea is to generate a motion in the Lie–bracket direction by first translating forward, then rotating clockwise, then translating backwards and finally rotating counter–clockwise. The first step corresponds to a shape oscillation with average π, for a few periods, the second step corresponds to a shape oscillation with average $\theta_{1,2}^{r=0}$, the third step corresponds to a shape oscillation with average 0 and the final step corresponds to a shape oscillation with average $-\theta_{1,2}^{r=0}$. This sequence of shape controls is shown in fig. 7.32.a, where, starting with a basic shape oscillation of the type of equation (7.1) with amplitude $\alpha_{1,2} = 0.1$, frequency $\omega_{1,2} = 1.0$ and period $T_{1,2} = \frac{2\pi}{\omega_{1,2}}$, we reset its average as was described above. The whole cycle lasts 30 $T_{1,2}$, after which we restart at π (shown as $-\pi$ in fig. 7.32.a).

The system trajectory (x_1, y_1, θ_1) is shown in fig. 7.31, while the shape control $\theta_{1,2}$ and the corresponding nonholonomic momentum p are shown in fig. 7.32.

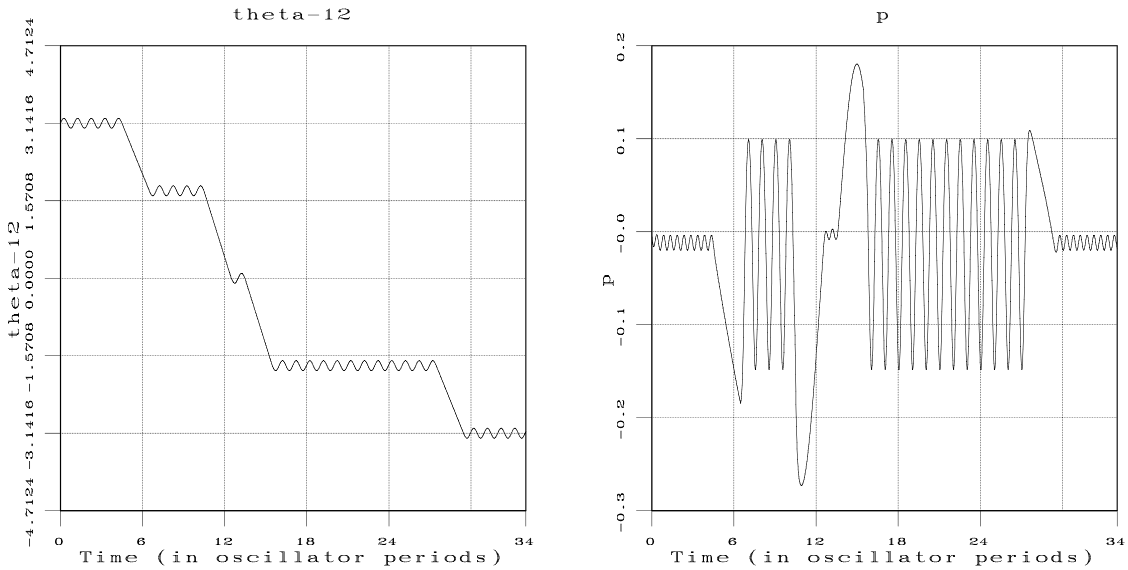


(a) (x_1, y_1)–trajectory



(b) θ_1–trajectory

Fig. 7.31: Model with Friction: Parallel Parking



(a) $\theta_{1,2}$–trajectory (b) p–trajectory

Fig. 7.32: Model with Friction: Parallel Parking

8 Conclusions

The present work is aimed at revealing some of the rich mathematical and physical structure associated with a specific mechanical system that is underactuated. Part of our fascination with this system derives from the drive to understand how it works at all! As shown in this paper, the interplay between the symmetries and the constraints is crucial to this understanding. Additionally, Lie algebraic analysis reveals both the capabilities and the limitations of such an underactuated system. The first draft of this paper was provided to the organizers of a workshop at the IEEE Conference on Decision and Control in Kobe in December 1996. (After the first draft of this paper was completed, in the Summer of 1997 we received a preprint *the Energy Momentum Method for the Stability of Nonholonomic Systems, by Zenkov, Bloch and Marsden* which investigates the stability of relative equilibria of the *unforced* Roller Racer as an application of a general theory of stability of nonholonomic systems.)

The present paper also explores via simulation certain motion control questions: specifically, controls for generating translation and curved motions, as well as parking maneuvers. The influence of dissipation is also considered in some detail. Much remains to be done to understand the problem of constructive control for the Roller Racer. Models of the type used here may prove to be of interest in understanding problems of locomotion in biology and in bio-mimetic robotic systems.

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