International Journal of Approximate Reasoning ••• (••••) •••–•••

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

A comprehensive study of argumentation frameworks with sets of attacking arguments

Giorgos Flouris [a], Antonis Bikakis [b]

[a] *Institute of Computer Science, Foundation for Research and Technology – Hellas (FORTH), N. Plastira 100, Vassilika Vouton, Heraklion, GR-70013, Greece*
[b] *Department of Information Studies, University College London (UCL), Gower Street, London WC1E 6BT, United Kingdom*

ARTICLE INFO

Article history:
Received 24 July 2018
Received in revised form 19 March 2019
Accepted 23 March 2019
Available online xxxx

Keywords:
Computational argumentation
Abstract argumentation frameworks
Labellings
Sets of attacking arguments
SETAF
Joint attacks
Acceptability semantics

ABSTRACT

It has been argued that Dung's classical Abstract Argumentation Framework (AAF) model is not appropriate for capturing "joint attacks", a feature that is inherent in several contexts and applications. The model proposed by Nielsen and Parsons in [1], often referred to as "framework with sets of attacking arguments" (SETAF), fills this gap by introducing joint attacks as a generalisation of the standard attack relationship of AAFs, thus constituting a faithful generalization of Dung's model. Building on that work, we provide a more complete characterization of these frameworks, which includes the treatment of various semantics not considered in the original publication, a more fine-grained representation of all acceptability semantics using labellings, and two functions allowing the transition between extensions and labellings along with their properties. Moreover, we show that a variety of well-known results that apply to AAF can be migrated to the SETAF setting. To further associate the two frameworks, we provide a natural way to represent a SETAF as a Dung-style AAF, and show how the generated AAF behaves.

© 2019 Elsevier Inc. All rights reserved.

1. Introduction

1.1. Motivation and problem statement

Abstract argumentation frameworks (AAF) constitute a simple and very successful model for representing arguments and their counter-arguments and for evaluating the acceptability of arguments based on their relationships with other arguments. AAFs were originally introduced by Dung [2], who defined their semantics in terms of extensions, and showed that many frameworks for non-monotonic reasoning, such as logic programming, default logic and others, are instances of abstract argumentation. Due to their generality and simplicity, AAFs have been applied to various domains such as law [3], medicine [4], social networks [5], the Web [6,7], surveillance systems [8], engineering design [9] and others, to support tasks such as inconsistency handling, non-monotonic reasoning, judgement aggregation, decision making, inter-agent communication, etc.

In a nutshell, an AAF is a directed graph, whose nodes correspond to arguments and whose edges correspond to attacks, which essentially represent the fact that a certain argument invalidates another. AAFs are given semantics through *extensions*, which are sets of arguments (nodes) that are non-conflicting (i.e., they do not attack each other) and, as a group, "shield" themselves from attack by other arguments (which are not in the extension). The exact formal meaning given to the

E-mail addresses: fgeo@ics.forth.gr (G. Flouris), a.bikakis@ucl.ac.uk (A. Bikakis).

https://doi.org/10.1016/j.ijar.2019.03.006
0888-613X/© 2019 Elsevier Inc. All rights reserved.

UNCORRECTED PROOF

term "shield" gives rise to a multitude of different semantics (complete, preferred, stable, etc.) which have been considered in the literature (e.g., see [10]).

A richer model for representing the semantics of abstract argumentation frameworks using labellings was proposed in [11] and was then used in [12] to describe the acceptability semantics defined by Dung. In that model, the arguments are classified into three groups, the `in`-arguments, `out`-arguments and `undec`-arguments, via a function called *labelling function*. Intuitively, `in`-arguments correspond to arguments that are accepted (i.e., are in the extension, in the classical sense described above), `out`-arguments are those that are invalidated by `in`-arguments (through an attack), whereas `undec`-arguments are "undecided", i.e., they are arguments that, even though there is no explicit attack by an `in`-argument rendering them invalid, their inclusion in the `in`-arguments would invalidate the "shielding" requirement mentioned above. The labelling semantics are more fine-grained, in the sense that they discriminate between different types of arguments that fail to be included in an extension into `out`-arguments and `undec`-arguments. This is important in several applications that try to "make sense" out of a debate, and need to understand what an agent should believe after the debate is over. In this sense, the difference between `out` and `undec` is analogous to the difference between believing that something is not true and not believing that something is true. In particular, `out`-arguments are rejected, whereas `undec`-arguments are simply not convincing enough to be incorporated in the agent's knowledge.

In a separate branch of research, it has been argued that it is necessary in many contexts to be able to represent and reason with "joint attacks", i.e., attacks where many arguments, as a group, attack a single one [1]. This should not be confused with the case where each argument attacks another individually; instead, the proposal of [1] refers to attacks by sets of attacking arguments. This essentially would turn the graph into a hypergraph, and constitutes a direct extension of Dung's work, where all original results of [2] can be recast in the richer model [1]. This richer model is often referred to as "*Framework with Sets of Attacking Arguments*" or, briefly, *SETAF*.

To comprehend the usefulness of the notion of "joint attacks", consider the following example from the legal domain (visualised in Fig. 1). According to the UK legislation about marriage and civil partnerships,[1] in the UK you are not allowed to marry or form a civil partnership with your partner in the following cases: (a) you are under 16; (b) you are closely related with your partner; (c) you are not single; (d) you are under 18 and you don't have the permission to marry from your parents or guardians. We can represent this part of the legislation as an AAF, where the right to get married is represented as an argument (M), which is attacked by four other arguments, each representing a different case from (a)–(d). The argument representing (d), however, is more complex that the rest of the arguments, as it combines two different statements ("you are under 18" and "you don't have the permission to marry from your parents or guardians"). Therefore, if we combine both statements into a single argument, we may run into trouble if either of the statements is used independently in other parts of the legislation, e.g., if acting as an argument by itself or if it is combined with other statements to form another argument.

Indeed, according to the legislation about voting,[2] you are not allowed to vote in the UK if you are under 18; in this case,"you are under 18" acts as an argument by itself against the eligibility to vote. Moreover, according to the legislation about alcohol and young people,[3] in the UK you are not allowed to drink alcohol in public if you are under 16, or if you are under 18 and you are either not accompanied by an adult or you are not having a meal. In the latter case, "you are under 18" is combined with one of two other statements to form an argument against the eligibility to drink alcohol in public. Using AAF, we would need to create a new argument for every different combination of simple statements that is needed to argue about something. This would also require specifying the associations among all these arguments, because, e.g., the validity of "being under 18" could affect the validity of "being under 18 and not having permission by parent or guardian", "being under 18 and not being accompanied by an adult", and "being under 18 and not having a meal". Furthermore, to represent some more complicated parts of the legislation involving many more conditions, we would need to create much more complex arguments that combine such conditions, making their understandability difficult. Indeed, Section 6 of this paper studies how organising arguments into sets can allow the representation of such cases in an AAF, albeit at an exponential cost (in terms of graph size) and a more complicated computation of the related extensions.

A framework supporting the notion of "joint attacks" provides a much more elegant and simple way to model this situation (see also Fig. 1). In particular, using SETAF, (d) could be represented as two separate arguments ($A18$ and NP), which jointly attack M. In the voting case, $A18$ by itself attacks V. In the alcohol case, $A18$ and NA, as well as $A18$ and NM jointly attack Alc.

Since its introduction, SETAF has already been used as the basis for supporting other useful features, such as higher-level attacks [13], evidence-based reasoning [14] and coalition formation in multiagent systems [15]. However, to the best of our knowledge, there has been no effort for a complete formal description of SETAF, both in terms of the different kinds of extensions that have been used to describe AAF (apart from the semantics described by Dung himself in [2], which was done in [1]), and in terms of the more fine-grained labelling-based semantics. Also, the relationship between AAF and SETAF has only been recently considered [16] under specific assumptions.

[1] https://www.gov.uk/marriages-civil-partnerships.
[2] https://www.gov.uk/elections-in-the-uk.
[3] https://www.gov.uk/alcohol-young-people-law.

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

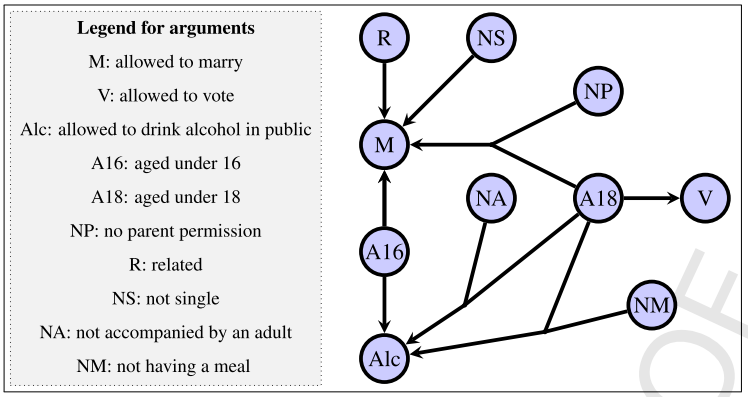

Fig. 1. Example of a SETAF, encoding a part of UK legislation.

Table 1
References to definitions of extension- and labelling-based semantics for abstract argumentation frameworks with single attacks and joint attacks. √ indicates that the corresponding definition is a novel contribution of the present paper.

Type of semantics	Dung's AAF [2]		SETAF [1]	
	Extensions	Labellings	Extensions	Labellings
Conflict-free	[2]	[12]	[1]	√
Admissible	[2]	[12]	[1]	√
Complete	[2]	[12]	[1]	√
Grounded	[2]	[12]	[1]	√
Preferred	[2]	[12]	[1]	√
Stable	[2]	[12]	[1]	√
Naive	[17]	√	√	√
Semi-stable	[18]	[12]	√	√
Eager	[19]	[20]	√	√
Ideal	[21]	[22]	√	√
Stage	[23]	[22]	√	√

1.2. Contributions and paper summary

In this paper, we provide a complete formal characterization of SETAF. Although our main focal point was the SETAF case, most of our results also apply for Dung's AAF as special cases, some of them being novel or alternative characterisations of known notions, thereby completing the picture in certain areas. Specifically:

- We introduce definitions for extension semantics that have been defined for AAF, but not for SETAF, i.e., naive, semi-stable, eager, ideal and stage semantics (Section 3).
- We introduce labelling-based definitions for all the semantics that were originally defined in [1] using extensions (i.e., conflict-free, admissible, complete, preferred, grounded and stable semantics), as well as for the new SETAF semantics defined in this paper (Section 4). Table 1 provides an overview of the semantics considered in this paper. For each semantics, the table either provides a reference to the work that the semantics was originally defined, or an indication that its definition for the specific setting is an original contribution of this paper.
- We show the association between extensions and labellings for all the SETAF semantics considered in this paper, in a similar way that [12] described the same association for AAFs (Section 5). Apart from their value with regard to understanding SETAFs, these results also apply for AAFs, and consider more types of semantics than the original work in [12]. Furthermore, we introduce a novel perspective to the problem, by studying the concept of proper labellings, thereby giving a more complete answer to the question of associating extensions and labellings (in both AAFs and SETAFs).
- We show how a SETAF can be modelled as an AAF, with an exponential increase in the number of arguments considered, and how the extensions of the two frameworks relate to each other (Section 6). Our approach circumvents the limitation formally proven in [16], where it was shown that SETAFs are strictly more expressive than AAFs when considering a fixed set of arguments.
- We present results on the inclusion relationships and the multiplicity of extensions and labellings for the different SETAF semantics, inspired by similar results on AAFs (Section 7).

The proofs for all new results are presented in Appendix A. Section 8 summarises the results of our study and discusses the related work.

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

2. Setting the groundwork: AAF and SETAF

It is well-known that AAFs constitute a very strong and versatile formalism. An AAF was defined in [2] as a pair $AF^D = \langle \mathbf{Args}, \rightarrowtail \rangle$ consisting of a (possible infinite) set of arguments **Args** and a binary attack relation $\rightarrowtail$ on this set. The same paper, as well as subsequent ones in the same area, defined different acceptability semantics for AAFs in terms of extensions, i.e., sets of arguments that can be considered acceptable. Table 1 presents 11 prominent semantics for AAFs and the papers that they were originally defined. Informally, $\mathbf{A} \subseteq \mathbf{Args}$ is: (i) a *conflict-free* extension of AF^D iff it contains no arguments attacking each other; (ii) an *admissible* extension iff it is conflict-free and defends all its elements (i.e., for each argument $b \in \mathbf{Args}$ attacking an argument in **A**, there is an argument in **A** counter-attacking b); (iii) a *complete* extension iff it is admissible and contains all the arguments it defends; (iv) a *grounded* extension iff it is minimal (w.r.t. set inclusion) among the complete extensions; (v) a *preferred* extension iff it is maximal among the complete extensions; (vi) a *stable* extension iff it is conflict-free and attacks all the arguments that it does not contain (i.e., all arguments in $\mathbf{Args} \setminus \mathbf{A}$); (vii) a *naive* extension iff it is maximal among the conflict-free extensions; $(viii)$ a *semi-stable* extension iff its union with the set of arguments it attacks is maximal among the complete extensions; (ix) an *eager* extension iff it is maximal among the complete extensions that are subsets of every semi-stable extension; (x) an *ideal* extension iff it is maximal among the complete extensions that are subsets of every preferred extension; and (xi) a *stage* extension iff its union with the set of arguments it attacks is maximal among the conflict-free extensions.

Dung's definition for argumentation frameworks was extended in [1] for the case where an argument can be attacked by a set of other arguments:

Definition 2.1 *(Definition 1 of [1]).* A *Framework with Sets of Attacking Arguments* (*SETAF* for short) is a pair $AF^S = \langle \mathbf{Args}, \triangleright \rangle$ such that **Args** is a set of arguments and $\triangleright \subseteq (2^{\mathbf{Args}} \setminus \{\emptyset\}) \times \mathbf{Args}$ is the *attack* relation.

It is interesting to note the asymmetry in the above definition of attack: a group of arguments can be the attacker, but not the recipient of an attack. The reason for this asymmetry is justified in [1], where it is shown that allowing a set of arguments to be jointly attacked by another does not add to the expressiveness of the proposed model. The same paper, however, uses the term "attack" also to describe the relationship between two sets of arguments, **A** and **B**, such that a subset **C** of **A** attacks an argument $b \in \mathbf{B}$. To avoid confusion, for describing such relationships we define here a new relation, $\blacktriangleright \subseteq (2^{\mathbf{Args}} \setminus \emptyset) \times (2^{\mathbf{Args}} \setminus \emptyset)$, such that $\mathbf{A} \blacktriangleright \mathbf{B}$ iff there exist $\mathbf{C} \subseteq \mathbf{A}, b \in \mathbf{B}$ such that $\mathbf{C} \triangleright b$.

We write $\mathbf{A} \not\triangleright b$ when it is not the case that $\mathbf{A} \triangleright b$, and $\mathbf{A} \not\blacktriangleright \mathbf{B}$ when it is not the case that $\mathbf{A} \blacktriangleright \mathbf{B}$. For singleton sets, we often write $\mathbf{A} \blacktriangleright b$ to denote $\mathbf{A} \blacktriangleright \{b\}$. We say that **A** *defends* an argument b from a set of arguments $\mathbf{A}'$ that attacks b, iff $\mathbf{A} \blacktriangleright \mathbf{A}'$.

3. Semantics for SETAF through extensions

In this section we provide definitions for the different acceptability semantics that were originally defined for AAFs (as informally described in Section 2) for the case of SETAF. Definitions 3.1–3.6 have been adopted from [1], whereas Definitions 3.7-3.11 constitute novel contributions of this paper and adapt, for the SETAF formalism, definitions of semantics originally proposed for the AAF case in various works.

In all the following definitions, we consider a fixed SETAF $AF^S = \langle \mathbf{Args}, \triangleright \rangle$ and a set of arguments $\mathbf{A} \subseteq \mathbf{Args}$.

Definition 3.1 *(Definition 2 of [1]).* **A** is said to be *conflict-free* iff it does not attack itself. Formally, **A** is conflict-free iff $\mathbf{A} \not\blacktriangleright \mathbf{A}$.

Definition 3.2 *(Definition 3 of [1]).* An argument $a \in \mathbf{Args}$ is said to be *acceptable* with respect to **A**, iff **A** defends a from all attacking sets of arguments in **Args**. Formally, a is acceptable with respect to **A** iff $\mathbf{A} \blacktriangleright \mathbf{B}$ for all $\mathbf{B} \subseteq \mathbf{Args}$ such that $\mathbf{B} \blacktriangleright a$. **A** is said to be *admissible* iff it is conflict-free and each argument in **A** is acceptable with respect to **A**. Formally, **A** is admissible iff $\mathbf{A} \not\blacktriangleright \mathbf{A}$ and $\mathbf{A} \blacktriangleright \mathbf{B}$ for all $\mathbf{B} \subseteq \mathbf{Args}$ such that $\mathbf{B} \blacktriangleright \mathbf{A}$.

Definition 3.3 *(Definition 8 of [1]).* An admissible set **A** is called a *complete extension* of AF^S, iff all arguments that are acceptable with respect to **A** are in **A**. Formally, **A** is a complete extension of AF^S iff all the following conditions hold: (a) $\mathbf{A} \not\blacktriangleright \mathbf{A}$; (b) $\mathbf{A} \blacktriangleright \mathbf{B}$ for all $\mathbf{B} \subseteq \mathbf{Args}$ such that $\mathbf{B} \blacktriangleright \mathbf{A}$; (c) If, for some $a \in \mathbf{Args}$, $\mathbf{A} \blacktriangleright \mathbf{B}$ for all $\mathbf{B} \subseteq \mathbf{Args}$ such that $\mathbf{B} \blacktriangleright a$, then $a \in \mathbf{A}$.

Definition 3.4 *(Definition 4 and Theorem 2 of [1]).* **A** is called a *preferred extension* of AF^S, iff it is a complete extension and there is no other complete extension $\mathbf{A}'$ such that $\mathbf{A} \subset \mathbf{A}'$.

Definition 3.5 *(Definition 7 and Theorem 2 of [1]).* **A** is called a *grounded extension* of AF^S, iff it is a complete extension and there is no other complete extension $\mathbf{A}'$ such that $\mathbf{A}' \subset \mathbf{A}$.

Definition 3.6 *(Definition 5 of [1]).* **A** is called a *stable extension* of AF^S, iff it is conflict-free and attacks all arguments in $\mathbf{Args} \setminus \mathbf{A}$.

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

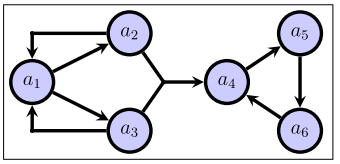

Fig. 2. SETAF for Example 3.12.

Table 2
Extensions for the SETAF of Fig. 2.

Extension type	Extensions
Conflict-free	$\emptyset$, $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, $\{a_4\}$, $\{a_5\}$, $\{a_6\}$, $\{a_1, a_4\}$, $\{a_1, a_5\}$, $\{a_1, a_6\}$, $\{a_2, a_3\}$, $\{a_2, a_3, a_5\}$, $\{a_2, a_3, a_6\}$, $\{a_2, a_4\}$, $\{a_2, a_5\}$, $\{a_2, a_6\}$, $\{a_3, a_4\}$, $\{a_3, a_5\}$, $\{a_3, a_6\}$
Admissible	$\emptyset$, $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, $\{a_2, a_3\}$, $\{a_2, a_3, a_5\}$
Complete	$\emptyset$, $\{a_1\}$, $\{a_2, a_3, a_5\}$
Preferred	$\{a_1\}$, $\{a_2, a_3, a_5\}$
Grounded	$\emptyset$
Stable	$\{a_2, a_3, a_5\}$
Naive	$\{a_1, a_4\}$, $\{a_1, a_5\}$, $\{a_1, a_6\}$, $\{a_2, a_4\}$, $\{a_2, a_6\}$, $\{a_3, a_4\}$, $\{a_3, a_6\}$, $\{a_2, a_3, a_5\}$
Semi-stable	$\{a_2, a_3, a_5\}$
Eager	$\{a_2, a_3, a_5\}$
Ideal	$\emptyset$
Stage	$\{a_2, a_3, a_5\}$

Definition 3.7 *(adapted from [17]).* **A** is called a *naive extension* of AF^S, iff it is conflict-free and is maximal w.r.t. set inclusion among the conflict-free subsets of **Args**.

Definition 3.8 *(adapted from [18]).* **A** is called a *semi-stable extension* of AF^S, iff it is a complete extension and the set $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal w.r.t. set inclusion among all complete extensions of AF^S.

Definition 3.9 *(adapted from [19]).* **A** is called an *eager extension* of AF^S, iff it is a maximal (with respect to set inclusion) complete extension that is a subset of each semi-stable extension of AF^S.

Definition 3.10 *(adapted from [21]).* **A** is called an *ideal extension* of AF^S, iff it is a maximal (with respect to set inclusion) complete extension that is a subset of each preferred extension of AF^S.

Definition 3.11 *(adapted from [23]).* **A** is called a *stage extension* of AF^S, iff it is conflict free and $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal among all conflict-free subsets of **Args**.

The following example helps to illustrate the concept of extensions and the intuition behind the different semantics:

Example 3.12. Consider the SETAF shown in Fig. 2. Its various extensions are shown in Table 2. Let us consider in more detail the complete extensions, which are: $\emptyset$, $\{a_1\}$, $\{a_2, a_3, a_5\}$. Note that, for example, $\{a_2, a_3\}$ is admissible and conflict-free but not complete, because it leaves out a_5, which is acceptable with respect to $\{a_2, a_3\}$. Similarly, $\{a_1, a_2\}$ is not a complete extension because it is not conflict-free, whereas $\{a_5\}$ and $\{a_1, a_5\}$ are not complete extensions because they are not admissible (a_5 is not acceptable with respect to the corresponding set in either case).

In the following, we use shorthands to refer to the various semantics and extension types, in particular: **cf** for conflict-free, **ad** for admissible, **co** for complete, **pr** for preferred, **gr** for grounded, **st** for stable, **na** for naive, **se** for semi-stable, **ea** for eager, **id** for ideal and **sg** for stage. We also use σ as a catch-all variable to indicate any of these semantics. For example, we write **co**-extension to refer to a complete extension, and σ-extension to refer to an extension of the type denoted by σ.

4. Semantics for SETAF through labellings

Here we introduce a different characterisation of acceptability semantics for SETAF, using the notion of labellings. The following definitions are similar to the corresponding definitions of [12,20,22], but adapted to SETAF. We start our analysis by defining labellings in general:

Definition 4.1 *(adapted from [12]).* Consider a SETAF $AF^S = \langle \mathbf{Args}, \triangleright \rangle$. A *labelling* for AF^S is a total function $\lambda : \mathbf{Args} \mapsto \{\mathtt{in}, \mathtt{out}, \mathtt{undec}\}$.

UNCORRECTED PROOF

Table 3
Complete extensions and complete labellings for the SETAF of Fig. 2.

Complete extensions	Complete labellings
$\mathbf{A_1} = \emptyset$	$\lambda_1(a_1) = \texttt{undec}$, $\lambda_1(a_2) = \texttt{undec}$, $\lambda_1(a_3) = \texttt{undec}$, $\lambda_1(a_4) = \texttt{undec}$, $\lambda_1(a_5) = \texttt{undec}$, $\lambda_1(a_6) = \texttt{undec}$
$\mathbf{A_2} = \{a_1\}$	$\lambda_2(a_1) = \texttt{in}$, $\lambda_2(a_2) = \texttt{out}$, $\lambda_2(a_3) = \texttt{out}$, $\lambda_2(a_4) = \texttt{undec}$, $\lambda_2(a_5) = \texttt{undec}$, $\lambda_2(a_6) = \texttt{undec}$
$\mathbf{A_3} = \{a_2, a_3, a_5\}$	$\lambda_3(a_1) = \texttt{out}$, $\lambda_3(a_2) = \texttt{in}$, $\lambda_3(a_3) = \texttt{in}$, $\lambda_3(a_4) = \texttt{out}$, $\lambda_3(a_5) = \texttt{in}$, $\lambda_3(a_6) = \texttt{out}$

Note that the labellings of a SETAF are defined over arguments (just like in [12]), not sets of arguments. For simplicity, for a given (fixed) label λ, we use the terms in-argument, out-argument, undec-argument to denote arguments that are labelled in, out, undec respectively. We also write $\texttt{in}(\lambda)$, $\texttt{out}(\lambda)$ and $\texttt{undec}(\lambda)$ to denote, respectively, the sets of in-, out- and undec-arguments with respect to λ.

We now define various special classes of labellings (i.e., labelling semantics):

Definition 4.2 *(adapted from [12,20,22]).* Let $AF^S = \langle \mathbf{Args}, \triangleright \rangle$ be a SETAF. A labelling $\lambda : \mathbf{Args} \mapsto \{\texttt{in}, \texttt{out}, \texttt{undec}\}$ of AF^S is called:

1. *conflict-free* iff for all $a \in \mathbf{Args}$:
   (a) if $\lambda(a) = \texttt{in}$ then $\texttt{in}(\lambda) \not\blacktriangleright a$
   (b) if $\lambda(a) = \texttt{out}$ then $\texttt{in}(\lambda) \blacktriangleright a$
2. *admissible* iff for all $a \in \mathbf{Args}$:
   (a) if $\lambda(a) = \texttt{in}$ then $\forall \mathbf{B} \blacktriangleright a, \exists b \in \mathbf{B} : \lambda(b) = \texttt{out}$
   (b) if $\lambda(a) = \texttt{out}$ then $\texttt{in}(\lambda) \blacktriangleright a$
3. *complete* iff for all $a \in \mathbf{Args}$:
   (a) $\lambda(a) = \texttt{in}$ if and only if $\forall \mathbf{B} \blacktriangleright a, \exists b \in \mathbf{B} : \lambda(b) = \texttt{out}$
   (b) $\lambda(a) = \texttt{out}$ if and only if $\texttt{in}(\lambda) \blacktriangleright a$
4. *preferred* iff it is complete and $\texttt{in}(\lambda)$ is maximal w.r.t. set inclusion among all complete labellings of AF^S
5. *grounded* iff it is complete and $\texttt{in}(\lambda)$ is minimal w.r.t. set inclusion among all complete labellings of AF^S
6. *stable* iff it is conflict-free and $\texttt{undec}(\lambda) = \emptyset$
7. *naive* iff it is conflict-free and $\texttt{in}(\lambda)$ is maximal w.r.t. set inclusion among the conflict-free labellings of AF^S
8. *semi-stable* iff it is complete and $\texttt{undec}(\lambda)$ is minimal w.r.t. set inclusion among all complete labellings of AF^S
9. *eager* iff
   a. it is complete
   b. $\texttt{in}(\lambda) \subseteq \texttt{in}(\lambda')$ for every semi-stable labelling λ' of AF^S
   c. $\texttt{in}(\lambda)$ is maximal w.r.t. set inclusion among all labellings of AF^S satisfying conditions (a), (b)
10. *ideal* iff
    a. it is complete
    b. $\texttt{in}(\lambda) \subseteq \texttt{in}(\lambda')$ for every preferred labelling λ' of AF^S
    c. $\texttt{in}(\lambda)$ is maximal w.r.t. set inclusion among all labellings of AF^S satisfying conditions (a), (b)
11. *stage* iff it is conflict-free and $\texttt{undec}(\lambda)$ is minimal w.r.t. set inclusion among all conflict-free labellings of AF^S

In the following, we use the same shorthands that we introduced for extensions (**cf**, **ad**, **co**, etc.) to refer also to labellings (e.g., a **co**-labelling is a complete labelling).

Example 4.3. Continuing Example 3.12, Table 3 shows the complete labellings that correspond to the SETAF of Fig. 2. Comparing complete extensions with complete labellings, we see that, e.g., the third labelling explicitly rejects a_6 (because it is attacked by a_5, which is accepted), but the second one makes no explicit decision on a_6, as the agent cannot make up its mind on how to resolve the cyclic attack among a_4, a_5, a_6. This distinction cannot be made with the corresponding complete extensions (first column of Table 3). Regarding the other semantics: λ_1 is the unique grounded and ideal labelling, λ_2 and λ_3 are preferred labellings, and λ_3 is a stable, semi-stable, stage and eager labelling. By comparing these with the extensions presented in Table 2, one can see that the results obtained with labellings seem to correspond exactly to those obtained with the extension-based approach. We further study this phenomenon in Section 5.

UNCORRECTED PROOF

5. Relating extensions and labellings in SETAF

5.1. From extensions to labellings and vice-versa

As already mentioned, for any given SETAF, there is a way to go from sets of arguments (i.e., extensions) to labellings and vice-versa. Before formalising this transition process, let us introduce some terminology. Given a SETAF $AF^S = \langle \mathbf{Args}, \triangleright \rangle$, we denote by $\mathcal{E}^{AF^S}$ the power-set of **Args** (i.e., all possible extensions, $\mathcal{E}^{AF^S} = 2^{\mathbf{Args}}$), and by $\mathcal{L}^{AF^S}$ the set of all possible labellings that can be defined over AF^S. When the SETAF is irrelevant or obvious from the context, we use the simpler notations $\mathcal{E}, \mathcal{L}$ respectively. We can now define the following functions for formalising the transition process:

Definition 5.1. Consider a SETAF $AF^S = \langle \mathbf{Args}, \triangleright \rangle$. We define the function $Lab : \mathcal{E} \mapsto \mathcal{L}$ such that, for $\mathbf{A} \in \mathcal{E}$, $\lambda = Lab(\mathbf{A})$:

- $\lambda(a) = \mathtt{in}$ for all $a \in \mathbf{A}$
- $\lambda(a) = \mathtt{out}$ for all $a \notin \mathbf{A}$, $\mathbf{A} \blacktriangleright a$
- $\lambda(a) = \mathtt{undec}$ for all $a \notin \mathbf{A}$, $\mathbf{A} \not\blacktriangleright a$

For some $\mathbf{A} \in \mathcal{E}$, we call $Lab(\mathbf{A})$ the *labelling generated by* **A**.

Definition 5.2. Consider a SETAF $AF^S = \langle \mathbf{Args}, \triangleright \rangle$. We define the function $Ext : \mathcal{L} \mapsto \mathcal{E}$ such that, for $\lambda \in \mathcal{L}$, $Ext(\lambda) = \{a \in \mathbf{Args} \mid \lambda(a) = \mathtt{in}\}$. For some $\lambda \in \mathcal{L}$, we call $Ext(\lambda)$ the *extension generated by* λ.

Clearly, $Ext(\lambda) = \mathtt{in}(\lambda)$ for all λ. Note that these definitions are very similar to the definition of *Ext2Lab* and *Lab2Ext* functions in Definition 8 of [12], although the domain of application here is different. Note also that both *Lab* and *Ext* are well-defined.

5.2. Properties of the transition from extensions to labellings

Now let's study the interaction between the two transition functions, *Lab* and *Ext*. In particular, we prove the conditions under which *Lab* is the inverse of *Ext* (and vice-versa), and whether these functions are injective, surjective and/or bijective.[4] In the following, we assume a fixed, but arbitrary, SETAF $AF^S = \langle \mathbf{Args}, \triangleright \rangle$.

We start our analysis by showing some properties of *Ext* and *Lab*:

Theorem 5.3. *The following are true:*

1. *Ext is surjective.*
2. *Lab is injective.*

Note that the reverse of Theorem 5.3 is not true (i.e., *Ext* is not injective and *Lab* is not surjective), not even in the more restrictive AAF case, as [12] showed.

Moreover, we can show that when a set of arguments **A** is transformed into a labelling ($\lambda = Lab(\mathbf{A})$), the information on **A** is not "lost", in the sense that we can revert the result back to the same set of arguments using *Ext* (i.e., $Ext(\lambda) = \mathbf{A}$). Formally:

Theorem 5.4. *For all* $\mathbf{A} \in \mathcal{E}$ *it holds that* $Ext(Lab(\mathbf{A})) = \mathbf{A}$.

Note that the reverse counterpart of Theorem 5.4 does not hold, i.e., in general, $Lab(Ext(\lambda)) \neq \lambda$. This is expected because labellings are more expressive than extensions, so one naturally expects $\mathcal{L}$ to be richer than $\mathcal{E}$. To complete the picture, we need the notion of *proper labellings*:

Definition 5.5. A labelling λ is called *proper* if and only if it holds that:

$$\lambda(a) = \mathtt{out} \text{ if and only if } \mathtt{in}(\lambda) \blacktriangleright a.$$

We denote by $\widehat{\mathcal{L}}$ the set of all proper labellings.

Proper labellings are important, because it can be shown that they correspond exactly to extensions. Indeed, the following is the counterpart of Theorem 5.4:

[4] Formally, a function $f : S_1 \mapsto S_2$ is called injective if and only if $f(x) = f(y)$ implies $x = y$; it is called surjective if and only if for any $y \in S_2$ there exists $x \in S_1$ such that $f(x) = y$; and it is called bijective if and only if it is both injective and surjective.

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

Theorem 5.6. *It holds that* $Lab(Ext(\lambda)) = \lambda$ *if and only if* $\lambda \in \widehat{\mathcal{L}}$.

Based on Theorems 5.4 and 5.6, we can easily prove the following important corollary, which essentially clarifies the connection between extensions and labellings, by showing that $\mathcal{E}$ and $\widehat{\mathcal{L}}$ are in fact isomorphic, i.e., there is a bijection between them:

Corollary 5.7. *The following points hold:*

1. $Ext\restriction_{\widehat{\mathcal{L}}}$ *is injective and surjective*
2. *Lab is injective and surjective in* $\widehat{\mathcal{L}}$
3. $(Ext\restriction_{\widehat{\mathcal{L}}})^{-1} = Lab$
4. $Lab^{-1} = Ext\restriction_{\widehat{\mathcal{L}}}$
5. $\widehat{\mathcal{L}}$ *is isomorphic to* $\mathcal{E}$

In Corollary 5.7, the notation $Ext\restriction_{\widehat{\mathcal{L}}}$ denotes the restriction of the function Ext in $\widehat{\mathcal{L}}$; formally, $Ext\restriction_{\widehat{\mathcal{L}}}$ is a function defined over $\widehat{\mathcal{L}}$, such that $Ext\restriction_{\widehat{\mathcal{L}}}(\lambda) = Ext(\lambda)$ for all $\lambda \in \widehat{\mathcal{L}}$. The -1 superscript is used to denote the inverse function, under the standard definition.

Now it remains to see which of the different labelling types introduced in Section 4 are proper. As expected (see [12]), complete labellings (and their subtypes, namely, grounded, preferred, stable, semi-stable, eager and ideal) are proper, whereas admissible, conflict-free and naive are not. Moreover, stage labellings are also proper:

Theorem 5.8. *For* $\sigma \in \{$**co**, **gr**, **pr**, **st**, **se**, **ea**, **id**, **sg**$\}$ *it holds that a* σ*-labelling is proper.*

The following counterexample shows that Theorem 5.8 does not apply to other types of semantics. Also, the same example shows that there are proper extensions that are neither complete nor stage:

Example 5.9. Consider a SETAF with two arguments a, b such that $\{a\} \blacktriangleright b$. Then:

- The labelling $\lambda(a) = \mathtt{in}$, $\lambda(b) = \mathtt{undec}$ is conflict-free, naive and admissible, but not proper.
- The labelling $\lambda(a) = \lambda(b) = \mathtt{undec}$ is conflict-free, admissible and proper, but neither complete nor stage.

Theorem 5.8 and Corollary 5.7, are the counterparts of Theorem 11 of [12] for the SETAF setting. In particular, we have shown that *Ext* and *Lab* are essentially two different ways to express the same thing, if we restrict ourselves to those semantics that are proper.

Note that, these results can be trivially applied to AAFs as well, since an AAF is also a SETAF. Under this viewpoint, our results are a direct extension of Theorem 11 of [12], handling more types of extensions, and identifying the important class of proper labellings, which provides a more complete answer with regard to the relation between $\mathcal{E}$ and $\mathcal{L}$.

5.3. Preservation of semantics during transitioning

The next step is to prove that each type of labelling corresponds to the respective type of extension and vice-versa. The following series of theorems prove these points.

Theorem 5.10. *Let* $AF^S = \langle$**Args**$, \triangleright\rangle$ *be a SETAF and* **A** $\subseteq$ **Args** *a* σ*-extension of* AF^S*, where* $\sigma \in \{$**cf**, **ad**, **co**, **pr**, **gr**, **st**, **na**, **se**, **ea**, **id**, **sg**$\}$*. Then, Lab*(**A**) *is a* σ*-labelling of* AF^S.

Theorem 5.11. *Let* $AF^S = \langle$**Args**$, \triangleright\rangle$ *be a SETAF and* λ : **Args** $\mapsto \{\mathtt{in}, \mathtt{out}, \mathtt{undec}\}$ *a* σ*-labelling of* AF^S*, where* $\sigma \in \{$**cf**, **ad**, **co**, **pr**, **gr**, **st**, **na**, **se**, **ea**, **id**, **sg**$\}$*. Then, Ext*(λ) *is a* σ*-extension of* AF^S.

The above theorems show that σ-labellings and σ-extensions are essentially analogous ways to define the semantics of a SETAF. For those types of labellings that are proper (i.e., for all considered semantics except **cf**, **ad**, **na**; see Theorem 5.8), these two ways are also isomorphic, as Corollary 5.7 shows. Note that most of these results are direct generalisations (for the SETAF case) of previous results that have appeared elsewhere for AAFs.

Example 5.12. Referring to Example 3.12 again (visualised in Fig. 2), and Table 3 (showing complete extensions and labellings), we can easily verify that: (a) the labellings can be generated through the corresponding extensions, using Definition 5.1; (b) the labellings are all complete labellings (under Definition 4.2); (c) the extensions could be generated from the labellings, using Definition 5.2.

UNCORRECTED PROOF

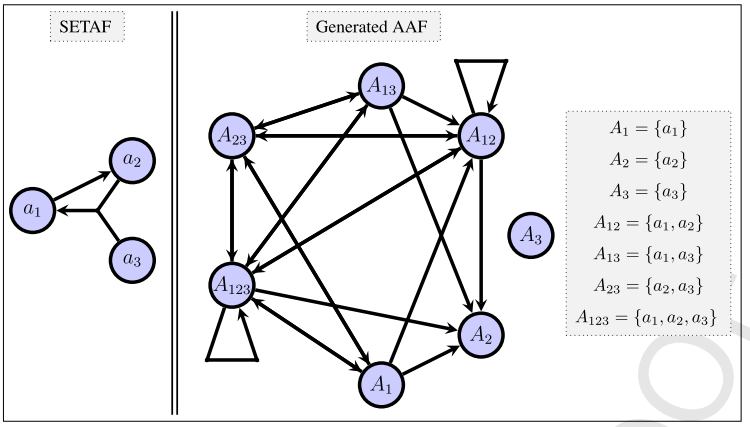

Fig. 3. A SETAF (left) and its generated AAF (right).

6. Associating SETAF with Dung-style argumentation

6.1. Generated argumentation frameworks

Since AAF is a special case of a SETAF, one would expect that the SETAF formalism is strictly more expressive than AAF, in the sense that one can create SETAF argumentation graphs, whose semantics (extensions) cannot be captured by any AAF. Recent work [16] has shown that this is indeed the case, under the important assumption that the set of arguments is fixed. In other words, [16] showed that, given a set of arguments, one can create a SETAF whose semantics cannot be captured by any AAF (using the same set of arguments).

In this section, we show that this limitation can be overcome when dropping this assumption, i.e., by considering a richer set of arguments for the AAF. More precisely, we show that, given a SETAF, one can generate an AAF and use it to compute the various extensions of the original SETAF; the caveat is that the AAF will be of exponentially bigger size, i.e., it will include an exponential number of arguments compared to the original SETAF.

In the rest of this section, the following notation will prove useful: given a set of arguments $\mathbf{A}$, we denote by $\overline{\mathbf{A}}$ the set consisting of all non-empty subsets of $\mathbf{A}$, i.e., $\overline{\mathbf{A}} = \{\mathbf{B} \mid \emptyset \subset \mathbf{B} \subseteq \mathbf{A}\}$.

The following definition shows one possible way to generate an AAF that represents the same information as a SETAF. In the rest of this section, we will study the relationship between the original SETAF and its generated AAF.

Definition 6.1. Consider a SETAF $AF^S = \langle \mathbf{Args}, \triangleright \rangle$. The AAF $AF^D = \langle \overline{\mathbf{Args}}, \rightarrowtail \rangle$, where $\mathbf{A} \rightarrowtail \mathbf{B}$ if and only if $\mathbf{A} \blacktriangleright \mathbf{B}$ is called the *generated AAF of the SETAF* AF^S.

Note that the arguments of the generated AAF are all the non-empty sets of arguments in the original SETAF, whereas the attack relation of the AAF ($\rightarrowtail$) is in fact equal with the generalised attack relation of the SETAF ($\blacktriangleright$); nevertheless, we chose to use a different symbol to avoid confusion in the discussion that follows. As a result of this transformation, a set of arguments (or a σ-extension) in the generated AAF essentially corresponds to a set of sets of arguments in the SETAF.

Example 6.2. Fig. 3 shows an example of application of Definition 6.1. As already mentioned, the arguments in the generated AAF are essentially all the subsets of arguments in the SETAF (for readability we use A_1 to denote $\{a_1\}$, A_{12} to denote $\{a_1, a_2\}$ etc.). As a result, extensions in the generated AAF are sets of sets of arguments, e.g., $\{A_2, A_3, A_{23}\}$ (which corresponds to $\{\{a_2\}, \{a_3\}, \{a_2, a_3\}\}$) is a complete extension of the generated AAF, whereas $\{a_2, a_3\}$ is a complete extension of the SETAF.

It can be observed that, as already hinted above, the generated AAF (under Definition 6.1) contains an exponentially larger number of arguments (compared to the SETAF). This is not an issue for this study, because our purpose is not to provide an efficient mapping of a SETAF to an AAF, but to show that one can always determine the extensions of the SETAF by examining the AAF, and vice-versa. More efficient transformations are considered in Subsection 6.10.

In the rest of this section, we assume an arbitrary SETAF $\langle \mathbf{Args}, \triangleright \rangle$ and its generated AAF $\langle \overline{\mathbf{Args}}, \rightarrowtail \rangle$. We also denote by $\mathcal{E}_S^\sigma$ the set of all σ-extensions of $\langle \mathbf{Args}, \triangleright \rangle$ and by $\mathcal{E}_D^\sigma$ the set of all σ-extensions of $\langle \overline{\mathbf{Args}}, \rightarrowtail \rangle$.

6.2. Relating extensions of a SETAF and its generated AAF

Definition 6.1 is based on a natural correspondence between a SETAF and an AAF, in which a set of arguments $\mathbf{A}$ in the SETAF corresponds to a set of arguments $\overline{\mathbf{A}}$ in the generated AAF. Thus, we expect the σ-extensions of the SETAF ($\mathcal{E}_S^\sigma$) and its generated AAF ($\mathcal{E}_D^\sigma$) to be faithful to this correspondence, for any σ. In particular, we expect that $\mathbf{A} \in \mathcal{E}_S^\sigma$ if and only if

UNCORRECTED PROOF

$\overline{\mathbf{A}} \in \mathcal{E}_D^\sigma$, and that, if $\mathcal{E} \in \mathcal{E}_D^\sigma$, then $\mathcal{E} = \overline{\mathbf{A}}$ for some $\mathbf{A} \in \mathcal{E}_S^\sigma$. Perhaps surprisingly, this is not, generally, the case. The following results will show that, although there are certain semantics (including the extension semantics defined by Dung [21]) in which these relations hold, there are other semantics in which the relation is more complex, as well as semantics in which $\mathcal{E}_S^\sigma, \mathcal{E}_D^\sigma$ are not related.

Our aim in this section is to make precise the relationship between $\mathcal{E}_S^\sigma$ and $\mathcal{E}_D^\sigma$ for each σ, by providing results showing how to determine the extensions of a SETAF through the generated AAF, and vice-versa, i.e., how one can determine extensions of the generated AAF through the SETAF. Importantly, we are not interested in showing how to compute the extensions of the SETAF or of the generated AAF in isolation.

Our results show that, although there is not always a relationship between $\mathcal{E}_S^\sigma$ and $\mathcal{E}_D^\sigma$, one can always determine the extensions of the SETAF by examining the AAF, and vice-versa. This validates our claim, i.e., that the extension semantics of a SETAF can be computed through the generated AAF (which is of exponentially bigger size), and vice-versa. Nevertheless, the use of SETAF provides a simpler, more intuitive, and exponentially more compact way to represent the information related to joint attacks.

6.3. Conflict-free sets

For conflict-free sets, it is true that $\mathbf{A}$ is conflict-free (in the SETAF) if and only if $\overline{\mathbf{A}}$ is conflict-free (in the generated AAF). However, there are additional conflict-free sets in the generated AAF (that are not of the form $\overline{\mathbf{A}}$); these are generated by combining appropriate conflict-free sets of the SETAF (say $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}$) into one conflict-free set of the generated AAF ($\{\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}\}$). More precisely:

Proposition 6.3. *For any set of arguments* $\mathbf{A} \subseteq \mathbf{Args}$ *in the SETAF, the following points are equivalent:*

1. $\mathbf{A} \in \mathcal{E}_S^{\mathbf{cf}}$
2. $\{\mathbf{A}\} \in \mathcal{E}_D^{\mathbf{cf}}$
3. $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{cf}}$
4. *If* $\mathcal{E} \subseteq \overline{\mathbf{A}}$, *then* $\mathcal{E} \in \mathcal{E}_D^{\mathbf{cf}}$

Proposition 6.3 establishes the relationship between conflict-free sets in the SETAF and its generated AAF. However, point #4 shows that $\mathcal{E}_D^{\mathbf{cf}}$ is much richer, and its elements are not restricted to sets of the form $\overline{\mathbf{A}}$. The following proposition completes the picture, showing that, in order for a set to be conflict-free in the generated AAF, it must consist entirely of sets that are conflict-free in the SETAF:

Proposition 6.4. *If* $\mathcal{E} \in \mathcal{E}_D^{\mathbf{cf}}$ *and* $\mathbf{A} \in \mathcal{E}$, *then* $\mathbf{A} \in \mathcal{E}_S^{\mathbf{cf}}$.

A direct corollary of Proposition 6.4 is that if $\mathcal{E} \in \mathcal{E}_D^{\mathbf{cf}}$, then $\mathcal{E} \subseteq \mathcal{E}_S^{\mathbf{cf}}$. These results lead to the following precise characterisation of $\mathcal{E}_S^{\mathbf{cf}}$, $\mathcal{E}_D^{\mathbf{cf}}$:

Theorem 6.5. *With regard to conflict-free sets, the following points hold:*

1. $\mathcal{E}_S^{\mathbf{cf}} = \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{cf}}\}$
2. $\mathcal{E}_D^{\mathbf{cf}} = \{\mathcal{E} \mid \mathcal{E} \subseteq \mathcal{E}_S^{\mathbf{cf}}$, *and* $\mathbf{A} \not\rightarrow \mathbf{B}$ *for all* $\mathbf{A}, \mathbf{B} \in \mathcal{E}\}$

Interestingly, the characterisation of $\mathcal{E}_S^{\mathbf{cf}}$ is simpler than the one of $\mathcal{E}_D^{\mathbf{cf}}$, i.e., it is simpler to compute the conflict-free extensions of the SETAF through the AAF, rather than the opposite. This is a pattern that will also appear in other types of extensions.

Note that the side condition that $\mathbf{A} \not\rightarrow \mathbf{B}$ for all $\mathbf{A}, \mathbf{B} \in \mathcal{E}_S^{\mathbf{cf}}$ (second bullet in Theorem 6.5) is crucial. To see this, consider Example 6.2: although $\{a_1\} \in \mathcal{E}_S^{\mathbf{cf}}$ and $\{a_2, a_3\} \in \mathcal{E}_S^{\mathbf{cf}}$, the set $\{\{a_1\}, \{a_2, a_3\}\}$ is not conflict-free in the generated AAF (i.e., $\{\{a_1\}, \{a_2, a_3\}\} \notin \mathcal{E}_D^{\mathbf{cf}}$).

6.4. Admissible sets

For admissible sets, the situation is similar. The following proposition shows how one can create various admissible sets of the generated AAF, given some admissible set of the generated AAF, $\mathcal{E}$:

Proposition 6.6. *If* $\mathcal{E} \in \mathcal{E}_D^{\mathbf{ad}}$, *then:*

1. *If* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \in \mathcal{E}$, *then* $\mathcal{E} \cup \{\mathbf{A}\} \in \mathcal{E}_D^{\mathbf{ad}}$

UNCORRECTED PROOF

2. *If* $\mathbf{A} \in \mathcal{E}$, *then* $\mathcal{E} \cup \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{ad}}$
3. *If* $\mathbf{A}, \mathbf{B} \in \mathcal{E}$, *then* $\mathcal{E} \cup \{\mathbf{A} \cup \mathbf{B}\} \in \mathcal{E}_D^{\mathbf{ad}}$
4. $\mathcal{E} \cup \{\bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}\} \in \mathcal{E}_D^{\mathbf{ad}}$
5. $\overline{\bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}} \in \mathcal{E}_D^{\mathbf{ad}}$
6. $\{\bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}\} \in \mathcal{E}_D^{\mathbf{ad}}$

Of particular interest is point #6, which shows that, once we have an admissible set $\mathcal{E}$, we can create a singleton set consisting of just the union of the sets in $\mathcal{E}$ and still have an admissible set. The following proposition establishes the connection between elements of $\mathcal{E}_S^{\mathbf{ad}}$ and $\mathcal{E}_D^{\mathbf{ad}}$, in a manner similar to the one established between conflict-free sets in Proposition 6.3:

Proposition 6.7. *The following points are equivalent for all* $\mathbf{A}$*:*

1. $\mathbf{A} \in \mathcal{E}_S^{\mathbf{ad}}$
2. $\{\mathbf{A}\} \in \mathcal{E}_D^{\mathbf{ad}}$
3. $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{ad}}$

Combining these results, we get the following precise characterisation of $\mathcal{E}_S^{\mathbf{ad}}$, $\mathcal{E}_D^{\mathbf{ad}}$:

Theorem 6.8. *With regard to admissible sets, the following points hold:*

1. $\mathcal{E}_S^{\mathbf{ad}} = \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{ad}}\}$
2. $\mathcal{E}_D^{\mathbf{ad}} = \{\mathcal{E} \mid \bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A} \in \mathcal{E}_S^{\mathbf{ad}}$*, and if* $\mathbf{C} \blacktriangleright \bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}$*, then there is* $\mathbf{B} \in \mathcal{E}$ *such that* $\mathbf{B} \blacktriangleright \mathbf{C}\}$

6.5. Complete, grounded, preferred and stable extensions

Complete, grounded, preferred and stable semantics exhibit a very "canonical" behaviour, as Theorem 6.10 shows. Before proving that theorem, we start by showing the following proposition, which is largely a corollary of Proposition 6.6:

Proposition 6.9. *If* $\mathcal{E} \in \mathcal{E}_D^{\mathbf{co}}$*, then:*

1. *If* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \in \mathcal{E}$*, then* $\mathbf{A} \in \mathcal{E}$
2. *If* $\mathbf{A} \in \mathcal{E}$*, then* $\overline{\mathbf{A}} \subseteq \mathcal{E}$
3. *If* $\mathbf{A}, \mathbf{B} \in \mathcal{E}$*, then* $\{\mathbf{A} \cup \mathbf{B}\} \in \mathcal{E}$
4. $\bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A} \in \mathcal{E}$
5. $\mathcal{E} = \overline{\bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}}$

The critical point of Proposition 6.9 is #5, which essentially mandates that a complete extension must be of the form $\overline{\mathbf{B}}$ for some $\mathbf{B}$; this is unlike admissible sets. Using this result, we can now show our main theorem related to complete, grounded, preferred and stable semantics:

Theorem 6.10. *For* $\sigma \in \{\mathbf{co}, \mathbf{pr}, \mathbf{gr}, \mathbf{st}\}$*, the following points hold:*

1. $\mathcal{E}_S^{\sigma} = \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\sigma}\}$
2. $\mathcal{E}_D^{\sigma} = \{\overline{\mathbf{A}} \mid \mathbf{A} \in \mathcal{E}_S^{\sigma}\}$

The following results are interesting corollaries of Theorem 6.10:

Corollary 6.11. *For* $\sigma \in \{\mathbf{co}, \mathbf{pr}, \mathbf{gr}, \mathbf{st}\}$, $\mathbf{A} \in \mathcal{E}_S^{\sigma}$ *if and only if* $\overline{\mathbf{A}} \in \mathcal{E}_D^{\sigma}$.

Corollary 6.12. *For* $\sigma \in \{\mathbf{co}, \mathbf{pr}, \mathbf{gr}, \mathbf{st}\}$*, if* $\mathcal{E} \in \mathcal{E}_D^{\sigma}$*, then there is some* $\mathbf{A}$ *such that* $\mathcal{E} = \overline{\mathbf{A}}$ *and* $\mathbf{A} \in \mathcal{E}_S^{\sigma}$.

6.6. Semi-stable and stage extensions

We now study the case of semi-stable and stage extensions, both of which aim at maximising the union of the arguments in the extension with those that are attacked by the extension. The difference between the two cases is that semi-stable extensions consider maximisation over alternative complete extensions, whereas stage extensions consider conflict-free sets.

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

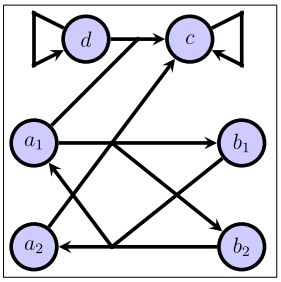

Fig. 4. Associating semi-stable/stage extensions of SETAF and generated AAF (prototypical example).

Before studying this case in detail, we introduce some simplifying notations. For any given $\mathbf{A} \subseteq \mathbf{Args}$, we denote: $\mathbf{A}^{\blacktriangleright} = \{c \mid \mathbf{A} \blacktriangleright c\}$ and $\mathbf{A}^{\rightarrowtail} = \{\mathbf{C} \mid \mathbf{A} \rightarrowtail \mathbf{C}\}$. Note that $\mathbf{A}^{\rightarrowtail} = \{\mathbf{C} \mid \mathbf{A}' \rightarrowtail \mathbf{C}$ for some $\mathbf{A}' \in \overline{\mathbf{A}}\}$ (see also Lemma 1 in Appendix A.2); these notions are obviously related to the maximisation requirements of Definitions 3.8 and 3.11 with regard to the semi-stable and stage extensions of the generated AAF.

It turns out that the situation in semi-stable and stage semantics is not as simple as with the family of complete, grounded, preferred and stable semantics. The prototypical example that shows the problem associated with semi-stable and stage semantics is the following:

Example 6.13. Consider a SETAF $\langle \mathbf{Args}, \triangleright \rangle$ defined as follows (see also Fig. 4):

- $\mathbf{Args} = \{a_1, a_2, b_1, b_2, c, d\}$
- $\{a_1, a_2\} \triangleright b_1$
- $\{a_1, a_2\} \triangleright b_2$
- $\{a_1, a_2\} \triangleright c$
- $\{b_1, b_2\} \triangleright a_1$
- $\{b_1, b_2\} \triangleright a_2$
- $\{c\} \triangleright c$
- $\{d\} \triangleright d$
- $\{a_1, d\} \triangleright c$

It is easy to note that the only complete extensions of this SETAF are $\{a_1, a_2\}$ and $\{b_1, b_2\}$. If we add any additional argument to any of these sets we will end up with a conflict, whereas if we remove any argument we will end up with a non-admissible set. With regard to semi-stable extensions, clearly $\{a_1, a_2\}$ is the only one, because it attacks c (whereas $\{b_1, b_2\}$ does not).

On the other hand, for the generated AAF, the set $\overline{\{b_1, b_2\}}$ becomes a semi-stable extension too. The reason for this is that any set that includes d and at least one of a_1, a_2 (e.g., $\{a_1, d\}$) is attacked by $\{b_1, b_2\}$ but not by $\{a_1, a_2\}$ (and is also not included in $\overline{\{a_1, a_2\}}$).

For stage extensions, the conclusion is similar: $\{a_1, a_2\}$ is a stage extension of the SETAF, whereas both $\overline{\{a_1, a_2\}}$ and $\overline{\{b_1, b_2\}}$ are stage extensions of the generated AAF.

As a more extreme case, consider the following example, where the generated AAF has an infinite number of semi-stable/stage extensions, whereas the SETAF has none:

Example 6.14. Consider a SETAF $\langle \mathbf{Args}, \triangleright \rangle$ defined as follows:

- $\mathbf{Args} = \{a_1, a_2, \dots\} \cup \{b_1, b_2, \dots\}$
- $a_i \triangleright a_j$ whenever $i \neq j$
- $a_i \triangleright b_j$ whenever $i \geq j$
- $b_i \triangleright b_i$ for all i

Note that there are no semi-stable (or stage) extensions for this SETAF. To see this, note that a semi-stable or stage extension should not contain any b_i or more than one a_i (or it would not be conflict-free). Moreover, we note that for any a_i it holds that $\{a_i\} \cup \{a_i\}^{\blacktriangleright} = \{a_1, a_2, \dots\} \cup \{b_1, \dots, b_i\}$. As a result $\{a_i\} \cup \{a_i\}^{\blacktriangleright} \subset \{a_{i+1}\} \cup \{a_{i+1}\}^{\blacktriangleright}$, and therefore there is no semi-stable or stage extension.

On the other hand, for all i, $\{\{a_i\}\}$ is both a semi-stable and a stage extension for the generated AAF, so the generated AAF contains an infinite number of semi-stable/stage extensions.

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

The above examples show that there is no direct relationship between the semi-stable/stage extensions of the SETAF/generated AAF. The cause of the problem is not in the SETAF formalism per se (note that the SETAF of Example 6.14 does not in fact contain joint attacks, so it is an AAF), but in the process of mapping the SETAF to its generated AAF. In more technical terms, the cause of the problem is evident in the following result, in particular point #3:

Proposition 6.15. *Consider two conflict-free sets* $\mathbf{A}, \mathbf{B}$*. Then:*

1. *If* $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$*, then* $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$.
2. *If* $\mathbf{A} \not\blacktriangleright \mathbf{B}$ *and* $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$ *then* $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$.
3. *If* $\mathbf{A} \blacktriangleright \mathbf{B}$ *and* $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} \neq \mathbf{Args}$ *then* $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \not\subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$.

Point #1 of Proposition 6.15 shows that an inclusion in the generated AAF carries over nicely in the SETAF. The second point deals with the opposite direction, but requires an additional hypothesis, namely that $\mathbf{A} \not\blacktriangleright \mathbf{B}$. The reason for this hypothesis is evident in point #3: if $\mathbf{A} \blacktriangleright \mathbf{B}$ then it cannot be the case that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$, unless of course $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} = \mathbf{Args}$ (in which case it also holds that $\overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail} = \overline{\mathbf{Args}}$ and thus $\mathbf{B}$, $\overline{\mathbf{B}}$ are stable extensions in the SETAF/AAF respectively). As a result of this, it may be the case that $\overline{\mathbf{A}}$ is a semi-stable (or stage) extension of the generated AAF, while $\mathbf{A}$ is not a semi-stable (or stage) extension of the SETAF (as was also the case in the above examples).

Before we study the problem in its generality, we start with a simple special case, which also helps us establish the relationship between stable and semi-stable semantics. A first obvious corollary of Theorem 6.10 is that $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$ if and only if $\mathcal{E}_D^{\mathbf{st}} \neq \emptyset$. Also, the following holds:

Proposition 6.16. *If* $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$ *then* $\mathcal{E}_S^{\mathbf{st}} = \mathcal{E}_S^{\mathbf{se}}$ *and* $\mathcal{E}_D^{\mathbf{st}} = \mathcal{E}_D^{\mathbf{se}}$.

Combining this with Theorem 6.10, we get the following corollary:

Corollary 6.17. *If* $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$*, then:*

1. $\mathcal{E}_S^{\mathbf{se}} = \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{st}}\}$
2. $\mathcal{E}_D^{\mathbf{se}} = \{\overline{\mathbf{A}} \mid \mathbf{A} \in \mathcal{E}_S^{\mathbf{st}}\}$

We can now study the more challenging case where $\mathcal{E}_S^{\mathbf{st}} = \emptyset$. For this, we need a new notion, which we call *domination*, defined in two flavours, one for the SETAF (*S-domination*) and one for the generated AAF (*D-domination*):

Definition 6.18. Consider two sets of arguments $\mathbf{A}$,$\mathbf{B}$. Then, we say that $\mathbf{A}$ *S-dominates* $\mathbf{B}$ if and only if:

- $\mathbf{B} \setminus \mathbf{A} \subseteq \mathbf{A}^{\blacktriangleright}$
- $\mathbf{B}^{\blacktriangleright} \cap (\mathbf{Args} \setminus (\mathbf{A} \cup \mathbf{B})) \subseteq \mathbf{A}^{\blacktriangleright} \cap (\mathbf{Args} \setminus (\mathbf{A} \cup \mathbf{B}))$

Similarly, we say that $\mathbf{A}$ *D-dominates* $\mathbf{B}$ if and only if:

- $\overline{\mathbf{B}} \setminus \overline{\mathbf{A}} \subseteq \mathbf{A}^{\rightarrowtail}$
- $\mathbf{B}^{\rightarrowtail} \cap \overline{\mathbf{Args} \setminus (\mathbf{A} \cup \mathbf{B})} \subseteq \mathbf{A}^{\rightarrowtail} \cap \overline{\mathbf{Args} \setminus (\mathbf{A} \cup \mathbf{B})}$

As usual, we say that $\mathbf{A}$ *strictly S-dominates* $\mathbf{B}$ if and only if $\mathbf{A}$ S-dominates $\mathbf{B}$, but $\mathbf{B}$ does not S-dominate $\mathbf{A}$ (similarly for *strict D-domination*).

Practically, S-domination (and D-domination) requires that $\mathbf{A}$ attacks "all of" $\mathbf{B}$ (except from their common elements of course) and that $\mathbf{A}$ attacks "more of" external arguments (arguments neither in $\mathbf{A}$ nor in $\mathbf{B}$) than $\mathbf{B}$. With regard to the second bullet of D-domination, note that $\overline{\mathbf{Args} \setminus (\mathbf{A} \cup \mathbf{B})}$ is not the complement of $\overline{\mathbf{A} \cup \mathbf{B}}$ (e.g., consider $\mathbf{C} = \{c, d\}$ where $c \in \mathbf{A} \cup \mathbf{B}$ and $d \notin \mathbf{A} \cup \mathbf{B}$).

In Example 6.13, $\{a_1, a_2\}$ strictly S-dominates (and also strictly D-dominates) $\{b_1, b_2\}$, because $\{a_1, a_2\}$ attacks both b_1 and b_2 and also it attacks c (whereas $\{b_1, b_2\}$ does not attack any argument not in $\{a_1, a_2\}$).

This property, not surprisingly, is closely related to our purposes; the following proposition shows that the above two forms of domination are in fact equivalent, and equivalent with the relation $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$:

Proposition 6.19. *The following points are equivalent for all* $\mathbf{A}, \mathbf{B} \in \mathcal{E}_S^{\mathbf{cf}}$*:*

1. $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$
2. $\mathbf{B}$ *S-dominates* $\mathbf{A}$
3. $\mathbf{B}$ *D-dominates* $\mathbf{A}$

UNCORRECTED PROOF

A glaring omission from Proposition 6.19 is the relation $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$, which is not equivalent to the rest. For this, we need an extra precondition, as hinted by Proposition 6.15 already:

Proposition 6.20. *If* $\mathbf{A}, \mathbf{B} \in \mathcal{E}_S^{\mathbf{cf}}$ *and* $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} \neq \mathbf{Args}$*, then the following points are equivalent:*

1. $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$
2. $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$ *and* $\mathbf{A} \not\blacktriangleright \mathbf{B}$
3. $\mathbf{B}$ *S-dominates* $\mathbf{A}$ *and* $\mathbf{A} \not\blacktriangleright \mathbf{B}$
4. $\mathbf{B}$ *D-dominates* $\mathbf{A}$ *and* $\mathbf{A} \not\rightarrowtail \mathbf{B}$

The following is an interesting consequence of the above propositions:

Proposition 6.21. *If* $\mathbf{A}, \mathbf{B} \in \mathcal{E}_S^{\mathbf{cf}}$ *and* $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \neq \mathbf{Args}$*, then the following points are equivalent:*

1. $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} = \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$
2. $\mathbf{A} = \mathbf{B}$

We can now move on showing the main result regarding semi-stable semantics. The following is an interesting prelude:

Proposition 6.22. *If* $\mathbf{A} \in \mathcal{E}_S^{\mathbf{se}}$ *then* $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{se}}$.

Note that the reverse of Proposition 6.22 does not hold, as Examples 6.13, 6.14 testify. These results allow us to prove the following theorem that clarifies the situation with respect to semi-stable semantics in the case where no stable extension exists:

Theorem 6.23. *The following points hold:*

1. $\mathcal{E}_S^{\mathbf{se}} = \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{se}}$*, and if* $\mathbf{B}$ *strictly D-dominates* $\mathbf{A}$ *then* $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{co}}\}$
2. *If* $\mathcal{E}_S^{\mathbf{st}} = \emptyset$ *then* $\mathcal{E}_D^{\mathbf{se}} = \{\overline{\mathbf{A}} \mid \mathbf{A} \in \mathcal{E}_S^{\mathbf{pr}}, \mathbf{A} \blacktriangleright \mathbf{B}$ *whenever* $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subset \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$ *and* $\mathbf{B} \in \mathcal{E}_S^{\mathbf{pr}}\}$

To understand better Theorem 6.23, let's apply it on Example 6.13. As described in Example 6.13, $\overline{\{a_1, a_2\}}$ and $\overline{\{b_1, b_2\}}$ are the only semi-stable extensions of the generated AAF. However, $\{a_1, a_2\}$ is the only semi-stable extension of the SETAF; $\{b_1, b_2\}$ is not, because it is strictly dominated by $\{a_1, a_2\}$, which is a complete extension (so $\{b_1, b_2\}$ cannot be a semi-stable extension, per the first bullet of Theorem 6.23). For the second bullet of the theorem, note that $\overline{\{a_1, a_2\}}$, $\overline{\{b_1, b_2\}}$ are the only preferred extensions of the SETAF, and, although $\{a_1, a_2\} \cup \{a_1, a_2\}^{\blacktriangleright} \supset \{b_1, b_2\} \cup \{b_1, b_2\}^{\blacktriangleright}$ and $\{a_1, a_2\} \in \mathcal{E}_S^{\mathbf{pr}}$, it happens that $\{b_1, b_2\} \blacktriangleright \{a_1, a_2\}$, so both $\overline{\{a_1, a_2\}}$ and $\overline{\{b_1, b_2\}}$ are semi-stable extensions of the generated AAF.

Note that the first part of Theorem 6.23 holds even when $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$. This result, combined with Proposition 6.16 also gives an alternative, but more cumbersome, characterisation of stable extensions for the SETAF. Moreover, the second bullet implies that if $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{se}}$ then $\mathbf{A} \in \mathcal{E}_S^{\mathbf{pr}}$. Thus, although we have no guarantee that $\mathbf{A}$ will be a semi-stable extension, we can at least restrict our search to preferred extensions.

With regard to stage semantics, we start by noting that the problems associated with Examples 6.13, 6.14 are also applicable to stage extensions. However, for stage semantics, the situation is more complex. The main reason is that stage extensions have as a starting point conflict-free sets, whose form is more complex in the generated AAF, compared to complete extensions, which are the starting point for semi-stable semantics (see Theorems 6.5 and 6.10). As a result, more complex versions of the above results are necessary.

For the simpler case where $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$, a result similar to Proposition 6.16 holds:

Proposition 6.24. *If* $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$ *then* $\mathcal{E}_S^{\mathbf{st}} = \mathcal{E}_S^{\mathbf{sg}}$ *and* $\mathcal{E}_D^{\mathbf{st}} = \mathcal{E}_D^{\mathbf{sg}}$.

Combining this with Theorem 6.10, we get the following corollary:

Corollary 6.25. *If* $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$*, then:*

1. $\mathcal{E}_S^{\mathbf{sg}} = \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{st}}\}$
2. $\mathcal{E}_D^{\mathbf{sg}} = \{\overline{\mathbf{A}} \mid \mathbf{A} \in \mathcal{E}_S^{\mathbf{st}}\}$

For the more challenging case where $\mathcal{E}_S^{\mathbf{st}} = \emptyset$, we start with the characterisation of $\mathcal{E}_S^{\mathbf{sg}}$, which does not require any additional tools:

UNCORRECTED PROOF

Theorem 6.26. *The following holds:*
$\mathcal{E}_S^{\mathbf{sg}} = \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{cf}}$, *and if* $\mathbf{B}$ *strictly D-dominates* $\mathbf{A}$ *then* $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{cf}}\}$

It is interesting to note the similarity of Theorem 6.26 (for stage semantics) with its corresponding Theorem 6.23 (for semi-stable semantics). No such similarity exists for the characterisations of $\mathcal{E}_D^{\mathbf{se}}$, $\mathcal{E}_D^{\mathbf{sg}}$, as we will soon prove.

To properly characterise stage semantics, we need to deal with families of sets of arguments. Recall (Theorem 6.5) that the conflict-free sets of the generated AAF are essentially families of conflict-free sets of the SETAF. Along these lines, the cornerstone for the characterisation of $\mathcal{E}_D^{\mathbf{sg}}$ when $\mathcal{E}_S^{\mathbf{st}} = \emptyset$, is the following result:

Proposition 6.27. *Consider two families of sets of arguments* $\{\mathbf{A_i}\}$, $\{\mathbf{B_i}\}$, *such that* $\mathbf{A_j} \not\blacktriangleright \mathbf{A_k}$, $\mathbf{B_j} \not\blacktriangleright \mathbf{B_k}$ *for all* j, k, *and* $\mathcal{E}_S^{\mathbf{st}} = \emptyset$. *Then* $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) \subseteq \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$ *if and only if all of the following points hold:*

- $\bigcup(\mathbf{A_i} \cup \mathbf{A_i}^{\blacktriangleright}) \subseteq \bigcup(\mathbf{B_i} \cup \mathbf{B_i}^{\blacktriangleright})$
- *For all* j, k, $\mathbf{A_j} \not\blacktriangleright \mathbf{B_k}$
- *For all* j *there is* k *such that* $\mathbf{A_j} \setminus \mathbf{B_k} \subseteq \bigcup \mathbf{B_i}^{\blacktriangleright}$

This result is critical; recall that conflict-free sets (the starting point for stage semantics) in the generated AAF can be any family of conflict-free sets from $\mathcal{E}_S^{\mathbf{cf}}$, so we need a way to compare the quantities $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail})$ and $\bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$. Essentially, Proposition 6.27 states that, in order for $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) \subseteq \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$ to hold, the corresponding relation in the SETAF must hold ($\bigcup(\mathbf{A_i} \cup \mathbf{A_i}^{\blacktriangleright}) \subseteq \bigcup(\mathbf{B_i} \cup \mathbf{B_i}^{\blacktriangleright})$), there should be no attack on elements of $\{\mathbf{B_i}\}$ by any element of $\{\mathbf{A_i}\}$, and for every set in $\{\mathbf{A_i}\}$, say $\mathbf{A_j}$, all elements of $\mathbf{A_j}$ should either be attacked by $\{\mathbf{B_i}\}$ or be contained in some, fixed, member of the family $\{\mathbf{B_i}\}$.

We also need the notion of covering. Intuitively, a family $\{\mathbf{A_i}\}$ covers a family $\{\mathbf{B_i}\}$ if the former contains "larger" sets than the latter. Formally:

Definition 6.28. Consider two families $\{\mathbf{A_i}\}$, $\{\mathbf{B_i}\}$. We say that $\{\mathbf{A_i}\}$ *covers* $\{\mathbf{B_i}\}$ if and only if for all j there is k such that $\mathbf{A_j} \supseteq \mathbf{B_k}$.

As an example, for the families $\{\mathbf{A_i}\} = \{\{a_1, a_2\}, \{a_3, a_4\}\}$, $\{\mathbf{B_i}\} = \{\{a_1, a_3\}, \{a_2, a_4\}\}$, it is not the case that $\{\mathbf{A_i}\}$ covers $\{\mathbf{B_i}\}$ (or vice-versa). Moreover, $\{\mathbf{A_i}\}$ covers $\{\{a_1\}, \{a_3\}\}$, but $\{\mathbf{B_i}\}$ does not.

The following proposition can be immediately derived:

Proposition 6.29. *If* $\{\mathbf{B_i}\}$ *covers* $\{\mathbf{A_i}\}$ *then* $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) \subseteq \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$.

Clearly, the reverse of Proposition 6.29 is not true. Interestingly, when both inclusions are known to hold (i.e., $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) = \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$), then we can show that families $\{\mathbf{A_i}\}$, $\{\mathbf{B_i}\}$ cover each other (under certain additional hypotheses):

Proposition 6.30. *Consider two families of sets of arguments* $\{\mathbf{A_i}\}$, $\{\mathbf{B_i}\}$, *such that* $\mathbf{A_j} \not\blacktriangleright \mathbf{A_k}$, $\mathbf{B_j} \not\blacktriangleright \mathbf{B_k}$ *for all* j, k *and* $\mathcal{E}_S^{\mathbf{st}} = \emptyset$. *Then, the following points are equivalent:*

1. $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) = \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$
2. $\{\mathbf{A_i}\}$ *covers* $\{\mathbf{B_i}\}$ *and* $\{\mathbf{B_i}\}$ *covers* $\{\mathbf{A_i}\}$

We now have all the necessary tools to show our final result related to stage semantics:

Theorem 6.31. *If* $\mathcal{E}_S^{\mathbf{st}} = \emptyset$, *then* $\mathcal{E}_D^{\mathbf{sg}} = \{\bigcup\overline{\mathbf{A_i}} \mid$ *for all* j, k $\mathbf{A_j} \not\blacktriangleright \mathbf{A_k}$, *and, if there is* $\{\mathbf{B_i}\}$ *such that* $\bigcup(\mathbf{A_i} \cup \mathbf{A_i}^{\blacktriangleright}) \subseteq \bigcup(\mathbf{B_i} \cup \mathbf{B_i}^{\blacktriangleright})$, $(\bigcup \mathbf{B_i}) \bigcap (\bigcup \mathbf{A_i}^{\blacktriangleright} \bigcup \mathbf{B_i}^{\blacktriangleright}) = \emptyset$, *and for all* j *there is some* k *such that* $\mathbf{A_j} \setminus \mathbf{B_k} \subseteq \bigcup \mathbf{B_i}^{\blacktriangleright}$, *then* $\{\mathbf{A_i}\}$ *covers* $\{\mathbf{B_i}\}\}$.

Some clarifications with regard to Theorem 6.31 are necessary. First, the theorem implies that all stage extensions of AF^D are of the form $\bigcup\overline{\mathbf{A_i}}$, where each $\mathbf{A_i}$ is conflict-free, and the various $\mathbf{A_i}$ do not attack each other. This is similar to the requirement for conflict-free sets in $\mathcal{E}_D^{\mathbf{cf}}$, except that here we take $\bigcup\overline{\mathbf{A_i}}$, rather than $\bigcup\mathbf{A_i}$.

Second, the theorem does not imply any connection between the SETAF and the AAF stage extensions. In fact, the elements of $\mathcal{E}_D^{\mathbf{sg}}$ are not necessarily of the form $\overline{\mathbf{A}}$ for some $\mathbf{A}$.

Third, the requirement associated with Theorem 6.31 essentially looks for families that exhibit the maximality property required by the definition of stage semantics. However, this maximality requirement is expressed in terms of the SETAF. In particular, for a given family $\{\mathbf{B_i}\}$ it checks whether it is "better" than $\{\mathbf{A_i}\}$ (by essentially reiterating the related condition proven in Proposition 6.27). If this is indeed the case, it requires that $\{\mathbf{A_i}\}$ covers $\{\mathbf{B_i}\}$ (which in turn implies that $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) \supseteq \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$, i.e., both families are stage extensions).

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

6.7. Ideal extensions

Ideal semantics are based on complete and preferred extensions, which behave quite nicely (see Theorem 6.10). As a result, the corresponding characterisation for ideal extensions is simple:

Theorem 6.32. *The following points hold:*

1. $\mathcal{E}_S^{\mathbf{id}} = \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{id}}\}$
2. $\mathcal{E}_D^{\mathbf{id}} = \{\overline{\mathbf{A}} \mid \mathbf{A} \in \mathcal{E}_S^{\mathbf{id}}\}$

6.8. Naive extensions

For naive semantics, the situation is a bit more complex, because they correspond to maximal conflict-free sets, whose characterisation is more convoluted. The following result characterises naive extensions, by just forbidding "better" alternatives of a candidate naive extension in each case:

Theorem 6.33. *The following points hold:*

1. $\mathcal{E}_S^{\mathbf{na}} = \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{cf}}$ *and* $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{cf}}$ *whenever* $\overline{\mathbf{A}} \subset \overline{\mathbf{B}}\}$
2. $\mathcal{E}_D^{\mathbf{na}} = \{\mathcal{E} \mid \mathcal{E} \subseteq \mathcal{E}_S^{\mathbf{cf}}$ *and* $\mathbf{A} \not\blacktriangleright \mathbf{B}$ *for* $\mathbf{A}, \mathbf{B} \in \mathcal{E}$, *and for* $\mathcal{E}'$ *for which* $\mathcal{E} \subset \mathcal{E}' \subseteq \mathcal{E}_S^{\mathbf{cf}}$ *there is some* $\mathbf{A}, \mathbf{B} \in \mathcal{E}'$ *such that* $\mathbf{A} \blacktriangleright \mathbf{B}\}$

6.9. Eager extensions

With regard to eager, the situation is even more complex. This semantics requires an extension to be maximal among all complete extensions that are subsets of all semi-stable extensions. To simplify presentation, we use the following sets:

$$\mathcal{E}^{\cap} = \bigcap \{\overline{\mathbf{A}} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{se}}, \text{ and if } \mathbf{B} \text{ strictly D-dominates } \mathbf{A} \text{ then } \overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{co}}\}$$

$$\mathbf{S}^{\cap} = \bigcap \{\mathbf{A} \mid \mathbf{A} \in \mathcal{E}_S^{\mathbf{pr}}, \mathbf{A} \blacktriangleright \mathbf{B} \text{ whenever } \mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subset \mathbf{B} \cup \mathbf{B}^{\blacktriangleright} \text{ and } \mathbf{B} \in \mathcal{E}_S^{\mathbf{pr}}\}.$$

Comparing the definitions of $\mathbf{S}^{\cap}$, $\mathcal{E}^{\cap}$ with Theorem 6.23 (on **se**-extensions), it is easy to show the following:

Proposition 6.34. *The following points are equivalent:*

1. $\mathbf{A} \in \mathcal{E}^{\cap}$
2. $\mathbf{A} \subseteq \mathbf{B}$ *for all* $\mathbf{B} \in \mathcal{E}_S^{\mathbf{se}}$

Proposition 6.35. *If* $\mathcal{E}_S^{\mathbf{st}} = \emptyset$, *then the following points are equivalent:*

1. $\mathbf{A} \subseteq \mathbf{S}^{\cap}$
2. $\overline{\mathbf{A}} \subseteq \mathcal{E}$ *for all* $\mathcal{E} \in \mathcal{E}_D^{\mathbf{se}}$

The above propositions clarify the intuition behind the rather complex definitions of $\mathcal{E}^{\cap}$, $\mathbf{S}^{\cap}$. In particular, $\mathcal{E}^{\cap}$ contains all the sets of arguments from the SETAF that are subsets of all semi-stable extensions of the SETAF; whereas $\mathbf{S}^{\cap}$ is itself a maximal subset of all the semi-stable extensions of the generated AAF. Given the requirement (for eager extensions) to be subsets of all semi-stable extensions, the relevance of these sets is obvious.

Based on Propositions 6.34, 6.35, we can use $\mathcal{E}^{\cap}$, $\mathbf{S}^{\cap}$ as intermediate notions for showing the characterisation of eager semantics:

Theorem 6.36. *The following points hold:*

1. $\mathcal{E}_S^{\mathbf{ea}} = \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}, \mathbf{A} \in \mathcal{E}^{\cap}$, *and* $\mathbf{A} \subset \mathbf{B}$ *implies* $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{co}}$ *or* $\mathbf{B} \notin \mathcal{E}^{\cap}\}$
2. *If* $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$ *then* $\mathcal{E}_D^{\mathbf{ea}} = \{\overline{\mathbf{A}} \mid \mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}, \mathbf{A} \subseteq \mathbf{B}$ *for all* $\mathbf{B} \in \mathcal{E}_S^{\mathbf{st}}$, *and* $\mathbf{A} \subset \mathbf{C}$ *implies* $\mathbf{C} \notin \mathcal{E}_S^{\mathbf{co}}$ *or there is some* $\mathbf{D} \in \mathcal{E}_S^{\mathbf{st}}$ *such that* $\mathbf{C} \not\subseteq \mathbf{D}\}$
3. *If* $\mathcal{E}_S^{\mathbf{st}} = \emptyset$ *then* $\mathcal{E}_D^{\mathbf{ea}} = \{\overline{\mathbf{A}} \mid \mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}, \mathbf{A} \subseteq \mathbf{S}^{\cap}$, *and* $\mathbf{A} \subset \mathbf{B}$ *implies* $\mathbf{B} \notin \mathcal{E}_S^{\mathbf{co}}$ *or* $\mathbf{B} \not\subseteq \mathbf{S}^{\cap}\}$

Again, the idea behind the characterisation of eager semantics is to disallow "better" extensions by requiring that, whenever a "better" candidate exists, this would fail the other requirements of the definition (being complete, or being a subset of all semi-stable extensions).

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

6.10. Alternative transformations

In the definition of the generated AAF (Definition 6.1), we assumed that all non-empty subsets of **Args** are included as arguments in the generated AAF. Alternatively, one could consider various schemes for including sets of arguments "as needed", i.e., different schemes for choosing which sets of arguments from $\overline{\textbf{Args}}$ will be considered in the generated AAF. In other words, the arguments of the generated AAF would include only some of the subsets of **Args**, whereas the attacks of the generated AAF would be the attacks that apply among the chosen sets of arguments.

A particularly effective scheme for choosing the arguments to include in the generated AAF would be to include only sets of arguments that minimally attack other arguments but one could also consider more complex alternatives. Although we have not exhaustively explored the space of possibilities and alternative transformations, we conjecture that such ideas will not produce any better (i.e., more elegant) results compared to the ones described previously in this section. The main reason behind this conjecture is that one can always create examples where specific sets of arguments will be "forced" to be included in the generated AAF, thereby breaking the elegance of the correspondence.

To visualise this, let us reconsider Example 6.13. The existence of the attack $\{a_1, d\} \triangleright c$ forces us to include $\{a_1, d\}$ in the arguments of the generated AAF. This inclusion is critical because $\{a_1, d\}$ becomes part of $\overline{\{b_1, b_2\}}^{\blacktriangleright}$, while no other set of arguments in the generated AAF is both conflict-free and includes $\{a_1, d\}$, leading to the conclusion that $\overline{\{b_1, b_2\}}$ is a semi-stable extension of the generated AAF, despite the fact that $\{b_1, b_2\}$ is not a semi-stable extension of the SETAF. And this is true regardless of the actual arguments we decide to include in the generated AAF so long as these arguments include all the minimally attacking/attacked sets of arguments from the original SETAF. Interestingly, if we had considered the minimal transformation (where only the minimally attacking/attacked arguments are considered) and if the attack $\{a_1, d\} \triangleright c$ was missing, then $\{a_1, d\}$ would not have been included in the arguments of the generated AAF and $\overline{\{a_1, a_2\}}$ would have been the only semi-stable extension of the generated AAF. However, this "elegant" behaviour cannot be generalised as explained above.

Despite these discouraging remarks, it is still important to consider these alternative transformations as a means to reduce the average size (number of arguments) of the generated AAF. A more detailed study of these aspects goes beyond the scope of this paper and is left for future work.

7. Recasting Dung-style results for SETAF

In the literature, there are various theorems on the properties of and relationships between different acceptability semantics for Dung-style argumentation frameworks [2,12,10,24]. Here, we show how these results extend to the SETAF setting, filling the gaps from previous work (most notably, [1]). Specifically, we focus on the inclusion relationships between the different semantics (Theorem 7.1) and the existence and multiplicity of extensions and labellings (Theorem 7.2).

Fig. 5 shows an overview of the results of this section. Each arrow in the graph pointing from semantics σ to σ' indicates that every σ-extension (or σ-labelling) of a SETAF is also a σ'-extension (resp. σ'-labelling) of the same SETAF (e.g., every stable extension is also a stage extension). The number (possibly followed by $+$) that appears next to each semantics indicates the multiplicity of extensions and labellings for the specific semantics (e.g., every SETAF has at least one preferred extension). Similarly to Dung-style AAFs, for certain semantics, the multiplicity of extensions (labellings) is different between finite and infinite SETAF, i.e., between SETAF with a finite and SETAF with an infinite number of arguments. All such arrows are strict, i.e., no semantics is equivalent to another.

Theorem 7.1. *All the inclusion relationships between the different acceptability semantics of SETAF depicted in Fig. 5 hold for any SETAF.*

Theorem 7.2. *Any SETAF AF^S has: (i) at least one conflict-free extension (labelling); (ii) at least one admissible extension (labelling); (iii) at least one complete extension (labelling); (iv) exactly one grounded extension (labelling); (v) at least one preferred extension*

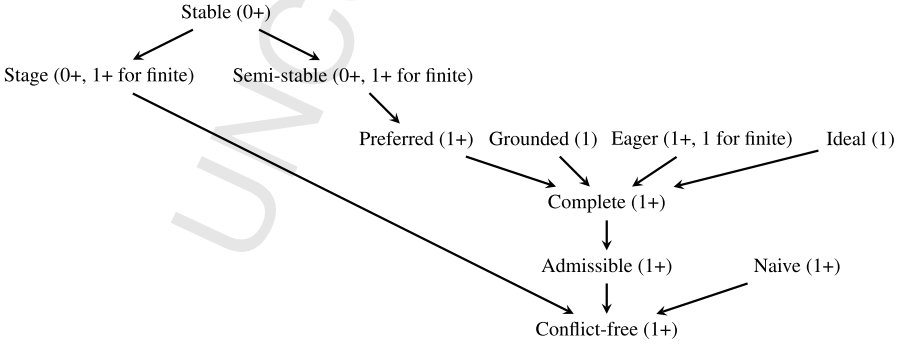

Fig. 5. Inclusion relations and multiplicity of extensions and labellings for SETAF acceptability semantics.

UNCORRECTED PROOF

(labelling); (vi) zero or more stable extensions (labellings); (vii) at least one naive extension (labelling); (viii) zero or more semi-stable extensions (labellings), and at least one if AF^S is finite; (ix) exactly one ideal extension (labelling); (x) at least one eager extension (labelling), and exactly one if AF^S is finite; (xi) zero or more stage extensions (labellings), and at least one if AF^S is finite.

8. Discussion

In this paper, we provided a complete formal characterization of Frameworks with Sets of Attacking Arguments (SETAF) by defining different kinds of acceptability semantics in terms of extensions and labellings and showing how they relate to each other. This way, our work provides labelling semantics for SETAF, something that was totally missing from the related literature (to the best of the authors' knowledge). In addition, our work studied the relationships between SETAF and AAF (as well as their extensions), and showed how a SETAF (and its extensions) can be expressed in terms of an AAF (albeit in a less compact and natural manner), complementing previous work [16]. Further, we have proven that several important results that have been shown for Dung-style AAFs also apply for SETAF. Our work is based on the definition of SETAF given in [1], and can be seen as a point of reference for SETAF, as it generalises different semantics (originally proposed in various papers) for SETAF and shows various properties. We claim that this is the most complete study of SETAF so far, properly positioning SETAF within the realm of the literature on computational argumentation.

SETAF is not the only argumentation model that has been proposed for formalising the notion of joint-attacks. The collective argumentation frameworks proposed in [25] support the notion of attacks between sets of arguments. The meta-argumentation frameworks that are used to represent bipolar argumentation frameworks as Dung-style frameworks in [26], use the notions of coalitions of arguments and attacks among coalitions. In both frameworks, the attacks are collective in the sense that they invalidate all arguments in the attacked set. As shown in [1], collective attacks from a set of arguments **A** on a set of arguments **B** can be represented in SETAF as a series of attacks from **A** on each of the arguments in **B**. Furthermore, compared to the frameworks proposed in [25], the semantics of SETAF generalise the semantics provided by Dung, sticking closer to the original definitions of abstract argumentation frameworks. Argumentation Frameworks with Necessities (AFNs) [27] are another kind of bipolar argumentation frameworks that support interactions between single arguments and sets of arguments but in a different way: a necessity relation between a set of arguments **B** and an argument a means that the acceptance of a requires the acceptance of at least one argument in **B**. The framework presented in [28] also considers sets of arguments, but as recipients of disjunctive attacks from single arguments. In this framework, the result of an attack from an argument a that is labelled `in` to a set of arguments **A** is that at least one of the arguments in **A** must be labelled `out`. Definition 2.8 and Theorem 2.9 of the same paper show how a finite disjunctive framework can be converted to a Dung-style AAF with the same set of extensions, which, combined with the results on the relationship between SETAF and AAF we present in Section 6, provide a way to associate SETAF with disjunctive argumentation frameworks. Finally, there is another line of research studying the accrual of arguments (e.g., see [29–31]), where the strength of a conclusion or an argument is determined by the number of the independent reasons or arguments that support or attack it. As also explained in [1], SETAF and AAF do not provide a way to quantify the strength of arguments or attacks and, therefore, cannot support the notion of accrual. It would be interesting though to study how different approaches applied to AAF that capture certain features of accrual (e.g., reasoning about preferences on arguments [32] or the graded acceptability semantics proposed in [33]) can also be applied to SETAF.

With regard to the relationship between SETAF and structured argumentation systems, [34] studies the formalisation of SETAF as well as other extensions of AAF in the *ASPIC*$^+$ framework [35], where arguments are structured as trees whose leaf nodes are facts and whose non-leaf nodes are either defeasible or strict inference rules. The paper concludes that such mapping may be ambiguous as there is often uncertainty in how to reify abstract relations of such frameworks to the elements of the logic, e.g., joint attacks of SETAF to binary attacks supported by *ASPIC*$^+$. A possible way to instantiate a SETAF to a formal logic is by generating an AAF (e.g., using Definition 6.1) and then mapping the generated AAF to the logic. This, however would result in an exponentially larger number of arguments. Moreover, as argued in [34], SETAF and other extensions of AAF, should not be viewed as abstractions of underlying theories in some formal logics, but rather as models of human reasoning.

Other research literature related to SETAF is mostly concerned with more practical matters. For example, [36] deals with the computation of preferred extensions. In [13], SETAF is leveraged for representing other notions such as higher-level attacks (or else, attacks on attacks); in that work, the model for representing higher-level attacks uses the machinery of SETAF for representing joint attacks. Further, [14] uses SETAF as the underlying framework for representing evidence against an argument in order to support evidence-based reasoning, whereas [15] extends SETAF to develop a formal argument-based framework for coalition formation.

As next steps of this work, one could consider enhancing the expressiveness of SETAF by extending its basic model with features similar to the ones used in extensions of the AAF model, such as the introduction of a joint support relation, weights on (joint) attacks, values promoted by (sets of) arguments, or a preference relation among (sets of) arguments. This would allow associating SETAFs with the corresponding AAF extensions, i.e. frameworks for bipolar argumentation [37,38], graded [33] or weighted argumentation [39] (along with the introduction of ranking-based semantics [40]), value-based [41], or preference-based argumentation [42] respectively. Given the almost perfect transferability of results from AAFs to SETAFs in the classical case, we expect the proposed extensions to be also smooth. Moreover, one could also consider studying al-

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

ternative (more compact) translation schemes (from SETAFs to AAFs), in the spirit discussed in Subsection 6.10, and examine the extent to which the results presented in this paper apply under these translation schemes.

Conflict of interest statement

None declared.

Acknowledgements

Work partly performed while Giorgos Flouris was on sabbatical leave in University College London (UCL), United Kingdom. The authors would like to thank the anonymous reviewers for helpful comments that helped improve the quality of the paper, and also for providing the ideas that motivated the inclusion of Subsection 6.10.

Appendix A. Proofs

A.1. Proofs for Section 5

Proof of Theorem 5.3.

(1) Take some $\mathbf{A} \in \mathcal{E}$ and take some λ such as $\lambda(a) = \texttt{in}$ if and only if $a \in \mathbf{A}$, $\lambda(a) = \texttt{undec}$ otherwise. Clearly, $Ext(\lambda) = \mathbf{A}$, so Ext is surjective.
(2) Suppose that $Lab(\mathbf{A}) = Lab(\mathbf{B}) = \lambda$. Then, by the definition of Lab, $a \in \mathbf{A}$ if and only if $\lambda(a) = \texttt{in}$ and also $a \in \mathbf{B}$ if and only if $\lambda(a) = \texttt{in}$, so $\mathbf{A} = \mathbf{B}$. Thus, Lab is injective. □

Proof of Theorem 5.4. Set $\lambda = Lab(\mathbf{A})$. Then: $a \in Ext(Lab(\mathbf{A})) \Leftrightarrow \lambda(a) = \texttt{in} \Leftrightarrow a \in \mathbf{A}$. □

Proof of Theorem 5.6. ($\Rightarrow$) We will show that λ is proper. Suppose that $\lambda(a) = \texttt{out}$. Then, by the definition of Lab, this is true if and only if $Ext(\lambda) \blacktriangleright a$, which, by the definition of Ext, is equivalent to $\texttt{in}(\lambda) \blacktriangleright a$. So λ is proper.

($\Leftarrow$) Let λ be a proper labelling and set $\lambda' = Lab(Ext(\lambda))$. We will show that $\lambda' = \lambda$. Indeed, $\lambda'(a) = \texttt{in}$ if and only if $a \in Ext(\lambda)$ which is true if and only if $a \in \texttt{in}(\lambda) \Leftrightarrow \lambda(a) = \texttt{in}$.

Similarly, $\lambda'(a) = \texttt{out}$ if and only if $a \notin Ext(\lambda)$, $Ext(\lambda) \blacktriangleright a$; using Definition 5.5 and the fact that $Ext(\lambda) = \texttt{in}(\lambda)$, this is equivalent to $\lambda(a) = \texttt{out}$.

From the above equivalences we can easily also conclude that $\lambda'(a) = \texttt{undec}$ if and only if $\lambda(a) = \texttt{undec}$, which means that $\lambda' = \lambda$ and concludes the proof. □

Proof of Theorem 5.8. Take a **co**-labelling λ. We note that, by definition, $\lambda(a) = \texttt{out}$ if and only if $\texttt{in}(\lambda) \blacktriangleright a$, so a **co**-labelling is proper.

Clearly, **gr**, **pr**, **se**, **ea** and **id**-labellings are complete, so the result carries over to these cases as well.

Now take a **st**-labelling λ. We note that a **st**-labelling is also a **cf**-labelling, so $\lambda(a) = \texttt{out}$ implies that $\texttt{in}(\lambda) \blacktriangleright a$, so it suffices to show the opposite implication. Indeed, suppose that $\texttt{in}(\lambda) \blacktriangleright a$. If $\lambda(a) = \texttt{in}$, we get that λ is not conflict-free, a contradiction; also, by the definition of **st**-labellings, $\texttt{undec}(\lambda) = \emptyset$, which forces us to accept that $\lambda(a) = \texttt{out}$. So λ is proper.

Now take a **sg**-labelling λ. We note that a **sg**-labelling is also a **cf**-labelling, so $\lambda(a) = \texttt{out}$ implies that $\texttt{in}(\lambda) \blacktriangleright a$, so it suffices to show the opposite implication. Indeed, suppose that $\texttt{in}(\lambda) \blacktriangleright a$. If $\lambda(a) = \texttt{in}$, we get that λ is not conflict-free, a contradiction. If $\lambda(a) = \texttt{undec}$, then we create a new labelling, λ', such that $\lambda'(a) = \texttt{out}$ and $\lambda'(x) = \lambda(x)$ for $x \neq a$. Clearly, λ' is conflict-free, and contains less elements marked as $\texttt{undec}$, which contradicts the minimality criterion of stage labellings, i.e., the hypothesis that λ is stage. We conclude that $\lambda(a) = \texttt{out}$. This shows that λ is a proper labelling. □

Theorems 5.10 and 5.11 consist of 11 parts, one for each of the semantics considered in the theorems. The proof of each part relies on the preceding parts of both theorems. For example, the proofs of the 9th parts of both theorems, i.e. the proofs for eager semantics, rely on parts 1–8 of both theorems.

Proof of Theorem 5.10.

1. *For* $\sigma = \mathbf{cf}$:
We set $\lambda = Lab(\mathbf{A})$. For any argument $a \in \mathbf{Args}$ the following conditions hold:
(i) $\lambda(a) = \texttt{in} \Leftrightarrow a \in \mathbf{A}$ (Definition 5.1) $\Rightarrow$
$\nexists \mathbf{B}$ such that $\forall b \in \mathbf{B} : b \in \mathbf{A}$ and $\mathbf{B} \blacktriangleright a$ (conflict-freeness) $\Rightarrow$
$\nexists \mathbf{B}$ such that $\forall b \in \mathbf{B} : \lambda(b) = \texttt{in}$ and $\mathbf{B} \blacktriangleright a$ (Definition 5.1) (1)

UNCORRECTED PROOF

(ii) $\lambda(a) = \texttt{out} \Leftrightarrow \mathbf{A} \blacktriangleright a$ and $a \notin \mathbf{A}$ (Definition 5.1) $\Rightarrow$
$\exists \mathbf{B}$ such that $\forall b \in \mathbf{B} : b \in \mathbf{A}$ and $\mathbf{B} \blacktriangleright a \Rightarrow$
$\exists \mathbf{B}$ such that $\forall b \in \mathbf{B} : \lambda(b) = \texttt{in}$ and $\mathbf{B} \blacktriangleright a$ (Definition 5.1) (2)
From (1), (2) and Definition 4.2, we conclude that λ is a conflict-free labelling of AF^S.

2. *For* $\sigma =$ **ad**:
We set $\lambda = Lab(\mathbf{A})$. For any argument $a \in$ **Args** the following conditions hold:
(i) $\lambda(a) = \texttt{in} \Leftrightarrow a \in \mathbf{A}$ (Definition 5.1) $\Rightarrow$
for all $\mathbf{B} \blacktriangleright a$: $\mathbf{A} \blacktriangleright \mathbf{B}$ (Definition 3.2) $\Rightarrow$
for all $\mathbf{B} \blacktriangleright a$: $\exists b \in \mathbf{B}$ such that $\lambda(b) = \texttt{out}$ (Definition 5.1) (1)
(ii) $\lambda(a) = \texttt{out} \Leftrightarrow \mathbf{A} \blacktriangleright a$ and $a \notin \mathbf{A}$ (Definition 5.1) $\Rightarrow$
$\texttt{in}(\lambda) \blacktriangleright a$ (Definition 5.1) (2)
From (1), (2) and Definition 4.2, we conclude that λ is an admissible labelling of AF^S.

3. *For* $\sigma =$ **co**:
We set $\lambda = Lab(\mathbf{A})$. For any argument $a \in$ **Args** the following conditions hold:
(i) $\lambda(a) = \texttt{in} \Leftrightarrow a \in \mathbf{A}$ (Definition 5.1) $\Leftrightarrow$
for all $\mathbf{B} \blacktriangleright a$: $\mathbf{A} \blacktriangleright \mathbf{B}$ (Definition 3.3) $\Leftrightarrow$
for all $\mathbf{B} \blacktriangleright a$: $\exists b \in \mathbf{B}$ such that $\mathbf{A} \blacktriangleright b$ (by the definition of $\blacktriangleright$) $\Leftrightarrow$
for all $\mathbf{B} \blacktriangleright a$: $\exists b \in \mathbf{B}$ such that $\lambda(b) = \texttt{out}$ (Definition 5.1) (1)
(ii) $\lambda(a) = \texttt{out} \Leftrightarrow \mathbf{A} \blacktriangleright a$ and $a \notin \mathbf{A}$ (Definition 5.1) $\Leftrightarrow$
$\mathbf{A} \blacktriangleright a$ (since by Definitions 3.2 and 3.3, $\mathbf{A}$ is conflict-free) $\Leftrightarrow$
$\texttt{in}(\lambda) \blacktriangleright a$ (Definition 5.1) (2)
From (1), (2) and Definition 4.2, we conclude that λ is a complete labelling of AF^S.

4. *For* $\sigma =$ **pr**:
By definition, every preferred extension of AF^S is also a complete extension of AF^S, therefore $\mathbf{A}$ is a complete extension of $AF^S \Rightarrow$
$\lambda = Lab(\mathbf{A})$ is a complete labelling of AF^S (by part 3 of this theorem for $\sigma =$ **co**)
Moreover, by the definition of preferred extensions, $\mathbf{A}$ is maximal w.r.t. set inclusion among the complete extensions of AF^S (1)
Suppose there is another complete labelling λ' such that $\texttt{in}(\lambda) \subset \texttt{in}(\lambda')$. Then, $\mathbf{A}' = Ext(\lambda') = \texttt{in}(\lambda')$ would be a complete extension of AF^S (by Theorem 5.11-3) and $\mathbf{A} = \texttt{in}(\lambda) \subset \mathbf{A}'$, which violates (1). Therefore, $\texttt{in}(\lambda)$ is maximal among the complete labellings of AF^S. Therefore, λ is a preferred labelling of AF^S.

5. *For* $\sigma =$ **gr**:
By definition, every grounded extension of AF^S is also a complete extension of AF^S, therefore $\mathbf{A}$ is a complete extension of $AF^S \Rightarrow$
$\lambda = Lab(\mathbf{A})$ is a complete labelling of AF^S (by part 3 of this theorem for $\sigma =$ **co**)
Moreover, by the definition of grounded extensions, $\mathbf{A}$ is minimal w.r.t. set inclusion among the complete extensions of AF^S (1)
Suppose there is another complete labelling λ' such that $\texttt{in}(\lambda') \subset \texttt{in}(\lambda)$. Then, $\mathbf{A}' = Ext(\lambda') = \texttt{in}(\lambda')$ would be a complete extension of AF^S (by Theorem 5.11-3) and $\mathbf{A}' \subset \texttt{in}(\lambda) = \mathbf{A}$, which violates (1). Therefore, $\texttt{in}(\lambda)$ is minimal among the complete labellings of AF^S. Therefore, λ is a grounded labelling of AF^S.

6. *For* $\sigma =$ **st**:
By definition, every stable extension of AF^S is a conflict-free subset of **Args**, therefore $\mathbf{A}$ is a conflict-free subset of **Args** $\Rightarrow$
$\lambda = Lab(\mathbf{A})$ is a conflict-free labelling of AF^S (by part 1 of this theorem for $\sigma =$ **cf**)
Moreover, for every argument $a \in$ **Args**: either $a \in \mathbf{A} \Rightarrow \lambda(a) = \texttt{in}$ (by Definition 5.1) or $a \notin \mathbf{A} \Rightarrow \mathbf{A} \blacktriangleright a$ (by Definition 3.6) $\Rightarrow \lambda(a) = \texttt{out}$ (by Definition 5.1). Therefore, $\texttt{undec}(\lambda) = \emptyset$ and λ is a conflict-free labelling of AF^S, which means that λ is a stable labelling of AF^S.

7. *For* $\sigma =$ **na**:
$\mathbf{A}$ is conflict-free (by definition of naive extensions), therefore $\lambda = Lab(\mathbf{A})$ is a conflict-free labelling of AF^S (by part 1 of this theorem for $\sigma =$ **cf**).
Moreover, by the definition of naive extensions, $\mathbf{A}$ is maximal w.r.t. set inclusion among the conflict-free subsets of **Args** (1)
Suppose there is another conflict-free labelling λ' such that $\texttt{in}(\lambda) \subset \texttt{in}(\lambda')$. Then, $\mathbf{A}' = Ext(\lambda') = \texttt{in}(\lambda')$ would be a conflict-free subset of **Args** (by Theorem 5.11-1) and $\mathbf{A} = \texttt{in}(\lambda) \subset \mathbf{A}'$, which violates (1). Therefore, $\texttt{in}(\lambda)$ is maximal among the conflict-free labellings of **Args**. Therefore, λ is a naive labelling of AF^S.

8. *For* $\sigma =$ **se**:
By definition, every semi-stable extension of AF^S is also a complete extension of AF^S, therefore $\mathbf{A}$ is a complete extension of $AF^S \Rightarrow$
$\lambda = Lab(\mathbf{A})$ is a complete labelling (by part 3 of this theorem for $\sigma =$ **co**)
Moreover, by the definition of semi-stable extensions, the set $\mathbf{A} \cup \{b \in \textbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal w.r.t. set inclusion among the complete extensions of AF^S (1)

UNCORRECTED PROOF

Suppose there is another complete labelling λ' such that $\mathtt{in}(\lambda) \cup \{b \in \mathbf{Args} \mid \mathtt{in}(\lambda) \blacktriangleright b\} \subset \mathtt{in}(\lambda') \cup \{b \in \mathbf{Args} \mid \mathtt{in}(\lambda') \blacktriangleright b\}$. Then, $\mathbf{A}' = Ext(\lambda') = \mathtt{in}(\lambda')$ would be a complete extension of AF^S (by Theorem 5.11-3) and $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A}) \blacktriangleright b\} = \mathtt{in}(\lambda) \cup \{b \in \mathbf{Args} \mid \mathtt{in}(\lambda) \blacktriangleright b\} \subset \mathbf{A}' \cup \{b \in \mathbf{Args} \mid \mathbf{A}' \blacktriangleright b\}$, which violates (1). Therefore, $\mathtt{in}(\lambda) \cup \{b \in \mathbf{Args} \mid \mathtt{in}(\lambda) \blacktriangleright b\}$ is maximal among the complete labellings of AF^S. Therefore, λ is a semi-stable labelling of AF^S.

9. *For* $\sigma = \mathbf{ea}$:
By definition, every eager extension of AF^S is also a complete extension of AF^S, therefore **A** is a complete extension of $AF^S \Rightarrow$
$\lambda = Lab(\mathbf{A})$ is a complete labelling of AF^S (by part 3 of this theorem for $\sigma = \mathbf{co}$)
Moreover, by the definition of eager extensions:
(1) $\mathbf{A} \subseteq \mathbf{B}$ for all semi-stable extensions **B** of AF^S and
(2) $\mathbf{C} \subseteq \mathbf{A}$ for all complete extensions **C** of AF^S that satisfy (1)
Suppose there is a semi-stable labelling λ' such that $\mathtt{in}(\lambda) \not\subseteq \mathtt{in}(\lambda')$. Then, $\mathbf{A}' = Ext(\lambda') = \mathtt{in}(\lambda')$ would be a semi-stable extension of AF^S (by Theorem 5.11-8) and $\mathbf{A}' \subset \mathtt{in}(\lambda) = \mathbf{A}$, which violates (1).
Suppose there is a complete labelling λ' such that $\mathtt{in}(\lambda') \subset \mathtt{in}(\lambda'')$ for all semi-stable labellings λ'' of AF^S and $\mathtt{in}(\lambda) \subset \mathtt{in}(\lambda')$. Then, $\mathbf{A}' = Ext(\lambda') = \mathtt{in}(\lambda')$ would be a complete extension of AF^S (by Theorem 5.11-3) and a subset of all semi-stable extensions of AF^S (since by Theorem 5.11-8 and part 8 of this theorem, all the semi-stable extensions of AF^S can be generated from the semi-stable labellings of AF^S using the function *Ext*) and $\mathbf{A} \subset \mathbf{A}'$, which violates (2).
Therefore, $\mathtt{in}(\lambda)$ is a complete labelling of AF^S, $\mathtt{in}(\lambda) \subseteq \mathtt{in}(\lambda')$ for all semi-stable labellings λ' of AF^S, and $\mathtt{in}(\lambda)$ is maximal w.r.t. set inclusion among all labellings satisfying the previous two conditions. Therefore λ is an eager labelling of AF^S.

10. *For* $\sigma = \mathbf{id}$:
By definition, every ideal extension of AF^S is also a complete extension of AF^S, therefore **A** is a complete extension of $AF^S \Rightarrow$
$\lambda = Lab(\mathbf{A})$ is a complete labelling of AF^S (by part 3 of this theorem for $\sigma = \mathbf{co}$)
Moreover, by the definition of ideal extensions:
(1) $\mathbf{A} \subseteq \mathbf{B}$ for all preferred extensions **B** of AF^S and
(2) $\mathbf{C} \subseteq \mathbf{A}$ for all complete extensions **C** of AF^S that satisfy (1)
Suppose there is a preferred labelling λ' such that $\mathtt{in}(\lambda) \not\subseteq \mathtt{in}(\lambda')$. Then, $\mathbf{A}' = Ext(\lambda') = \mathtt{in}(\lambda')$ would be a preferred extension of AF^S (by Theorem 5.11-4) and $\mathbf{A}' \subset \mathtt{in}(\lambda) = \mathbf{A}$, which violates (1).
Suppose there is a complete labelling λ' such that $\mathtt{in}(\lambda') \subset \mathtt{in}(\lambda'')$ for all preferred labellings λ'' of AF^S and $\mathtt{in}(\lambda) \subset \mathtt{in}(\lambda')$. Then, $\mathbf{A}' = Ext(\lambda') = \mathtt{in}(\lambda')$ would be a complete extension of AF^S (by Theorem 5.11-3) and a subset of all preferred extensions of AF^S (since by Theorem 5.11-4 and part 4 of this theorem, all the preferred extensions of AF^S can be generated from the preferred labellings of AF^S using the function *Ext*) and $\mathbf{A} \subset \mathbf{A}'$, which violates (2).
Therefore, $\mathtt{in}(\lambda)$ is a complete labelling of AF^S, $\mathtt{in}(\lambda) \subseteq \mathtt{in}(\lambda')$ for all preferred labellings λ' of AF^S, and $\mathtt{in}(\lambda)$ is maximal w.r.t. set inclusion among all labellings satisfying the previous two conditions. Therefore λ is an ideal labelling of AF^S.

11. *For* $\sigma = \mathbf{sg}$:
By the definition of stage extensions, **A** is conflict-free, therefore $\lambda = Lab(\mathbf{A})$ is a conflict-free labelling of AF^S (by part 1 of this theorem for $\sigma = \mathbf{cf}$). Moreover, by the definition of stage extensions, the set $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal w.r.t. set inclusion among the conflict-free subsets of **Args** (1)
Suppose there is another conflict-free labelling λ' such that $\mathtt{in}(\lambda) \cup \{b \in \mathbf{Args} \mid \mathtt{in}(\lambda) \blacktriangleright b\} \subset \mathtt{in}(\lambda') \cup \{b \in \mathbf{Args} \mid \mathtt{in}(\lambda') \blacktriangleright b\}$.
Then, $\mathbf{A}' = Ext(\lambda') = \mathtt{in}(\lambda')$ would be a conflict-free subset of **Args** (by Theorem 5.11-1) and $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A}) \blacktriangleright b\} = \mathtt{in}(\lambda) \cup \{b \in \mathbf{Args} \mid \mathtt{in}(\lambda) \blacktriangleright b\} \subset \mathbf{A}' \cup \{b \in \mathbf{Args} \mid \mathbf{A}' \blacktriangleright b\}$, which violates (1).
Therefore, $\mathtt{in}(\lambda) \cup \{b \in \mathbf{Args} \mid \mathtt{in}(\lambda) \blacktriangleright b\}$ is maximal among the conflict-free labellings of AF^S, which means that $\mathtt{undec}(\lambda)$ is minimal among the conflict-free labellings of AF^S. Therefore, λ is a stage labelling of AF^S. □

Proof of Theorem 5.11.

1. *For* $\sigma = \mathbf{cf}$:
We set $\mathbf{A} = Ext(\lambda)$. According to Definition 5.2:
$\mathbf{A} = \{a \in \mathbf{Args} \mid \lambda(a) = \mathtt{in}\} \Rightarrow$
$\mathbf{A} = \{a \in \mathbf{Args} \mid \nexists \mathbf{B} \subseteq \mathtt{in}(\lambda) : \mathbf{B} \blacktriangleright a\}$ (Definition 4.2: cf-labellings) $\Rightarrow$
$\mathbf{A} = \{a \in \mathbf{Args} \mid \nexists \mathbf{B} \subseteq Ext(\lambda) : \mathbf{B} \blacktriangleright a\}$ (Definition 5.2) $\Rightarrow$
A does not attack itself, therefore, by Definition 3.1, **A** is conflict-free.

2. *For* $\sigma = \mathbf{ad}$:
We set $\mathbf{A} = Ext(\lambda)$. λ is admissible and, therefore, by the definition of admissible labellings (Definition 4.2), conflict-free, and, therefore, by the first part of this theorem, **A** is conflict-free (1).
For any $\mathbf{B} \subseteq \mathbf{Args}$ such that $\mathbf{B} \blacktriangleright \mathbf{A}$: By the definition of $\blacktriangleright$, there is some $a \in \mathbf{A}$ such that $\mathbf{B} \blacktriangleright a$. Since $a \in \mathbf{A}$, $\lambda(a) = \mathtt{in}$. Combining these facts and the definition of admissible labellings (Definition 4.2), it follows that there is some $b \in \mathbf{B}$ such

UNCORRECTED PROOF

that $\lambda(b) = \texttt{out}$, so (by Definition 4.2 again) $\exists \mathbf{A}' \blacktriangleright b, \forall a' \in \mathbf{A}' : \lambda(a') = \texttt{in}$, so, by Definition 5.2, $\mathbf{A}' \subseteq \mathbf{A}$. We conclude that $\mathbf{A} \blacktriangleright b$. Therefore, for all $\mathbf{B} \blacktriangleright \mathbf{A}$: $\mathbf{A} \blacktriangleright \mathbf{B}$ (2).
From (1), (2) and Definition 3.2, we conclude that $\mathbf{A}$ is an admissible extension of **Args**.

3. *For* $\sigma =$ **co**:
We set $\mathbf{A} = Ext(\lambda)$. λ is complete and, therefore, by the definition of complete labellings (Definition 4.2), conflict-free, and, therefore, by the first part of this theorem, $\mathbf{A}$ is conflict-free (1).
For any $\mathbf{B} \subseteq$ **Args** such that $\mathbf{B} \blacktriangleright \mathbf{A}$: By the definition of $\blacktriangleright$, there is some $a \in \mathbf{A}$ such that $\mathbf{B} \blacktriangleright a$. Since $a \in \mathbf{A}$, $\lambda(a) = \texttt{in}$. Combining these facts and the definition of complete labellings (Definition 4.2), it follows that there is some $b \in \mathbf{B}$ such that $\lambda(b) = \texttt{out}$, so (by Definition 4.2 again) $\exists \mathbf{A}' \blacktriangleright b, \forall a' \in \mathbf{A}' : \lambda(a') = \texttt{in}$, so, by Definition 5.2, $\mathbf{A}' \subseteq \mathbf{A}$. We conclude that $\mathbf{A} \blacktriangleright b$. Therefore, for all $\mathbf{B} \blacktriangleright \mathbf{A}$: $\mathbf{A} \blacktriangleright \mathbf{B}$ (2).
Thirdly, suppose that for an argument $a \in$ **Args**, $\mathbf{A} \blacktriangleright \mathbf{B}$ for any $\mathbf{B} \subseteq$ **Args** such that $\mathbf{B} \blacktriangleright a \Rightarrow \exists b \in \mathbf{B} : \mathbf{A} \blacktriangleright b$, thus $\lambda(b) = \texttt{out}$ and $\lambda(a) = \texttt{in}$ (by Definitions 4.2 and 5.2). Therefore $a \in \mathbf{A}$ (by Definition 5.2) (3).
Combining (1)-(3) and Definition 3.3, we conclude that $\mathbf{A}$ is a complete extension of AF^S.

4. *For* $\sigma =$ **pr**:
By the definition of preferred labellings, λ is a complete labelling, and therefore, by part 3 of this theorem for $\sigma =$ **co**, $\mathbf{A} = Ext(\lambda) = \texttt{in}(\lambda)$ is a complete extension of AF^S. Moreover, $\texttt{in}(\lambda)$ is maximal w.r.t. set inclusion among the complete labellings of AF^S. Therefore, $\mathbf{A}$ is maximal among the complete extensions generated by the complete labellings of AF^S (as described in Definition 5.2). By Theorem 5.10-3 and Theorem 5.4, all the complete extensions of AF^S can be generated from the complete labellings of AF^S using the function *Ext*. Therefore, $\mathbf{A}$ is maximal w.r.t. set inclusion among all complete extensions of AF^S, and therefore, by Definition 3.4, a preferred extension of AF^S.

5. *For* $\sigma =$ **gr**:
By the definition of grounded labellings, λ is a complete labelling, and therefore, by part 3 of this theorem for $\sigma =$ **co**, $\mathbf{A} = Ext(\lambda) = \texttt{in}(\lambda)$ is a complete extension of AF^S. Moreover, $\texttt{in}(\lambda)$ is minimal w.r.t. set inclusion among the complete labellings of AF^S. Therefore, $\mathbf{A}$ is minimal among the complete extensions generated by the complete labellings of AF^S (as described in Definition 5.2). As shown in the proof above (for $\sigma =$ **pr**), there is no complete extension that is not generated by a complete labelling of AF^S as described in Definition 5.2. Therefore, $\mathbf{A}$ is minimal w.r.t. set inclusion among the complete extensions of AF^S, and therefore, by Definition 3.5, a grounded extension of AF^S.

6. *For* $\sigma =$ **st**:
We set $\mathbf{A} = Ext(\lambda) = \texttt{in}(\lambda)$. By the definition of stable labellings, for any argument $a \in$ **Args**, either:
(1) $\lambda(a) = \texttt{in} \Leftrightarrow a \in \mathbf{A}$ (by Definition 5.2) or
(2) $\lambda(a) = \texttt{out} \Rightarrow$
$\exists \mathbf{B} \subseteq$ **Args** such that $\mathbf{B} \blacktriangleright a$ and $\forall b \in \mathbf{B}$: $\lambda(b) = \texttt{in}$ (since λ is conflict-free) $\Rightarrow$
$\exists \mathbf{B} \subseteq$ **Args** such that $\mathbf{B} \blacktriangleright a$ and $\forall b \in \mathbf{B}$: $b \in \mathbf{A}$ (by Definition 5.2) $\Rightarrow \mathbf{A} \blacktriangleright a$
Summing up: $\forall a \in$ **Args**: $a \in \mathbf{A}$ or $\mathbf{A} \blacktriangleright a$ (3).
Moreover, by the definition of stable labellings, λ is also a conflict-free labelling of AF^S, and, therefore, by part 1 of this theorem for $\sigma =$ **cf**, $\mathbf{A}$ is conflict-free. By combining the latter fact with (3) and Definition 3.6, we conclude that $\mathbf{A}$ is a stable extension of AF^S.

7. *For* $\sigma =$ **na**:
By the definition of naive labellings, λ is a conflict-free labelling, and therefore, by part 1 of this theorem for $\sigma =$ **cf**, $\mathbf{A} = Ext(\lambda) = \texttt{in}(\lambda)$ is a conflict-free subset of **Args**. Moreover, $\texttt{in}(\lambda)$ is maximal w.r.t. set inclusion among the conflict-free labellings of AF^S. Therefore, $\mathbf{A}$ is maximal among the conflict free sets of arguments generated by the conflict-free labellings of AF^S (as described in Definition 5.2). By Theorem 5.10-1 and Theorem 5.4, all the conflict-free extensions of AF^S can be generated from the conflict-free labellings of AF^S using the function *Ext*. We, therefore, conclude that $\mathbf{A}$ is maximal w.r.t. set inclusion among all conflict-free subsets of **Args**, and therefore, by Definition 3.7 a naive extension of AF^S.

8. *For* $\sigma =$ **se**:
By the definition of semi-stable labellings, λ is a complete labelling, and therefore, by part 3 of this theorem for $\sigma =$ **co**, $\mathbf{A} = Ext(\lambda) = \texttt{in}(\lambda)$ is a complete extension of AF^S. Moreover, $\texttt{undec}(\lambda)$ is minimal, and therefore $\texttt{in}(\lambda) \cup \texttt{out}(\lambda)$ is maximal w.r.t. set inclusion among the complete labellings of AF^S. Therefore, by the definition of complete labellings, $\texttt{in}(\lambda) \cup \{b \in \textbf{Args} \mid \texttt{in}(\lambda) \blacktriangleright b\}$ is maximal w.r.t. set inclusion among the complete labellings of AF^S, and by Definition 5.2, $\mathbf{A} \cup \{b \in \textbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal among the complete extensions generated by the complete labellings of AF^S (as described in Definition 5.2). As already proved, there is no complete extension that is not generated by a complete labelling of AF^S as described in Definition 5.2. Therefore, $\mathbf{A} \cup \{b \in \textbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal w.r.t. set inclusion among the complete extensions of AF^S, and therefore, by Definition 3.8, a semi-stable extension of AF^S.

9. *For* $\sigma =$ **ea**:
By the definition of eager labellings, λ is a complete labelling, and therefore, by part 3 of this theorem for $\sigma =$ **co**, $\mathbf{A} = Ext(\lambda) = \texttt{in}(\lambda)$ is a complete extension of AF^S (1)
Moreover, $\texttt{in}(\lambda) \subseteq \texttt{in}(\lambda')$ for every semi-stable labelling λ' of AF^S. Therefore, by Definition 5.2 and part 8 of this theorem for $\sigma =$**se**, $\mathbf{A} \subseteq \mathbf{A}'$ for every semi-stable extension $\mathbf{A}' = Ext(\lambda')$ generated (as described in Definition 5.2) by a semi-stable labelling λ' of AF^S. By Theorem 5.10-8 and Theorem 5.4, all the semi-stable extensions of AF^S can be

UNCORRECTED PROOF

generated from the semi-stable labellings of AF^S using the function Ext. We, therefore, conclude that $\mathbf{A}$ is a subset of every semi-stable extension of AF^S (2)
Finally, $\mathtt{in}(\lambda)$ is maximal w.r.t. set inclusion among all labellings of AF^S satisfying conditions (1) and (2). Therefore, $\mathbf{A}$ is maximal w.r.t. set inclusion among all complete extensions generated by the complete labellings of AF^S (as in Definition 5.2), satisfying conditions (1) and (2). As already proved, there is no complete extension that is not generated by a complete labelling of AF^S as described in Definition 5.2. We, therefore, conclude that $\mathbf{A}$ is maximal w.r.t. set inclusion among all complete extensions of AF^S that are subsets of every semi-stable extension of AF^S. By Definition 3.9, it is, therefore, an eager extension of AF^S.

10. *For* $\sigma = \mathbf{id}$:
By the definition of ideal labellings, λ is a complete labelling, and therefore, by part 3 of this theorem for $\sigma = \mathbf{co}$, $\mathbf{A} = Ext(\lambda) = \mathtt{in}(\lambda)$ is a complete extension of AF^S (1)
Moreover, $\mathtt{in}(\lambda) \subseteq \mathtt{in}(\lambda')$ for every semi-stable labelling λ' of AF^S. Therefore, by Definition 5.2 and part 4 of this theorem for $\sigma = \mathbf{pr}$, $\mathbf{A} \subseteq \mathbf{A}'$ for every preferred extension $\mathbf{A}' = Ext(\lambda')$ generated (as described in Definition 5.2) by a preferred labelling λ' of AF^S. By Theorem 5.10-4 and Theorem 5.4, all the preferred extensions of AF^S can be generated from the preferred labellings of AF^S using the function Ext. We, therefore, conclude that $\mathbf{A}$ is a subset of every preferred extension of AF^S (2)
Finally, $\mathtt{in}(\lambda)$ is maximal w.r.t. set inclusion among all labellings of AF^S satisfying conditions (1) and (2). Therefore, $\mathbf{A}$ is maximal w.r.t. set inclusion among all complete extensions generated by the complete labellings of AF^S (as in Definition 5.2), satisfying conditions (1) and (2). As already proved, there is no complete extension that is not generated by a complete labelling of AF^S as described in Definition 5.2. We, therefore, conclude that $\mathbf{A}$ is maximal w.r.t. set inclusion among all complete extensions of AF^S that are subsets of every preferred extension of AF^S. By Definition 3.10, it is, therefore, an ideal extension of AF^S.

11. *For* $\sigma = \mathbf{sg}$:
By the definition of stage labellings, λ is a conflict-free labelling, and therefore, by part 1 of this theorem for $\sigma = \mathbf{cf}$, $\mathbf{A} = Ext(\lambda) = \mathtt{in}(\lambda)$ is a conflict-free subset of **Args**. Moreover, $\mathtt{undec}(\lambda)$ is minimal, and therefore $\mathtt{in}(\lambda) \cup \mathtt{out}(\lambda)$ is maximal w.r.t. set inclusion among the conflict-free labellings of AF^S. Therefore, by the definition of conflict-free labellings, $\mathtt{in}(\lambda) \cup \{b \in \mathbf{Args} \mid \mathtt{in}(\lambda) \blacktriangleright b\}$ is maximal w.r.t. set inclusion among the conflict-free labellings of AF^S, and by Definition 5.2, $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal among the conflict-free subsets of **Args** generated by the conflict-free labellings of AF^S (as described in Definition 5.2). As already proved, there is no conflict-free subset of **Args** that is not generated by a conflict-free labelling of AF^S as described in Definition 5.2. Therefore, $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal w.r.t. set inclusion among the conflict-free subsets of **Args**, and therefore, by Definition 3.11, a stage extension of AF^S. □

A.2. Proofs for Section 6

Lemma 1. *The following points hold:*

1. *If* $\mathbf{A} \rightarrowtail \mathbf{B}$, $\mathbf{A} \subseteq \mathbf{A}'$ *and* $\mathbf{B} \subseteq \mathbf{B}'$, *then* $\mathbf{A}' \rightarrowtail \mathbf{B}'$
2. *If* $\mathbf{A} \rightarrowtail \mathbf{B}$, *then there exists some* $\mathbf{A}' \subseteq \mathbf{A}$ *such that* $\mathbf{A}' \rightarrowtail \mathbf{B}$
3. *If* $\mathbf{A} \rightarrowtail \mathbf{B}$, *then there exists some* $b \in \mathbf{B}$ *such that* $\mathbf{A} \rightarrowtail \{b\}$
4. $\{\mathbf{C} \mid \mathbf{A}' \rightarrowtail \mathbf{C}$ *for some* $\mathbf{A}' \in \overline{\mathbf{A}}\} = \{\mathbf{C} \mid \mathbf{A} \rightarrowtail \mathbf{C}\}$.

Proof. Obvious from the properties of $\blacktriangleright$ and the definition of $\rightarrowtail$. □

Lemma 2. *If* $\mathcal{E} \in \mathcal{E}_D^{\mathbf{cf}}$, *then:*

1. *If* $\mathcal{E}' \subseteq \mathcal{E}$ *then* $\mathcal{E}' \in \mathcal{E}_D^{\mathbf{cf}}$
2. *If* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\mathbf{B} \in \mathcal{E}$, *then* $\mathcal{E} \cup \{\mathbf{A}\} \in \mathcal{E}_D^{\mathbf{cf}}$
3. *If* $\mathbf{A} \in \mathcal{E}$, *then* $\mathcal{E} \cup \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{cf}}$.

Proof. #1 is obvious.
Regarding #2, consider some $\mathbf{C}$ such that $\mathbf{C} \rightarrowtail \mathbf{A}$. Then, $\mathbf{C} \rightarrowtail \mathbf{B}$ (Lemma 1). Since $\mathcal{E}$ is conflict-free and $\mathbf{B} \in \mathcal{E}$, $\mathbf{C} \notin \mathcal{E}$. If $\mathbf{C} = \mathbf{A}$ then $\mathbf{A} \rightarrowtail \mathbf{B}$, so $\mathbf{B} \rightarrowtail \mathbf{B}$ and $\mathbf{B} \in \mathcal{E}$, a contradiction since $\mathcal{E}$ is conflict-free. So $\mathbf{C} \notin \mathcal{E} \cup \{\mathbf{A}\}$, thus $\mathcal{E} \cup \{\mathbf{A}\}$ is conflict-free.
#3 is a direct corollary of #2. □

Proof of Proposition 6.3.

(1) $\Leftrightarrow$ (2): $\mathbf{A} \in \mathcal{E}_S^{\mathbf{cf}} \Leftrightarrow$ (by definition of $\mathcal{E}_S^{\mathbf{cf}}$)
$\mathbf{A} \not\blacktriangleright \mathbf{A} \Leftrightarrow$ (by Definition 6.1)

UNCORRECTED PROOF

$\mathbf{A} \not\rightarrowtail \mathbf{A} \Leftrightarrow$ (by the definition of conflict-free sets in AFs [2]) $\{\mathbf{A}\} \in \mathcal{E}_D^{\mathbf{cf}}$.
(2) $\Rightarrow$ (3): direct from Lemma 2, point #3.
(3) $\Rightarrow$ (2): direct from Lemma 2, point #1.
(3) $\Rightarrow$ (4): direct from Lemma 2, point #1.
(4) $\Rightarrow$ (3): obvious. □

Proof of Proposition 6.4. If $\mathbf{A} \notin \mathcal{E}_S^{\mathbf{cf}}$, then $\mathbf{A} \blacktriangleright \mathbf{A}$, i.e., $\mathbf{A} \rightarrowtail \mathbf{A}$, which contradicts the assumption. □

Proof of Theorem 6.5. #1 is immediate from the equivalence of points #1, #3 of Proposition 6.3.

For #2, take some $\mathcal{E} \in \mathcal{E}_D^{\mathbf{cf}}$. By Proposition 6.4, $\mathcal{E} \subseteq \mathcal{E}_S^{\mathbf{cf}}$. Moreover, by the fact that $\mathcal{E} \in \mathcal{E}_D^{\mathbf{cf}}$, we conclude that $\mathbf{A} \not\rightarrowtail \mathbf{B}$, i.e., $\mathbf{A} \not\blacktriangleright \mathbf{B}$ for all $\mathbf{A}, \mathbf{B} \in \mathcal{E}$. □

Proof of Proposition 6.6. For #1, note initially that $\mathcal{E} \cup \{\mathbf{A}\}$ is conflict-free (Lemma 2, point #2). Moreover, consider some $\mathbf{C}$ such that $\mathbf{C} \rightarrowtail \mathbf{A}$. Then, $\mathbf{C} \rightarrowtail \mathbf{B}$, so there is some $\mathbf{A}' \in \mathcal{E}$ such that $\mathbf{A}' \rightarrowtail \mathbf{C}$. We conclude that $\mathcal{E} \cup \{\mathbf{A}\}$ is admissible.

#2 is a direct corollary of #1.

Regarding #3, we must first show that $\mathcal{E} \cup \{\mathbf{A} \cup \mathbf{B}\}$ is conflict-free. Indeed, suppose that $\mathcal{E} \cup \{\mathbf{A} \cup \mathbf{B}\}$ is not conflict-free, i.e., there are $\mathbf{A_1}, \mathbf{A_2} \in \mathcal{E} \cup \{\mathbf{A} \cup \mathbf{B}\}$ such that $\mathbf{A_1} \rightarrowtail \mathbf{A_2}$. Then, there is some $a \in \mathbf{A_2}$ such that $\mathbf{A_1} \rightarrowtail \{a\}$. We can conclude that there is some $\mathbf{A_3} \in \mathcal{E}$ such that $a \in \mathbf{A_3}$; indeed, if $\mathbf{A_2} \in \mathcal{E}$, then take $\mathbf{A_3} = \mathbf{A_2}$, otherwise $\mathbf{A_2} = \mathbf{A} \cup \mathbf{B}$ so $a \in \mathbf{A}$ or $a \in \mathbf{B}$, and we can take $\mathbf{A_3} = \mathbf{A}$ or $\mathbf{A_3} = \mathbf{B}$ respectively. Therefore, $\mathbf{A_1} \rightarrowtail \mathbf{A_3}$ and $\mathbf{A_3} \in \mathcal{E}$. Since $\mathcal{E}$ is conflict-free, we conclude that $\mathbf{A_1} \notin \mathcal{E}$, thus $\mathbf{A_1} = \mathbf{A} \cup \mathbf{B}$, i.e., $\mathbf{A} \cup \mathbf{B} \rightarrowtail \mathbf{A_3}$. Since $\mathbf{A_3} \in \mathcal{E}$ and $\mathcal{E}$ is admissible, there is some $\mathbf{A_4} \in \mathcal{E}$ such that $\mathbf{A_4} \rightarrowtail \mathbf{A} \cup \mathbf{B}$. But then, there is some $a' \in \mathbf{A} \cup \mathbf{B}$ such that $\mathbf{A_4} \rightarrowtail \{a'\}$, i.e., $\mathbf{A_4} \rightarrowtail \mathbf{A}$ or $\mathbf{A_4} \rightarrowtail \mathbf{B}$. This leads to a contradiction, because, $\mathbf{A_4}, \mathbf{A}, \mathbf{B} \in \mathcal{E}$ and $\mathcal{E}$ is conflict-free. We conclude that $\mathcal{E} \cup \{\mathbf{A} \cup \mathbf{B}\}$ is conflict-free.

Now consider some $\mathbf{C}$ such that $\mathbf{C} \rightarrowtail \mathbf{A} \cup \mathbf{B}$, and take any $a \in \mathbf{A} \cup \mathbf{B}$ such that $\mathbf{C} \rightarrowtail \{a\}$. Suppose that $a \in \mathbf{A}$. Then $\mathbf{C} \rightarrowtail \mathbf{A}$, so (since $\mathcal{E}$ is admissible) there is some $\mathbf{A}' \in \mathcal{E}$ such that $\mathbf{A}' \rightarrowtail \mathbf{C}$. Same arguments can be used if $a \in \mathbf{B}$. Thus, $\mathbf{A} \cup \mathbf{B}$ is acceptable with respect to $\mathcal{E}$. This concludes the proof.

For #4 the proof follows the arguments of case #3.

For #5, set $\mathbf{B} = \bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}$. We note first that $\mathcal{E} \cup \{\mathbf{B}\} \in \mathcal{E}_D^{\mathbf{ad}}$ (case #4). Also, for any $\mathbf{C} \in \overline{\mathbf{B}}$, we get that $\mathcal{E} \cup \{\mathbf{B}\} \cup \{\mathbf{C}\} \in \mathcal{E}_D^{\mathbf{ad}}$ (case #1). Thus, $\mathcal{E} \cup \overline{\mathbf{B}} \in \mathcal{E}_D^{\mathbf{ad}}$. But, by construction, $\mathcal{E} \subseteq \overline{\mathbf{B}}$, which leads to the result.

For #6, set $\mathbf{B} = \bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}$. Now take any $\mathbf{C} \rightarrowtail \mathbf{B}$. Then, since $\overline{\mathbf{B}} \in \mathcal{E}_D^{\mathbf{ad}}$ (point #5) and $\mathbf{B} \in \overline{\mathbf{B}}$, we conclude that there is some $\mathbf{D} \in \overline{\mathbf{B}}$ such that $\mathbf{D} \rightarrowtail \mathbf{C}$. Given that $\mathbf{D} \in \overline{\mathbf{B}}$, we conclude that $\mathbf{D} \subseteq \mathbf{B}$ so $\mathbf{B} \rightarrowtail \mathbf{C}$, which leads to the result. □

Proof of Proposition 6.7. #1 $\Rightarrow$ #2: take some $\mathbf{A} \in \mathcal{E}_S^{\mathbf{ad}}$. By Proposition 6.3, $\{\mathbf{A}\} \in \mathcal{E}_D^{\mathbf{cf}}$. Now consider some $\mathbf{C}$ such that $\mathbf{C} \rightarrowtail \mathbf{A}$. Then $\mathbf{C} \blacktriangleright \mathbf{A}$, so $\mathbf{A} \blacktriangleright \mathbf{C}$ (since $\mathbf{A} \in \mathcal{E}_S^{\mathbf{ad}}$), i.e., $\mathbf{A} \rightarrowtail \mathbf{C}$. These facts prove that $\{\mathbf{A}\} \in \mathcal{E}_D^{\mathbf{ad}}$.

#2 $\Rightarrow$ #1: suppose that $\{\mathbf{A}\} \in \mathcal{E}_D^{\mathbf{ad}}$. By Proposition 6.3, $\mathbf{A} \in \mathcal{E}_S^{\mathbf{cf}}$. Now consider some $\mathbf{C}$ such that $\mathbf{C} \blacktriangleright \mathbf{A}$. Then $\mathbf{C} \rightarrowtail \mathbf{A}$, so $\mathbf{A} \rightarrowtail \mathbf{C}$ (since $\{\mathbf{A}\} \in \mathcal{E}_D^{\mathbf{ad}}$), i.e., $\mathbf{A} \blacktriangleright \mathbf{C}$. These facts prove that $\mathbf{A} \in \mathcal{E}_S^{\mathbf{ad}}$.

#2 $\Rightarrow$ #3: direct from Proposition 6.6, point #5.

#3 $\Rightarrow$ #2: direct from Proposition 6.6, point #6. □

Proof of Theorem 6.8. #1 is direct from Proposition 6.7, in particular by the equivalence of points #1, #3 of that proposition.

For #2, take some $\mathcal{E} \in \mathcal{E}_D^{\mathbf{ad}}$ and set $\mathbf{D} = \bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}$. By Proposition 6.6, $\{\mathbf{D}\} \in \mathcal{E}_D^{\mathbf{ad}}$, so, by Proposition 6.7, $\mathbf{D} \in \mathcal{E}_S^{\mathbf{ad}}$. Now take some $\mathbf{C} \blacktriangleright \mathbf{D}$. By construction, there is some $\mathbf{A} \in \mathcal{E}$ and $a \in \mathbf{A}$ such that $\mathbf{C} \blacktriangleright a$, so $\mathbf{C} \blacktriangleright \mathbf{A} \Rightarrow \mathbf{C} \rightarrowtail \mathbf{A}$. Given that $\mathcal{E}$ is admissible, we can now conclude that $\mathcal{E}_D^{\mathbf{ad}} \subseteq \{\mathcal{E} \mid \bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A} \in \mathcal{E}_S^{\mathbf{ad}}$, and if $\mathbf{C} \blacktriangleright \bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}$, then there is $\mathbf{B} \in \mathcal{E}$ such that $\mathbf{B} \blacktriangleright \mathbf{C}\}$.

For the opposite inclusion, let's take some $\mathcal{E}$ as required by the theorem and set $\mathbf{D} = \bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}$. First, we will show that $\mathcal{E} \in \mathcal{E}_D^{\mathbf{cf}}$. Indeed, suppose $\mathbf{A}, \mathbf{B} \in \mathcal{E}$ such that $\mathbf{A} \rightarrowtail \mathbf{B}$. Then, $\mathbf{A} \blacktriangleright \mathbf{B}$, so $\mathbf{D} \blacktriangleright \mathbf{D}$, a contradiction by the fact that $\mathbf{D} \in \mathcal{E}_S^{\mathbf{ad}}$. Further, assume some $\mathbf{C}$ such that $\mathbf{C} \rightarrowtail \mathbf{B}$ for some $\mathbf{B} \in \mathcal{E}$. Then, $\mathbf{C} \blacktriangleright \mathbf{B}$, so $\mathbf{C} \blacktriangleright \mathbf{D}$ (because $\mathbf{B} \subseteq \mathbf{D}$), so by the definition of $\mathcal{E}$, there is some $\mathbf{B}' \in \mathcal{E}$ such that $\mathbf{B}' \blacktriangleright \mathbf{C}$, i.e., $\mathbf{B}' \rightarrowtail \mathbf{C}$, so $\mathcal{E} \in \mathcal{E}_D^{\mathbf{ad}}$ and the proof is complete. □

Proof of Proposition 6.9. For #1, note that, by Proposition 6.6, point #1, $\mathbf{A}$ is acceptable with respect to $\mathcal{E}$, so $\mathbf{A} \in \mathcal{E}$.

Point #2 is a direct corollary of point #1.

Points #3 and #4 also follow from Proposition 6.6 (points #3, #4, respectively) by noting that $\mathbf{A} \cup \mathbf{B}$ and $\bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}$ are acceptable with respect to $\mathcal{E}$.

For point #5, note that $\mathbf{B} \in \mathcal{E}$ implies that $\mathbf{B} \in \overline{\bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}}$, so $\mathcal{E} \subseteq \overline{\bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}}$. The opposite inclusion follows by combining points #4 and #2 of this proposition. □

Proof of Theorem 6.10. We will first show the theorem for $\sigma = \mathbf{co}$.

For #1: Take $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$. First we note that $\mathbf{A} \in \mathcal{E}_S^{\mathbf{ad}}$, so $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{ad}}$ (Proposition 6.7). Moreover, we can deduce that for any $a \in \mathbf{Args} \setminus \mathbf{A}$, there is some $\mathbf{B}$ such that $\mathbf{B} \blacktriangleright a$ and $\mathbf{A} \not\blacktriangleright \mathbf{B}$, i.e., $\mathbf{B} \rightarrowtail \{a\}$ and $\mathbf{A} \not\rightarrowtail \mathbf{B}$. Now take any $\mathbf{C} \notin \overline{\mathbf{A}}$. Then, there is some

UNCORRECTED PROOF

$c \in \mathbf{C} \setminus \mathbf{A}$, and by the above observation it follows that there is some $\mathbf{B}$ such that $\mathbf{B} \rightarrowtail \{c\}$ and $\mathbf{A} \not\rightarrow \mathbf{B}$, i.e., $\mathbf{B} \rightarrowtail \mathbf{C}$ and $\mathbf{A} \not\rightarrow \mathbf{B}$. Thus, $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$, i.e., $\mathcal{E}_S^{\mathbf{co}} \subseteq \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}\}$.

For the opposite inclusion, take some $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$. It follows that $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{ad}}$, so $\mathbf{A} \in \mathcal{E}_S^{\mathbf{ad}}$ (Proposition 6.7). Moreover, for any $\mathbf{C} \notin \overline{\mathbf{A}}$ there is some $\mathbf{B}$ such that $\mathbf{B} \rightarrowtail \mathbf{C}$ and there is no $\mathbf{D} \in \overline{\mathbf{A}}$ such that $\mathbf{D} \rightarrowtail \mathbf{B}$, or, equivalently, $\mathbf{B} \rightarrowtail \mathbf{C}$ and $\mathbf{A} \not\rightarrow \mathbf{B}$, or, equivalently, $\mathbf{B} \blacktriangleright \mathbf{C}$ and $\mathbf{A} \not\blacktriangleright \mathbf{B}$. Now take any $c \notin \mathbf{A}$. Then $\{c\} \notin \overline{\mathbf{A}}$, so by the above conclusion, there is some $\mathbf{B} \blacktriangleright \{c\}$ and $\mathbf{A} \not\blacktriangleright \mathbf{B}$, which means that $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$.

For #2, take some $\mathcal{E} \in \mathcal{E}_D^{\mathbf{co}}$ and set $\mathbf{B} = \bigcup_{\mathbf{A} \in \mathcal{E}} \mathbf{A}$. Then, by Proposition 6.9, $\mathcal{E} = \overline{\mathbf{B}}$. But then, as already shown above, $\mathbf{B} \in \mathcal{E}_S^{\mathbf{co}}$, which proves that $\mathcal{E}_D^{\mathbf{co}} \subseteq \{\overline{\mathbf{A}} \mid \mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}\}$. The opposite inclusion is also trivial, because if $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$ then $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$ by the first point.

Now consider the case where $\sigma = \mathbf{pr}$.

If $\mathbf{A} \in \mathcal{E}_S^{\mathbf{pr}}$, then also $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$, so $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$. If we assume some $\mathcal{E} \in \mathcal{E}_D^{\mathbf{co}}$ such that $\mathcal{E} \supset \overline{\mathbf{A}}$, then by the case where $\sigma = \mathbf{co}$ we conclude that $\mathcal{E} = \overline{\mathbf{B}}$ for some $\mathbf{B} \supset \mathbf{A}$ and $\mathbf{B} \in \mathcal{E}_S^{\mathbf{co}}$, a contradiction. Thus, $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{pr}}$, i.e., $\mathcal{E}_S^{\mathbf{pr}} \subseteq \{\mathbf{A} \mid \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{pr}}\}$.

For the opposite inclusion, note that if $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{pr}}$ then $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$, so $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$ and if we assume some $\mathbf{B} \in \mathcal{E}_S^{\mathbf{co}}$ such that $\mathbf{B} \supset \mathbf{A}$ we end up with $\overline{\mathbf{B}} \in \mathcal{E}_D^{\mathbf{co}}$ and $\overline{\mathbf{B}} \supset \overline{\mathbf{A}}$, a contradiction.

For #2, if we consider an extension $\mathcal{E} \in \mathcal{E}_D^{\mathbf{pr}}$ then $\mathcal{E} \in \mathcal{E}_D^{\mathbf{co}}$, so $\mathcal{E} = \overline{\mathbf{A}}$ for some $\mathbf{A}$. But then, by the first case above, $\mathbf{A} \in \mathcal{E}_S^{\mathbf{pr}}$. The opposite inclusion is also direct from the above case.

The case where $\sigma = \mathbf{gr}$ is similar to the case where $\sigma = \mathbf{pr}$ and omitted.

Now consider the case where $\sigma = \mathbf{st}$.

We have: $\mathbf{A} \in \mathcal{E}_S^{\mathbf{st}} \Leftrightarrow$
$\mathbf{A} \not\blacktriangleright a$ for all $a \in \mathbf{A}$ and $\mathbf{A} \blacktriangleright b$ for all $b \in \mathbf{Args} \setminus \mathbf{A} \Leftrightarrow$
$\mathbf{A} \not\rightarrow \{a\}$ for all $\{a\} \subseteq \mathbf{A}$ and $\mathbf{A} \blacktriangleright \mathbf{B}$ for all $\mathbf{B} \not\subseteq \mathbf{A} \Leftrightarrow$
$\mathbf{A_1} \not\rightarrow \mathbf{A_2}$ for all $\mathbf{A_1}, \mathbf{A_2} \in \overline{\mathbf{A}}$ and $\mathbf{A} \rightarrowtail \mathbf{B}$ for all $\mathbf{B} \notin \overline{\mathbf{A}} \Leftrightarrow$
$\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{st}}$.

Based on the above equivalence, all the necessary inclusions for points #1 and #2 can be easily shown. □

Lemma 3. *For all* $\mathbf{A}$ *and* c, $c \in \mathbf{A} \cup \mathbf{A}^{\blacktriangleright}$ *if and only if* $\{c\} \in \overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail}$.

Proof of Lemma 3. We note that $c \in \mathbf{A}$ if and only if $\{c\} \in \overline{\mathbf{A}}$. Furthermore, $c \in \mathbf{A}^{\blacktriangleright} \Leftrightarrow \mathbf{A} \blacktriangleright c \Leftrightarrow \mathbf{A} \rightarrowtail \{c\} \Leftrightarrow \{c\} \in \mathbf{A}^{\rightarrowtail}$. Combining these two results, the proof follows trivially. □

Proof of Proposition 6.15. For point #1:

Take some $c \in \mathbf{A} \cup \mathbf{A}^{\blacktriangleright}$. Using Lemma 3: $\{c\} \in \overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$, so $c \in \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$.

For point #2:

Take some $\mathbf{C} \in \overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail}$. We consider two cases:

Case 1: If $\mathbf{C} \in \overline{\mathbf{A}}$ then $\mathbf{C} \subseteq \mathbf{A} \subseteq \mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$. If $\mathbf{C} \cap \mathbf{B}^{\blacktriangleright} = \emptyset$ then $\mathbf{C} \subseteq \mathbf{B}$, i.e., $\mathbf{C} \in \overline{\mathbf{B}} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$, so let's assume that $\mathbf{C} \cap \mathbf{B}^{\blacktriangleright} \neq \emptyset$ and take $c \in \mathbf{C} \cap \mathbf{B}^{\blacktriangleright}$. Then $c \in \mathbf{B}^{\blacktriangleright} \Rightarrow \{c\} \in \mathbf{B}^{\rightarrowtail} \Rightarrow \mathbf{B} \rightarrowtail \{c\} \Rightarrow \mathbf{B} \rightarrowtail \mathbf{C}$, i.e., $\mathbf{C} \in \mathbf{B}^{\rightarrowtail} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$.

Case 2: If $\mathbf{C} \in \mathbf{A}^{\rightarrowtail}$ then there is some $c \in \mathbf{C}$ such that $\mathbf{A} \rightarrowtail \{c\}$, so $\mathbf{A} \blacktriangleright c$, i.e., $c \in \mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$. If $c \in \mathbf{B}$ then $\mathbf{A} \rightarrowtail \mathbf{B}$, i.e., $\mathbf{A} \blacktriangleright \mathbf{B}$, a contradiction, so we have to assume that $c \in \mathbf{B}^{\blacktriangleright}$, which means that $\mathbf{B} \blacktriangleright \mathbf{C}$, i.e., $\mathbf{C} \in \mathbf{B}^{\rightarrowtail}$.

We conclude that in either case $\mathbf{C} \in \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$, so $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$.

For point #3:

Since $\mathbf{A} \blacktriangleright \mathbf{B}$, there is $b \in \mathbf{B}$ such that $\mathbf{A} \blacktriangleright b$. Consider also some $c \in \mathbf{Args} \setminus (\mathbf{B} \cup \mathbf{B}^{\blacktriangleright})$. Then, for the set $\mathbf{C} = \{b, c\}$, it holds that $\mathbf{A} \blacktriangleright \mathbf{C}$, so $\mathbf{A} \rightarrowtail \mathbf{C}$, i.e., $\mathbf{C} \in \overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail}$. However, $\mathbf{C} \not\subseteq \mathbf{B}$ (because $c \notin \mathbf{B}$) and $\mathbf{B} \not\rightarrow \mathbf{C}$ (because it does not attack b due to conflict-freeness, and it does not attack c by the choice of c). Thus, $\mathbf{C} \notin \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$, which leads to the desired result. □

Proof of Proposition 6.16. Take $\mathbf{A} \in \mathcal{E}_S^{\mathbf{st}}$. Then $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$ and $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} = \mathbf{Args}$, so $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright}$ is maximal, i.e., $\mathbf{A} \in \mathcal{E}_S^{\mathbf{se}}$. For the opposite, note that since $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$, there is at least one $\mathbf{B}$ such that $\mathbf{B} \in \mathcal{E}_S^{\mathbf{co}}$ and $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} = \mathbf{Args}$. Thus, if $\mathbf{A} \in \mathcal{E}_S^{\mathbf{se}}$ then $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$ and $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} = \mathbf{Args}$, thus $\mathbf{A} \in \mathcal{E}_S^{\mathbf{st}}$.

For the second relation, we note that $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$ implies that $\mathcal{E}_D^{\mathbf{st}} \neq \emptyset$, thus, following the same reasoning we can show that $\mathcal{E}_D^{\mathbf{st}} = \mathcal{E}_D^{\mathbf{se}}$. □

Proof of Proposition 6.19. (1) ⇒ (2) Take $c \in \mathbf{A} \setminus \mathbf{B}$; obviously $c \in \mathbf{A} \subseteq \mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$. Since $c \notin \mathbf{B}$ by construction, it follows that $c \in \mathbf{B}^{\blacktriangleright}$, which proves that $\mathbf{A} \setminus \mathbf{B} \subseteq \mathbf{B}^{\blacktriangleright}$.

Moreover, take $c \in \mathbf{A}^{\blacktriangleright} \cap (\mathbf{Args} \setminus (\mathbf{A} \cup \mathbf{B}))$. To prove the second condition of S-domination, it suffices to show that $\mathbf{B} \blacktriangleright c$. Indeed, $c \in \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$, and, by construction, $c \notin \mathbf{B}$, thus $c \in \mathbf{B}^{\blacktriangleright}$, which concludes this part of the proof.

(2) ⇒ (3) We will first show that $\overline{\mathbf{A}} \setminus \overline{\mathbf{B}} \subseteq \mathbf{B}^{\rightarrowtail}$. If $\overline{\mathbf{A}} \setminus \overline{\mathbf{B}} = \emptyset$ then the result follows trivially, so suppose that this is not the case and take $\mathbf{C} \in \overline{\mathbf{A}} \setminus \overline{\mathbf{B}}$. Then $\mathbf{C} \subseteq \mathbf{A}$ and $\mathbf{C} \not\subseteq \mathbf{B}$, so there is some $c \in \mathbf{C} \cap (\mathbf{A} \setminus \mathbf{B})$. By S-domination, $\mathbf{B} \blacktriangleright c$, so $\mathbf{B} \blacktriangleright \mathbf{C}$, i.e., $\mathbf{B} \rightarrowtail \mathbf{C}$, which proves that $\overline{\mathbf{A}} \setminus \overline{\mathbf{B}} \subseteq \mathbf{B}^{\rightarrowtail}$.

UNCORRECTED PROOF

Now take $\mathbf{C} \in \mathbf{A}^{\rightarrowtail} \cap \overline{\mathbf{Args} \setminus (\mathbf{A} \cup \mathbf{B})}$. To prove the second condition of D-domination, it suffices to show that $\mathbf{B} \rightarrowtail \mathbf{C}$. Indeed, by construction, $\mathbf{A} \rightarrowtail \mathbf{C}$, so there is $c \in \mathbf{C}$ such that $\mathbf{A} \rightarrowtail \{c\}$, i.e., $\mathbf{A} \blacktriangleright c$. But then $c \in \mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$. By the construction of c and $\mathbf{C}$, it follows that $c \notin \mathbf{B}$, so $c \in \mathbf{B}^{\blacktriangleright}$, which in turn implies that $\mathbf{B} \rightarrowtail \mathbf{C}$.

(3) $\Rightarrow$ (1) Take $c \in \mathbf{A} \cup \mathbf{A}^{\blacktriangleright}$. It suffices to show that $c \in \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$. This is obviously true if $c \in \mathbf{B}$, so we will assume that $c \notin \mathbf{B}$ and show that $\mathbf{B} \blacktriangleright c$ under these assumptions. We split the proof in two cases.

Case 1: If $c \in \mathbf{A}$, then $c \in \mathbf{A} \setminus \mathbf{B}$ so $\{c\} \in \overline{\mathbf{A}} \setminus \overline{\mathbf{B}} \subseteq \mathbf{B}^{\rightarrowtail}$, so $\mathbf{B} \rightarrowtail \{c\}$, i.e., $\mathbf{B} \blacktriangleright c$.

Case 2: If $c \in \mathbf{A}^{\blacktriangleright}$, then $\mathbf{A} \blacktriangleright c \Rightarrow \mathbf{A} \rightarrowtail \{c\} \Rightarrow \{c\} \in \mathbf{A}^{\rightarrowtail}$. Moreover, given that $\mathbf{A}$ is conflict-free, $c \notin \mathbf{A}$ and also $c \notin \mathbf{B}$ (by the hypothesis), so $\{c\} \in \overline{\mathbf{Args} \setminus (\mathbf{A} \cup \mathbf{B})}$. We conclude that $\{c\} \in \mathbf{A}^{\rightarrowtail} \cap \overline{\mathbf{Args} \setminus (\mathbf{A} \cup \mathbf{B})} \subseteq \mathbf{B}^{\rightarrowtail} \cap \overline{\mathbf{Args} \setminus (\mathbf{A} \cup \mathbf{B})} \subseteq \mathbf{B}^{\rightarrowtail}$, so $\mathbf{B} \rightarrowtail \{c\}$, i.e., $\mathbf{B} \blacktriangleright c$. □

Proof of Proposition 6.20. The equivalence between (2), (3) and (4) is obvious by Proposition 6.19. Also, (1) clearly implies (2), by Proposition 6.15, points #1 and #3. Further, (2) implies (1) by Proposition 6.15, point #2. □

Lemma 4. *For all* $\mathbf{A}$, $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} = \mathbf{Args}$ *if and only if* $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} = \overline{\mathbf{Args}}$.

Proof of Lemma 4. The result is obvious from Corollary 6.11, for the case where $\sigma = \mathbf{st}$. □

Proof of Proposition 6.21. Suppose that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} = \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$. Then, by Lemma 4 and the hypothesis, we conclude that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \neq \overline{\mathbf{Args}}$, $\overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail} \neq \overline{\mathbf{Args}}$ and $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} \neq \mathbf{Args}$. Moreover, $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$, so, by Proposition 6.20 and the fact that $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} \neq \mathbf{Args}$, $\mathbf{A} \not\blacktriangleright \mathbf{B}$; also (by S-domination), $\mathbf{A} \setminus \mathbf{B} \subseteq \mathbf{B}^{\blacktriangleright}$. Similarly, $\overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail} \subseteq \overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail}$, so $\mathbf{B} \not\blacktriangleright \mathbf{A}$ and $\mathbf{B} \setminus \mathbf{A} \subseteq \mathbf{A}^{\blacktriangleright}$. The combination of these relations can be true only if $\mathbf{A} \setminus \mathbf{B} = \mathbf{B} \setminus \mathbf{A} = \emptyset$, which means that $\mathbf{A} = \mathbf{B}$. The opposite implication is trivial, so this concludes the proof. □

Proof of Proposition 6.22. If $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$, then there is some $\mathbf{B} \in \mathcal{E}_S^{\mathbf{co}}$ such that $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} = \mathbf{Args}$, and, since, $\mathbf{A} \in \mathcal{E}_S^{\mathbf{se}}$, it follows that $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} = \mathbf{Args}$, i.e., $\mathbf{A} \in \mathcal{E}_S^{\mathbf{st}}$, which implies, by Theorem 6.10, that $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{st}}$. The latter means that $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$ and that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} = \overline{\mathbf{Args}}$ so it is maximal, which concludes the proof for this case.

Now let's assume that $\mathcal{E}_S^{\mathbf{st}} = \emptyset$. Then $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \neq \mathbf{Args}$. Since $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$, $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$ (Theorem 6.10). Also, consider some $\mathcal{E} \in \mathcal{E}_D^{\mathbf{co}}$ such that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subset \mathcal{E} \cup \mathcal{E}^{\rightarrowtail}$. By Theorem 11 of [2] there is some $\mathcal{E}' \in \mathcal{E}_D^{\mathbf{pr}}$ such that $\mathcal{E}' \supseteq \mathcal{E}$ and it is obviously the case that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subset \mathcal{E}' \cup \mathcal{E}'^{\rightarrowtail}$. Moreover, by Theorem 6.10, there is some $\mathbf{C} \in \mathcal{E}_S^{\mathbf{pr}}$ such that $\mathcal{E}' = \overline{\mathbf{C}}$, i.e., $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subset \overline{\mathbf{C}} \cup \mathbf{C}^{\rightarrowtail}$. But then, by Proposition 6.15, $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{C} \cup \mathbf{C}^{\blacktriangleright}$. Since $\mathbf{A} \in \mathcal{E}_S^{\mathbf{se}}$ it follows that $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} = \mathbf{C} \cup \mathbf{C}^{\blacktriangleright}$. If $\mathbf{C} \not\blacktriangleright \mathbf{A}$ then the latter relation would imply that $\overline{\mathbf{C}} \cup \mathbf{C}^{\rightarrowtail} \subseteq \overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail}$ (Proposition 6.15), a contradiction, so $\mathbf{C} \blacktriangleright \mathbf{A}$. But $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$ by construction, so the latter relation implies that $\mathbf{A} \blacktriangleright \mathbf{C}$. Combining the above results we have that $\mathbf{A} \blacktriangleright \mathbf{C}$ and $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{C}} \cup \mathbf{C}^{\rightarrowtail}$, so, by Proposition 6.15 we are forced to assume that $\mathbf{C} \cup \mathbf{C}^{\blacktriangleright} = \mathbf{Args}$, a contradiction by our hypothesis. □

Proof of Theorem 6.23. For #1, take some $\mathbf{A} \in \mathcal{E}_S^{\mathbf{se}}$. By Proposition 6.22, $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{se}}$. Moreover, if $\mathbf{B}$ strictly D-dominates $\mathbf{A}$, then, by Proposition 6.19, $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subset \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$, so, since $\mathbf{A} \in \mathcal{E}_S^{\mathbf{se}}$, we are forced to conclude that $\mathbf{B} \notin \mathcal{E}_S^{\mathbf{co}}$, thus $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{co}}$.

For the opposite inclusion, we note that, since $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{se}}$, $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$, so $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$. Now suppose that there is some $\mathbf{B} \in \mathcal{E}_S^{\mathbf{co}}$ such that $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subset \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$. Then, $\mathbf{B}$ strictly D-dominates $\mathbf{A}$ (by Proposition 6.19), so, by the hypothesis, $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{co}}$, thus $\mathbf{B} \notin \mathcal{E}_S^{\mathbf{co}}$, a contradiction by our hypothesis on $\mathbf{B}$.

For #2, take some $\mathcal{E} \in \mathcal{E}_D^{\mathbf{se}}$. Then, by Theorem 2 of [18], $\mathcal{E}$ is also preferred, so, by Theorem 6.10, it is of the form $\overline{\mathbf{A}}$ for some $\mathbf{A} \in \mathcal{E}_S^{\mathbf{pr}}$. Now take some $\mathbf{B} \in \mathcal{E}_S^{\mathbf{pr}}$ such that $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subset \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$. If $\mathbf{A} \not\blacktriangleright \mathbf{B}$, then by Proposition 6.20 and the above inclusion, it follows that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$. Since $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{se}}$ and $\mathbf{B} \in \mathcal{E}_S^{\mathbf{pr}} \subseteq \mathcal{E}_S^{\mathbf{co}}$, we are forced to assume that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} = \overline{\mathbf{B}} \cup \mathbf{B}^{\rightarrowtail}$, so $\mathbf{A} = \mathbf{B}$ (Proposition 6.21), which contradicts the fact that $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subset \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$. Thus $\mathbf{A} \blacktriangleright \mathbf{B}$, so: $\mathcal{E}_D^{\mathbf{se}} \subseteq \{\overline{\mathbf{A}} \mid \mathbf{A} \in \mathcal{E}_S^{\mathbf{pr}}, \mathbf{A} \blacktriangleright \mathbf{B}$ whenever $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subset \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}, \mathbf{B} \in \mathcal{E}_S^{\mathbf{pr}}\}$.

For the opposite inclusion, take some $\overline{\mathbf{A}}$ for which $\mathbf{A} \in \mathcal{E}_S^{\mathbf{pr}}$ and $\mathbf{A} \blacktriangleright \mathbf{B}$ whenever $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subset \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}, \mathbf{B} \in \mathcal{E}_S^{\mathbf{pr}}$. Suppose, for the sake of contradiction, that $\overline{\mathbf{A}} \notin \mathcal{E}_D^{\mathbf{se}}$, so there is some $\mathcal{E} \in \mathcal{E}_D^{\mathbf{co}}$ such that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subset \mathcal{E} \cup \mathcal{E}^{\rightarrowtail}$. By Theorem 11 of [2] there is some $\mathcal{E}' \in \mathcal{E}_D^{\mathbf{pr}}$ such that $\mathcal{E}' \supseteq \mathcal{E}$ and it is obviously the case that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subset \mathcal{E}' \cup \mathcal{E}'^{\rightarrowtail}$. Moreover, by Theorem 6.10, there is some $\mathbf{C} \in \mathcal{E}_S^{\mathbf{pr}}$ such that $\mathcal{E}' = \overline{\mathbf{C}}$, i.e., $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subset \overline{\mathbf{C}} \cup \mathbf{C}^{\rightarrowtail}$. But then, by Proposition 6.15, $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subseteq \mathbf{C} \cup \mathbf{C}^{\blacktriangleright}$. If $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} \subset \mathbf{C} \cup \mathbf{C}^{\blacktriangleright}$ then, by the initial hypothesis, $\mathbf{A} \blacktriangleright \mathbf{C}$. If $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} = \mathbf{C} \cup \mathbf{C}^{\blacktriangleright}$ and $\mathbf{C} \not\blacktriangleright \mathbf{A}$ then Proposition 6.15 leads to a contradiction, so $\mathbf{C} \blacktriangleright \mathbf{A}$. But $\mathbf{A} \in \mathcal{E}_S^{\mathbf{pr}}$ thus the above attack implies that $\mathbf{A} \blacktriangleright \mathbf{C}$. Thus, in any case, $\mathbf{A} \blacktriangleright \mathbf{C}$, so there is some $c \in \mathbf{C}$ such that $\mathbf{A} \blacktriangleright c$. Moreover, since $\mathcal{E}_S^{\mathbf{st}} = \emptyset$, $\mathbf{C} \notin \mathcal{E}_S^{\mathbf{st}}$, so there is some $d \in \mathbf{Args} \setminus (\mathbf{C} \cup \mathbf{C}^{\blacktriangleright})$. For the set $\mathbf{D} = \{c, d\}$, we have that $\mathbf{A} \blacktriangleright \mathbf{D}$, so $\mathbf{A} \rightarrowtail \mathbf{D}$, i.e., $\mathbf{D} \in \mathbf{A}^{\rightarrowtail} \subseteq \overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail}$. On the other hand, $\mathbf{D} \notin \overline{\mathbf{C}}$ (because $d \notin \mathbf{C}$) and $\mathbf{D} \notin \mathbf{C}^{\rightarrowtail}$ (because $\mathbf{C} \not\blacktriangleright d$, and $c \in \mathbf{C}$ so $\mathbf{C} \not\blacktriangleright c$). This contradicts with the assumption that $\overline{\mathbf{A}} \cup \mathbf{A}^{\rightarrowtail} \subset \mathbf{C} \cup \mathbf{C}^{\rightarrowtail}$, and the proof is complete. □

Proof of Proposition 6.24. Take $\mathbf{A} \in \mathcal{E}_S^{\mathbf{st}}$. Then $\mathbf{A} \in \mathcal{E}_S^{\mathbf{cf}}$ and $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} = \mathbf{Args}$, so $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright}$ is maximal, i.e., $\mathbf{A} \in \mathcal{E}_S^{\mathbf{sg}}$. For the opposite, note that, since $\mathcal{E}_S^{\mathbf{st}} \neq \emptyset$, there is at least one $\mathbf{B}$ such that $\mathbf{B} \in \mathcal{E}_S^{\mathbf{co}} \subseteq \mathcal{E}_S^{\mathbf{cf}}$ and $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} = \mathbf{Args}$. Thus, if $\mathbf{A} \in \mathcal{E}_S^{\mathbf{sg}}$ then $\mathbf{A} \in \mathcal{E}_S^{\mathbf{cf}}$ and

UNCORRECTED PROOF

$\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} = \mathbf{Args}$. It remains to show that $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$. Indeed, if $\mathbf{C} \blacktriangleright \mathbf{A}$, then $\mathbf{C} \not\subseteq \mathbf{A}$ (otherwise $\mathbf{A}$ would not be conflict free). Thus, there is some $c \in \mathbf{C} \setminus \mathbf{A}$, and by our hypothesis on $\mathbf{A}$ it follows that $\mathbf{A} \blacktriangleright c$, thus $\mathbf{A} \blacktriangleright \mathbf{C}$. Further, since $\mathbf{A} \cup \mathbf{A}^{\blacktriangleright} = \mathbf{Args}$, $\mathbf{A}$ cannot be expanded further, because for all $b \notin \mathbf{A}$, we get that $\mathbf{A} \blacktriangleright b$ and $\mathbf{A} \not\blacktriangleright \mathbf{A}$. Thus, $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$.

The second relation is straightforward from Theorems 2, 4 of [10]. □

Proof of Theorem 6.26. Take some $\mathbf{A} \in \mathcal{E}_S^{\mathbf{sg}}$. Then, by definition, $\mathbf{A} \in \mathcal{E}_S^{\mathbf{cf}}$, so $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{cf}}$ (Theorem 6.5). Moreover, if $\mathbf{B}$ strictly D-dominates $\mathbf{A}$ then $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} \supset \mathbf{A} \cup \mathbf{A}^{\blacktriangleright}$ (Proposition 6.19), and since $\mathbf{A} \in \mathcal{E}_S^{\mathbf{sg}}$ we are forced to conclude that $\mathbf{B} \notin \mathcal{E}_S^{\mathbf{cf}}$, thus $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{cf}}$.

For the opposite, since $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{cf}}$, it follows that $\mathbf{A} \in \mathcal{E}_S^{\mathbf{cf}}$ (Proposition 6.3). Moreover, if there is $\mathbf{B} \in \mathcal{E}_S^{\mathbf{cf}}$ such that $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} \supset \mathbf{A} \cup \mathbf{A}^{\blacktriangleright}$, then $\mathbf{B}$ strictly D-dominates $\mathbf{A}$, so $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{cf}}$ (by construction), i.e., $\mathbf{B} \notin \mathcal{E}_S^{\mathbf{cf}}$, a contradiction. Thus, $\mathbf{A} \in \mathcal{E}_S^{\mathbf{sg}}$. □

Lemma 5. *Consider some family* $\{\mathbf{A_i}\}$ *such that* $\mathbf{A_j} \not\blacktriangleright \mathbf{A_k}$ *for all* j, k. *If* $\mathcal{E}_S^{\mathbf{st}} = \emptyset$, *then either* $\bigcup(\mathbf{A_i} \cup \mathbf{A_i}^{\blacktriangleright}) \neq \mathbf{Args}$ *or* $\bigcup \mathbf{A_i} \notin \{\mathbf{A_i}\}$.

Proof of Lemma 5. Suppose that $\mathcal{E}_S^{\mathbf{st}} = \emptyset$. Assume, for the sake of contradiction, that $\bigcup(\mathbf{A_i} \cup \mathbf{A_i}^{\blacktriangleright}) = \mathbf{Args}$ and that $\bigcup \mathbf{A_i} \in \{\mathbf{A_i}\}$. We will show that, for $\mathbf{B} = \bigcup \mathbf{A_i}$, $\mathbf{B} \in \mathcal{E}_S^{\mathbf{st}}$, which is a contradiction by the hypothesis that $\mathcal{E}_S^{\mathbf{st}} = \emptyset$. Indeed, $\mathbf{B} \in \{\mathbf{A_i}\}$, so $\mathbf{B} \not\blacktriangleright \mathbf{B}$, i.e., $\mathbf{B} \in \mathcal{E}_S^{\mathbf{cf}}$. Furthermore, $\mathbf{B}^{\blacktriangleright} \supseteq \mathbf{A_i}^{\blacktriangleright}$ for all i, thus $\bigcup(\mathbf{A_i} \cup \mathbf{A_i}^{\blacktriangleright}) \subseteq \mathbf{B} \cup \mathbf{B}^{\blacktriangleright}$, so $\mathbf{B} \cup \mathbf{B}^{\blacktriangleright} = \mathbf{Args}$. Thus, $\mathbf{B} \in \mathcal{E}_S^{\mathbf{st}}$, a contradiction. □

Proof of Proposition 6.27. ($\Rightarrow$) For the first bullet, take $c \in \bigcup(\mathbf{A_i} \cup \mathbf{A_i}^{\blacktriangleright})$. Then $\{c\} \in \bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail})$, so $\{c\} \in \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$ which proves that $c \in \bigcup(\mathbf{B_i} \cup \mathbf{B_i}^{\blacktriangleright})$.

For the second bullet, suppose that $\mathbf{A_j} \blacktriangleright \mathbf{B_k}$. Since $\mathcal{E}_S^{\mathbf{st}} = \emptyset$, from Lemma 5 we get that either $\bigcup(\mathbf{B_i} \cup \mathbf{B_i}^{\blacktriangleright}) \neq \mathbf{Args}$ or $\bigcup \mathbf{B_i} \notin \{\mathbf{B_i}\}$. We split the proof in two cases:

If the former is true, take some $c \notin \bigcup(\mathbf{B_i} \cup \mathbf{B_i}^{\blacktriangleright})$. Also, there is some $b \in \mathbf{B_k}$ such that $\mathbf{A_j} \blacktriangleright b$, and set $\mathbf{C} = \{b, c\}$. Then, we note that $\mathbf{C} \not\subseteq \bigcup \mathbf{B_i}$ (by the construction of $c \in \mathbf{C}$), and that $\mathbf{C} \notin \bigcup \mathbf{B_i}^{\rightarrowtail}$ (because, if this was the case, then there would be some $\mathbf{B_m}$ attacking either b or c, both of which are impossible by the hypothesis on $\{\mathbf{B_i}\}$ and the choice of b, c). We conclude that $\mathbf{C} \notin \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$. On the other hand, $\mathbf{A_j} \blacktriangleright \mathbf{C}$, so $\mathbf{C} \in \bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail})$, which contradicts the hypothesis.

If the latter is true, then set $\mathbf{C} = \bigcup \mathbf{B_i}$. It follows that $\mathbf{C} \not\subseteq \mathbf{B_j}$ for all j, so $\mathbf{C} \notin \bigcup \overline{\mathbf{B_i}}$. Moreover, there is no $\mathbf{B_j}$ such that $\mathbf{B_j} \blacktriangleright \mathbf{C}$, because then $\mathbf{B_j} \blacktriangleright \mathbf{B_k}$ for some k, a contradiction. Thus, $\mathbf{C} \notin \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$. On the other hand, $\mathbf{A_j} \blacktriangleright \mathbf{C}$, so $\mathbf{C} \in \bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail})$, which contradicts the hypothesis.

For the third bullet, consider some j such that for all k, $\mathbf{A_j} \setminus \mathbf{B_k} \not\subseteq \bigcup \mathbf{B_i}^{\blacktriangleright}$. Then, for each k, there is some c_k such that $c_k \in \mathbf{A_j} \setminus \mathbf{B_k}$ and $c_k \notin \bigcup \mathbf{B_i}^{\blacktriangleright}$. Set $\mathbf{C} = \bigcup\{c_k\}$. Then, obviously, $\mathbf{C} \subseteq \mathbf{A_j}$, so $\mathbf{C} \in \bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail})$. On the other hand, if there is m such that $\mathbf{C} \in \overline{\mathbf{B_m}}$, then $\mathbf{C} \subseteq \mathbf{B_m}$, a contradiction because $c_m \in \mathbf{C}$ but $c_m \notin \mathbf{B_m}$ by construction. Similarly, if there is m such that $\mathbf{C} \in \mathbf{B_m}^{\rightarrowtail}$, then there would exist $c_n \in \mathbf{C}$ such that $\mathbf{B_m} \blacktriangleright c_n$, i.e., $c_n \in \mathbf{B_m}^{\blacktriangleright}$, a contradiction by construction again. Thus, $\mathbf{C} \notin \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$, which contradicts the hypothesis.

($\Leftarrow$) Take some $\mathbf{C} \in \bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail})$. We will show that $\mathbf{C} \in \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$. We consider the following cases:

Case 1: If $\mathbf{C} \in \bigcup \overline{\mathbf{A_i}}$ then there is some j such that $\mathbf{C} \subseteq \mathbf{A_j}$. Moreover, by the hypothesis, there is some k such that $\mathbf{A_j} \setminus \mathbf{B_k} \subseteq \bigcup \mathbf{B_i}^{\blacktriangleright}$. We consider two sub-cases:

Case 1a: If $\mathbf{C} \cap (\mathbf{A_j} \setminus \mathbf{B_k}) = \emptyset$, then, since $\mathbf{C} \subseteq \mathbf{A_j}$, we conclude that $\mathbf{C} \subseteq \mathbf{B_k}$, i.e., $\mathbf{C} \in \bigcup \overline{\mathbf{B_i}}$, thus $\mathbf{C} \in \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$.

Case 1b: If $\mathbf{C} \cap (\mathbf{A_j} \setminus \mathbf{B_k}) \neq \emptyset$, then take $c \in \mathbf{C} \cap (\mathbf{A_j} \setminus \mathbf{B_k})$. It follows that $c \in \bigcup \mathbf{B_i}^{\blacktriangleright}$, so $\mathbf{C} \in \bigcup \mathbf{B_i}^{\rightarrowtail}$, which implies, again, that $\mathbf{C} \in \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$.

Case 2: If $\mathbf{C} \in \bigcup \mathbf{A_i}^{\rightarrowtail}$, then there is j such that $\mathbf{A_j} \rightarrowtail \mathbf{C}$, so there is $c \in \mathbf{C}$ such that $\mathbf{A_j} \blacktriangleright c$, which means that $c \in \bigcup(\mathbf{A_i} \cup \mathbf{A_i}^{\blacktriangleright}) \subseteq \bigcup(\mathbf{B_i} \cup \mathbf{B_i}^{\blacktriangleright})$. However, by the hypothesis that $\mathbf{A_j} \not\blacktriangleright \mathbf{B_k}$ for all j, k, we conclude that $c \notin \bigcup \mathbf{B_i}$, thus $c \in \bigcup \mathbf{B_i}^{\blacktriangleright}$. The latter allows us to conclude that $\{c\} \in \bigcup \mathbf{B_i}^{\rightarrowtail}$, i.e., $\mathbf{C} \in \bigcup \mathbf{B_i}^{\rightarrowtail} \subseteq \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$, which concludes the proof. □

Proof of Proposition 6.29. Take $\mathbf{C} \in \bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail})$. If there is j such that $\mathbf{C} \in \overline{\mathbf{A_j}}$ then, since $\{\mathbf{B_i}\}$ covers $\{\mathbf{A_i}\}$, $\mathbf{C} \subseteq \mathbf{A_j} \subseteq \mathbf{B_k}$ for some k, i.e., $\mathbf{C} \in \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$. Similarly, if there is j such that $\mathbf{C} \in \mathbf{A_j}^{\rightarrowtail}$ then for the set $\mathbf{B_k}$ for which $\mathbf{B_k} \supseteq \mathbf{A_j}$ we get that $\mathbf{C} \in \mathbf{B_k}^{\rightarrowtail}$, i.e., $\mathbf{C} \in \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$, which concludes the proof. □

Proof of Proposition 6.30. (1) $\Rightarrow$ (2) Consider some j. Then, applying Proposition 6.27 for the inclusion $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) \subseteq \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$, we conclude that there is some k such that $\mathbf{A_j} \setminus \mathbf{B_k} \subseteq \bigcup \mathbf{B_i}^{\blacktriangleright}$. However, applying Proposition 6.27 again for the opposite inclusion, we conclude that for all j, m $\mathbf{B_m} \not\blacktriangleright \mathbf{A_j}$. The combination of these facts can only be true if $\mathbf{A_j} \setminus \mathbf{B_k} = \emptyset$. Thus, for all j there is k such that $\mathbf{A_j} \subseteq \mathbf{B_k}$, i.e., $\{\mathbf{B_i}\}$ covers $\{\mathbf{A_i}\}$.

The proof that $\{\mathbf{A_i}\}$ covers $\{\mathbf{B_i}\}$ is analogous.

(2) $\Rightarrow$ (1) Trivial by applying Proposition 6.29 twice. □

Lemma 6. *If* $\mathcal{E} \in \mathcal{E}_D^{\mathbf{sg}}$ *then there exists a family* $\{\mathbf{A_i}\}$ *such that* $\mathcal{E} = \bigcup \overline{\mathbf{A_i}}$.

UNCORRECTED PROOF

Proof of Lemma 6. Suppose that $\mathcal{E} = \{\mathbf{A_i}\}$. Set $\mathcal{E}' = \bigcup \overline{\mathbf{A_i}}$. It suffices to show that $\mathcal{E} = \mathcal{E}'$. Initially we note that for any $\mathbf{B_i}, \mathbf{B_j} \in \mathcal{E}'$, if $\mathbf{B_i} \rightarrowtail \mathbf{B_j}$, then there exist $\mathbf{A_i}, \mathbf{A_j}$ such that $\mathbf{B_i} \subseteq \mathbf{A_i}$, $\mathbf{B_j} \subseteq \mathbf{A_j}$, so $\mathbf{A_i} \rightarrowtail \mathbf{A_j}$, i.e., $\mathcal{E}$ is not conflict-free, a contradiction. Thus, $\mathcal{E}' \in \mathcal{E}_D^{\mathbf{cf}}$. Moreover, obviously, $\mathcal{E} \subseteq \mathcal{E}'$. Now suppose, for the sake of contradiction, that $\mathcal{E} \subset \mathcal{E}'$, and take $\mathbf{B} \in \mathcal{E}' \setminus \mathcal{E}$. Then, $\mathbf{B} \subseteq \mathbf{A_j}$ for some j. We observe that $\mathbf{B}$ is not attacked by some set in $\mathcal{E}$ (because then that set would attack $\mathbf{A_j}$ and thus $\mathcal{E}$ would not be conflict-free), i.e., $\mathbf{B} \notin \mathcal{E}^{\blacktriangleright}$. But then $\mathbf{B} \in \mathcal{E}'$, but $\mathbf{B} \notin \mathcal{E} \cup \mathcal{E}^{\blacktriangleright}$, a contradiction from the hypothesis that $\mathcal{E}$ is a stage extension. □

Proof of Theorem 6.31. Initially, take some $\mathcal{E} \in \mathcal{E}_D^{\mathbf{sg}}$. Then, by Lemma 6, $\mathcal{E} = \bigcup \overline{\mathbf{A_i}}$ for some family $\{\mathbf{A_i}\}$. Moreover, $\mathcal{E} \in \mathcal{E}_D^{\mathbf{cf}}$ so by Theorem 6.5, $\mathbf{A_j} \not\blacktriangleright \mathbf{A_k}$ for all j, k. Now suppose some $\{\mathbf{B_i}\}$ as required in the theorem. From the fact that $(\bigcup \mathbf{B_i}) \bigcap (\bigcup \mathbf{A_i}^{\blacktriangleright} \bigcup \mathbf{B_i}^{\blacktriangleright}) = \emptyset$, we conclude that for all j, k $\mathbf{A_j} \not\blacktriangleright \mathbf{B_k}$ and $\mathbf{B_j} \not\blacktriangleright \mathbf{B_k}$. Thus, Proposition 6.27 can be applied to conclude that $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) \subseteq \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$. But since $\mathcal{E} \in \mathcal{E}_D^{\mathbf{sg}}$, we conclude that $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) = \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$, so by Proposition 6.30 $\{\mathbf{A_i}\}$ covers $\{\mathbf{B_i}\}$.

For the opposite inclusion, consider some $\mathcal{E} = \bigcup \overline{\mathbf{A_i}}$ for some $\{\mathbf{A_i}\}$ as described by the theorem. From the hypothesis that $\mathbf{A_j} \not\blacktriangleright \mathbf{A_k}$ we conclude that $\mathcal{E} \in \mathcal{E}_D^{\mathbf{cf}}$. Suppose that $\mathcal{E} \notin \mathcal{E}_D^{\mathbf{sg}}$. Then, by the hypothesis that $\mathcal{E} \notin \mathcal{E}_D^{\mathbf{sg}}$ and Lemma 6 there is some other family, say $\{\mathbf{B_i}\}$, for which $\mathbf{B_j} \not\blacktriangleright \mathbf{B_k}$ for all j, k and $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) \subset \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$. Thus, Proposition 6.27 can be applied to conclude that: $\bigcup(\mathbf{A_i} \cup \mathbf{A_i}^{\blacktriangleright}) \subseteq \bigcup(\mathbf{B_i} \cup \mathbf{B_i}^{\blacktriangleright})$, $\mathbf{A_j} \not\blacktriangleright \mathbf{B_k}$ for all j, k, and for all j there is some k such that $\mathbf{A_j} \setminus \mathbf{B_k} \subseteq \bigcup \mathbf{B_i}^{\blacktriangleright}$. From these we can also conclude that $(\bigcup \mathbf{B_i}) \bigcap (\bigcup \mathbf{A_i}^{\blacktriangleright} \bigcup \mathbf{B_i}^{\blacktriangleright}) = \emptyset$, so by the construction of $\{\mathbf{A_i}\}$, we get that $\{\mathbf{A_i}\}$ covers $\{\mathbf{B_i}\}$, i.e., $\bigcup(\overline{\mathbf{A_i}} \cup \mathbf{A_i}^{\rightarrowtail}) \supseteq \bigcup(\overline{\mathbf{B_i}} \cup \mathbf{B_i}^{\rightarrowtail})$ (Proposition 6.29), which is a contradiction. □

Lemma 7. *For any family* $\{\mathbf{A_i}\}$, $\overline{\bigcap \mathbf{A_i}} = \bigcap \overline{\mathbf{A_i}}$.

Proof of Lemma 7. $\mathbf{C} \in \overline{\bigcap \mathbf{A_i}} \Leftrightarrow$
$\mathbf{C} \subseteq \bigcap \mathbf{A_i} \Leftrightarrow$
$\mathbf{C} \subseteq \mathbf{A_i}$ for all $i \Leftrightarrow$
$\mathbf{C} \in \overline{\mathbf{A_i}}$ for all $i \Leftrightarrow$
$\mathbf{C} \in \bigcap \overline{\mathbf{A_i}}$. □

Proof of Theorem 6.32. We note the following facts:
$\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}} \Leftrightarrow \overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$ (using Theorem 6.10)
$\mathbf{A} \subseteq \bigcap \mathcal{E}_S^{\mathbf{pr}} \Leftrightarrow \mathbf{A} \subseteq \bigcap \mathbf{A_i} \Leftrightarrow \overline{\mathbf{A}} \subseteq \overline{\bigcap \mathbf{A_i}} \Leftrightarrow \overline{\mathbf{A}} \subseteq \bigcap \overline{\mathbf{A_i}} \Leftrightarrow \overline{\mathbf{A}} \subseteq \bigcap \mathcal{E}_D^{\mathbf{pr}}$ (using Lemma 7, Theorem 6.10, and assuming, without loss of generality, that $\mathcal{E}_S^{\mathbf{pr}} = \{\mathbf{A_i}\}$)
$\mathbf{A} \subset \mathbf{B} \Leftrightarrow \overline{\mathbf{A}} \subset \overline{\mathbf{B}}$ (obvious).
Combining the above facts, and the definition of ideal extensions, we can easily prove the theorem.
For the first result: $\mathbf{A} \in \mathcal{E}_S^{\mathbf{id}} \Leftrightarrow$
$\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$, $\mathbf{A} \subseteq \bigcap \mathcal{E}_S^{\mathbf{pr}}$ and there is no $\mathbf{B}$ such that $\mathbf{B} \in \mathcal{E}_S^{\mathbf{co}}$, $\mathbf{B} \subseteq \bigcap \mathcal{E}_S^{\mathbf{pr}}$ and $\mathbf{A} \subset \mathbf{B} \Leftrightarrow$
$\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$, $\overline{\mathbf{A}} \subseteq \bigcap \mathcal{E}_D^{\mathbf{pr}}$ and there is no $\mathbf{B}$ such that $\overline{\mathbf{B}} \in \mathcal{E}_D^{\mathbf{co}}$, $\overline{\mathbf{B}} \subseteq \bigcap \mathcal{E}_D^{\mathbf{pr}}$ and $\overline{\mathbf{A}} \subset \overline{\mathbf{B}} \Leftrightarrow$
$\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$, $\overline{\mathbf{A}} \subseteq \bigcap \mathcal{E}_D^{\mathbf{pr}}$ and there is no $\mathcal{E}$ such that $\mathcal{E} \in \mathcal{E}_D^{\mathbf{co}}$, $\mathcal{E} \subseteq \bigcap \mathcal{E}_D^{\mathbf{pr}}$ and $\overline{\mathbf{A}} \subset \mathcal{E} \Leftrightarrow$
$\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{id}}$.
For the second result: $\mathcal{E} \in \mathcal{E}_D^{\mathbf{id}} \Leftrightarrow$
$\mathcal{E} \in \mathcal{E}_D^{\mathbf{co}}$, $\mathcal{E} \subseteq \bigcap \mathcal{E}_D^{\mathbf{pr}}$ and there is no $\mathcal{E}'$ such that $\mathcal{E}' \in \mathcal{E}_D^{\mathbf{co}}$, $\mathcal{E}' \subseteq \bigcap \mathcal{E}_D^{\mathbf{pr}}$ and $\mathcal{E} \subset \mathcal{E}' \Leftrightarrow$
there exists $\mathbf{A}$ such that $\mathcal{E} = \overline{\mathbf{A}}$, $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$, $\overline{\mathbf{A}} \subseteq \bigcap \mathcal{E}_D^{\mathbf{pr}}$ and there is no $\mathbf{B}$ such that $\mathcal{E}' = \overline{\mathbf{B}}$, $\overline{\mathbf{B}} \in \mathcal{E}_D^{\mathbf{co}}$, $\overline{\mathbf{B}} \subseteq \bigcap \mathcal{E}_D^{\mathbf{pr}}$ and $\overline{\mathbf{A}} \subset \overline{\mathbf{B}} \Leftrightarrow$
there exists $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$, $\mathbf{A} \subseteq \bigcap \mathcal{E}_S^{\mathbf{pr}}$ and there is no $\mathbf{B}$ such that $\mathbf{B} \in \mathcal{E}_S^{\mathbf{co}}$, $\mathbf{B} \subseteq \bigcap \mathcal{E}_S^{\mathbf{pr}}$ and $\mathbf{A} \subset \mathbf{B} \Leftrightarrow$
$\mathbf{A} \in \mathcal{E}_S^{\mathbf{id}}$. □

Proof of Theorem 6.33. For the first bullet note that if $\mathbf{A} \in \mathcal{E}_S^{\mathbf{na}}$ then $\mathbf{A} \in \mathcal{E}_S^{\mathbf{cf}}$, so $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{cf}}$. Moreover, if $\overline{\mathbf{A}} \subset \overline{\mathbf{B}}$ then $\mathbf{A} \subset \mathbf{B}$ so (since $\mathbf{A}$ is a naive extension) $\mathbf{B} \notin \mathcal{E}_S^{\mathbf{cf}}$, thus $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{cf}}$.

For the opposite inclusion, we first note that $\mathbf{A}$ is conflict-free, so it suffices to show that there is no $\mathbf{B} \in \mathcal{E}_S^{\mathbf{cf}}$ such that $\mathbf{B} \supset \mathbf{A}$. Suppose, for the sake of contradiction, that such a $\mathbf{B}$ exists. Then, $\overline{\mathbf{B}} \supset \overline{\mathbf{A}}$, so $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{cf}}$, thus $\mathbf{B} \notin \mathcal{E}_S^{\mathbf{cf}}$, a contradiction.

For the second bullet, take some $\mathcal{E} \in \mathcal{E}_D^{\mathbf{na}}$. Then, $\mathcal{E}$ is conflict-free, so $\mathcal{E} \subseteq \mathcal{E}_S^{\mathbf{cf}}$ and $\mathbf{A} \not\blacktriangleright \mathbf{B}$ for all $\mathbf{A}, \mathbf{B} \in \mathcal{E}$ (Theorem 6.5). Moreover, if $\mathcal{E}' \supset \mathcal{E}$, then $\mathcal{E}' \notin \mathcal{E}_D^{\mathbf{cf}}$, i.e., (by Theorem 6.5 again) it is either the case that $\mathcal{E}' \not\subseteq \mathcal{E}_S^{\mathbf{cf}}$ or there is some $\mathbf{A}, \mathbf{B} \in \mathcal{E}'$ for which $\mathbf{A} \blacktriangleright \mathbf{B}$. The latter means that if $\mathcal{E} \subset \mathcal{E}' \subseteq \mathcal{E}_S^{\mathbf{cf}}$ then there is some $\mathbf{A}, \mathbf{B} \in \mathcal{E}'$ for which $\mathbf{A} \blacktriangleright \mathbf{B}$.

For the opposite inclusion, take some $\mathcal{E}$ as required by the theorem. Theorem 6.5 and the definition of $\mathcal{E}$ implies that $\mathcal{E} \in \mathcal{E}_D^{\mathbf{cf}}$. Now assume that there is some $\mathcal{E}' \supset \mathcal{E}$, $\mathcal{E}' \in \mathcal{E}_D^{\mathbf{cf}}$. Then, $\mathcal{E} \subset \mathcal{E}' \subseteq \mathcal{E}_S^{\mathbf{cf}}$, so there is some $\mathbf{A}, \mathbf{B} \in \mathcal{E}'$ such that $\mathbf{A} \blacktriangleright \mathbf{B}$, a contradiction by the hypothesis that $\mathcal{E}'$ is conflict free. □

Proof of Proposition 6.34. $\mathbf{A} \in \mathcal{E}^{\cap} \Leftrightarrow$ (by the definition of $\mathcal{E}^{\cap}$)
$\mathbf{A} \subseteq \mathbf{B}$ for all $\mathbf{B} \in \{\mathbf{C} \mid \overline{\mathbf{C}} \in \mathcal{E}_D^{\mathbf{se}}$, and if $\mathbf{D}$ strictly D-dominates $\mathbf{C}$ then $\overline{\mathbf{D}} \notin \mathcal{E}_D^{\mathbf{co}}\} \Leftrightarrow$ (by Theorem 6.23)
$\mathbf{A} \subseteq \mathbf{B}$ for all $\mathbf{B} \in \mathcal{E}_S^{\mathbf{se}}$. □

UNCORRECTED PROOF

Proof of Proposition 6.35. $\mathbf{A} \subseteq \mathbf{S}^{\cap} \Leftrightarrow$ (by the definition of $\mathbf{S}^{\cap}$)
$\mathbf{A} \subseteq \mathbf{B}$ for all $\mathbf{B} \in \{\mathbf{C} \mid \mathbf{C} \in \mathcal{E}_S^{\mathbf{pr}}, \mathbf{C} \blacktriangleright \mathbf{D}$ whenever $\mathbf{C} \cup \mathbf{C}^{\blacktriangleright} \subset \mathbf{D} \cup \mathbf{D}^{\blacktriangleright}$ and $\mathbf{D} \in \mathcal{E}_S^{\mathbf{pr}}\} \Leftrightarrow$ (by definition)
$\overline{\mathbf{A}} \subseteq \overline{\mathbf{B}}$ whenever $\overline{\mathbf{B}} \in \{\overline{\mathbf{C}} \mid \mathbf{C} \in \mathcal{E}_S^{\mathbf{pr}}, \mathbf{C} \blacktriangleright \mathbf{D}$ whenever $\mathbf{C} \cup \mathbf{C}^{\blacktriangleright} \subset \mathbf{D} \cup \mathbf{D}^{\blacktriangleright}$ and $\mathbf{D} \in \mathcal{E}_S^{\mathbf{pr}}\} \Leftrightarrow$ (by Theorem 6.23)
$\overline{\mathbf{A}} \subseteq \mathcal{E}$ for all $\mathcal{E} \in \mathcal{E}_D^{\mathbf{se}}$. □

Proof of Theorem 6.36. (1) Take some $\mathbf{A} \in \mathcal{E}_S^{\mathbf{ea}}$. Then, by definition, $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$, i.e., $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$ (Theorem 6.10). Also, by Proposition 6.34, it must be the case that $\mathbf{A} \in \mathcal{E}^{\cap}$. Further, since $\mathbf{A}$ is an eager extension, if there is some $\mathbf{B}$ such that $\mathbf{B} \supset \mathbf{A}$, then $\mathbf{B} \notin \mathcal{E}_S^{\mathbf{co}}$ (thus $\overline{\mathbf{B}} \notin \mathcal{E}_D^{\mathbf{co}}$), or it is not the case that $\mathbf{B} \subseteq \mathbf{C}$ for all $\mathbf{C} \in \mathcal{E}_S^{\mathbf{se}}$ (thus $\mathbf{B} \notin \mathcal{E}^{\cap}$ by Proposition 6.34).

For the opposite inclusion, take some $\mathbf{A}$ as specified in the theorem. We will show that $\mathbf{A} \in \mathcal{E}_S^{\mathbf{ea}}$. Indeed, $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$ (Theorem 6.10) and $\mathbf{A} \subseteq \mathbf{B}$ for all $\mathbf{B} \in \mathcal{E}_S^{se}$ (Proposition 6.34). Further, $\mathbf{A}$ is the maximal set with these properties, because if $\mathbf{A} \subset \mathbf{B}$ then $\mathbf{B}$ is either not complete or is not a subset of all semi-stable extensions.

(2) The proof is obvious by definition, when considering the fact that $\mathcal{E}_S^{\mathbf{st}} = \mathcal{E}_S^{\mathbf{se}}$ (Proposition 6.16).

(3) Take some $\mathcal{E} \in \mathcal{E}_D^{\mathbf{ea}}$. Then $\mathcal{E} \in \mathcal{E}_D^{\mathbf{co}}$, so by Theorem 6.10, there must be some $\mathbf{A} \in \mathcal{E}_S^{\mathbf{co}}$ such that $\mathcal{E} = \overline{\mathbf{A}}$. Moreover, $\overline{\mathbf{A}}$ is a subset of all semi-stable extensions, so by Proposition 6.35 it follows that $\mathbf{A} \subseteq \mathbf{S}^{\cap}$. Further, assume that $\mathbf{A} \subset \mathbf{B}$. Then $\overline{\mathbf{A}} \subset \overline{\mathbf{B}}$. Since $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{ea}}$, it follows that $\overline{\mathbf{B}}$ is either not complete, or it is not a subset of all semi-stable extensions, so we get the result by considering Theorem 6.10 and Proposition 6.35.

For the opposite inclusion, consider $\overline{\mathbf{A}}$ with these properties. Then $\overline{\mathbf{A}} \in \mathcal{E}_D^{\mathbf{co}}$ (by Theorem 6.10) and $\overline{\mathbf{A}}$ is a subset of all semi-stable extensions (by Proposition 6.35). It remains to show that it is maximal with these properties. Indeed, suppose that there is some $\mathcal{E}'$ such that $\mathcal{E}' \in \mathcal{E}_D^{\mathbf{co}}$, $\mathcal{E}' \subseteq \mathcal{E}''$ for all $\mathcal{E}'' \in \mathcal{E}_D^{\mathbf{se}}$ and $\mathcal{E}' \supset \overline{\mathbf{A}}$. Then, since $\mathcal{E}' \in \mathcal{E}_D^{\mathbf{co}}$ there is some $\mathbf{B}$ such that $\mathcal{E}' = \overline{\mathbf{B}}$ and $\mathbf{B} \in \mathcal{E}_S^{\mathbf{co}}$. But then $\mathbf{A} \subset \mathbf{B}$, so by the definition of $\mathbf{A}$ it follows that $\mathbf{B} \notin \mathcal{E}_S^{\mathbf{co}}$ or that $\mathbf{B} \not\subseteq \mathbf{S}^{\cap}$, both of which contradict our hypotheses on $\mathcal{E}', \mathbf{B}$. This concludes the proof. □

A.3. Proofs for Section 7

Proof of Theorem 7.1. *Every stable extension of AF^S is also a stage extension of AF^S:*
Let $\mathbf{A}$ be a stable extension of AF^S. Then, by Definition 3.6, $\mathbf{A}$ is conflict-free and attacks all arguments in $\mathbf{Args} \setminus \mathbf{A}$. Therefore, $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\} = \mathbf{Args}$, which means that $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal among all conflict-free subsets of $\mathbf{Args}$. Therefore, $\mathbf{A}$ is a stage extension of AF^S.
Every stable labelling of AF^S is also a stage labelling of AF^S:
Let λ be a stable labelling of $AF^S \Rightarrow$
$Ext(\lambda)$ is a stable extension of AF^S (by Theorem 5.11) $\Rightarrow$
$Ext(\lambda)$ is a stage extension of AF^S (as proved above) $\Rightarrow$
$Lab(Ext(\lambda))$ is a stage labelling of AF^S (by Theorem 5.10) (1)
By Theorem 5.6 and the fact that λ is a proper labelling: $\lambda = Lab(Ext(\lambda))$ (2).
From (1) and (2): λ is a stage labelling of AF^S.
Every stable extension of AF^S is also a semi-stable extension of AF^S:
Let $\mathbf{A}$ be a stable extension of AF^S. Then, by Definition 3.6, $\mathbf{A}$ is conflict-free and attacks all arguments in $\mathbf{Args} \setminus \mathbf{A}$. Therefore, $\mathbf{A}$ is a complete extension of AF^S, as (i) it is conflict-free, (ii) for all $\mathbf{B} \subseteq \mathbf{Args}$ such that $\mathbf{B} \blacktriangleright \mathbf{A}$, $\mathbf{A} \blacktriangleright \mathbf{B}$ (since any such $\mathbf{B}$ must be a subset of $\mathbf{Args} \setminus \mathbf{A}$), and (iii) $\mathbf{Args} \setminus \mathbf{A}$ does not contain any arguments that are defended by $\mathbf{A}$ (since they are all attacked by $\mathbf{A}$). It also holds that $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\} = \mathbf{Args}$, therefore $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal among all complete extensions. Therefore, $\mathbf{A}$ is a semi-stable extension of AF^S.
Every stable labelling of AF^S is also a semi-stable labelling of AF^S:
The proof is similar with the stable – stage labellings case; just replace "stage" with "semi-stable".
Every semi-stable extension of AF^S is also a preferred extension of AF^S:
Let $\mathbf{A}$ be a semi-stable extension of AF^S. Then, by Definition 3.8, $\mathbf{A}$ is a complete extension of AF^S and $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\}$ is maximal w.r.t. set inclusion among all complete extensions of AF^S. Suppose that $\mathbf{A}$ is not a preferred extension of AF^S. Then, there is another complete extension $\mathbf{A}'$, such that:

$$\mathbf{A} \subset \mathbf{A}' \Rightarrow$$
$$\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\} \subset \mathbf{A}' \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\} \Rightarrow$$
$$\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\} \subset \mathbf{A}' \cup \{b' \in \mathbf{Args} \mid \mathbf{A}' \blacktriangleright b'\}$$

which violates the condition that $\mathbf{A}$ is maximal w.r.t. set inclusion among all complete extensions of AF^S. Therefore, $\mathbf{A}$ is a preferred extension of AF^S.
Every semi-stable labelling of AF^S is also a preferred labelling of AF^S:
The proof is similar with the stable – stage labellings case; just replace "stable" with "semi-stable" and "stage" with "preferred".
All other inclusion relationships are directly derived from the definitions of the respective semantics. □

Proof of Theorem 7.2. *Any SETAF AF^S has at least one preferred extension (labelling)*:

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

Let AF^D be the generated AAF of SETAF AF^S (Definition 6.1). As proved in [24] every AAF has at least one preferred extension, therefore AF^D has at least one preferred extension. By Theorem 6.10 and the fact that for any two sets of arguments $\mathbf{A}, \mathbf{B} \subseteq \mathbf{Args}$, it holds that $\mathbf{A} = \mathbf{B}$ if and only if $\overline{\mathbf{A}} = \overline{\mathbf{B}}$, AF^S has the same number of preferred extensions with AF^D. Therefore, AF^S has at least one preferred extension, and, by Theorem 5.10, at least one preferred labelling.

Any SETAF AF^S has at least one complete, admissible and conflict-free extension (labelling):

By Theorem 7.1, every preferred extension (labelling) of a SETAF AF^S is also a complete, admissible and conflict-free extension (labelling) of AF^S. As proved above, any SETAF has at least one preferred extension (labelling). Therefore, any SETAF has at least one complete, admissible and conflict-free extension (labelling).

Any SETAF AF^S has exactly one grounded extension (labelling):

Let AF^D be the generated AAF of SETAF AF^S (Definition 6.1). According to [2], every AAF has a unique grounded extension, therefore AF^D has a unique grounded extension. By Theorem 6.10 and the fact that for any two sets of arguments $\mathbf{A}, \mathbf{B} \subseteq \mathbf{Args}$, it holds that $\mathbf{A} = \mathbf{B}$ if and only if $\overline{\mathbf{A}} = \overline{\mathbf{B}}$, AF^S has the same number of grounded extensions with AF^D. Therefore, AF^S has a unique grounded extension. By Theorem 5.10, AF^S therefore has at least one grounded labelling. Suppose that AF^S has two grounded labellings, λ and λ'. By 5.11, we would then be able to generate two grounded extensions, $Ext(\lambda)$ and $Ext(\lambda')$. According to Theorem 5.8 and Corollary 5.7 *Ext* is injective for grounded labellings. Therefore, $Ext(\lambda)$ and $Ext(\lambda')$ can't be the same. This, however, contradicts that AF^S has a unique grounded extension. Therefore, our hypothesis that AF^S has two grounded labellings doesn't hold, which means that AF^S has a unique grounded labelling.

Any SETAF AF^S has zero or more stable extensions (labellings):

It suffices to find a SETAF with no stable extensions or labellings. Consider the SETAF $AF^S = \langle \mathbf{Args}, \triangleright \rangle$, where $\mathbf{Args} = \{a, b, c\}$ and $\triangleright = \{(a, b), (b, c), (c, a)\}$. AF^S has no stable extensions or labellings.

Any SETAF AF^S has at least one naive extension (labelling):

The proof procedure is similar to the one used in [43] for the preferred semantics of AAF. Consider a (possible infinite) sequence of increasing conflict-free subsets of **Args**, $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \ldots$ such that $\mathbf{A_1} \subset \mathbf{A_2} \subset \mathbf{A_3} \subset \ldots$. The union of these sets, $\bigcup \mathbf{A_i}$ is also conflict-free, as if this was not the case, then at least one of $\mathbf{A_i}$ would not be conflict-free. According to Zorn's Lemma [44], every non-empty partially ordered set (P) of which every totally ordered subset (T) has an upper bound contains at least one maximal element. Let P be the set of all conflict-free subsets of **Args** ordered according to the subset relation. Every sequence of conflict-free subsets of **Args**, $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \ldots$ such that $\mathbf{A_1} \subset \mathbf{A_2} \subset \mathbf{A_3} \subset \ldots$ is a totally ordered subset of P, and has an upper bound, i.e. their union, $\bigcup \mathbf{A_i}$. Therefore, according to Zorn's Lemma, P has at least one maximal element, which is a naive extension of AF^S. Therefore, by Theorem 5.10, AF^S at least one naive labelling.

Any SETAF AF^S has zero or more semi-stable extensions (labellings), and at least one if AF^S is finite:

Finite SETAF: As already proved, AF^S has at least one complete extension. Since **Args** is a finite set of arguments, it can only have a finite number of complete extensions. Therefore, the set of sets $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\}$, where $\mathbf{A}$ is a complete extension of AF^S, has at least one maximal element. Therefore, AF^S has at least one semi-stable extension, and, by Theorem 5.10, at least one semi-stable labelling.

Infinite SETAF: Example 6.14 shows an infinite SETAF with no semi-stable extensions or labellings.

Any SETAF AF^S has exactly one ideal extension:

Let AF^D be the generated AAF of SETAF AF^S (Definition 6.1). As proved in [24], every AAF has a unique ideal extension, therefore AF^D has a unique ideal extension. By Theorem 6.32 and the fact that for any two sets of arguments $\mathbf{A}, \mathbf{B} \subseteq \mathbf{Args}$, it holds that $\mathbf{A} = \mathbf{B}$ if and only if $\overline{\mathbf{A}} = \overline{\mathbf{B}}$, AF^S has the same number of ideal extensions with AF^D. Therefore, AF^S has a unique ideal extension. By Theorem 5.10, AF^S therefore has at least one ideal labelling. Suppose that AF^S has two ideal labellings, λ and λ'. By 5.11, we would then be able to generate two ideal extensions, $Ext(\lambda)$ and $Ext(\lambda')$. According to Theorem 5.8 and Corollary 5.7 *Ext* is injective for ideal labellings. Therefore, $Ext(\lambda)$ and $Ext(\lambda')$ can't be the same. This, however, contradicts that AF^S has a unique ideal extension. Therefore, our hypothesis that AF^S has two ideal labellings doesn't hold, which means that AF^S has a unique ideal labelling.

Any SETAF AF^S has at least one eager extension (labelling), and exactly one if AF^S is finite:

Finite SETAF: We call a set of arguments eager iff it admissible and is a subset of every semi-stable extension of AF^S. As already proved, a finite SETAF has at least one semi-stable extension. Consider two subsets of **Args**, **X**, **Y**, which are both eager. Then, $\mathbf{S} = \mathbf{X} \cup \mathbf{Y}$ is also a subset of every semi-stable extension of AF^S. Since **S** is a subset of every semi-stable extension, it is conflict-free. It also holds that for every set of arguments **B** attacking an argument in **S**, $\mathbf{S} \blacktriangleright \mathbf{B}$. Therefore, **S** is also admissible. There is, therefore, one maximal eager set, **A**, which is the union of all eager sets in AF^S. Suppose there is an argument $a \in \mathbf{Args} \setminus \mathbf{A}$, such that a is acceptable w.r.t. **A**. Since **A** is a subset of every semi-stable extension, a is acceptable w.r.t., and, therefore, contained in every semi-stable extension of AF^S. This implies that $\mathbf{A}' = \mathbf{A} \cup \{a\}$ is an eager set, since it is admissible and is a subset of every semi-stable extension of AF^S, which violates the maximality of **A**. Therefore, **A** contains all the arguments it defends. It is therefore a complete extension of AF^S, and maximal among the complete extensions of AF^S that are subsets of every semi-stable extension of AF^S, and therefore the unique eager extension of AF^S. By Theorem 5.10, AF^S has, therefore, at least one eager labelling. Suppose that AF^S has two eager labellings, λ and λ'. By 5.11, we would then be able to generate two eager extensions, $Ext(\lambda)$ and $Ext(\lambda')$. According to Theorem 5.8 and Corollary 5.7 *Ext* is injective for eager labellings. Therefore, $Ext(\lambda)$ and $Ext(\lambda')$ can't be the same. This, however, contradicts that AF^S has a unique eager extension. Therefore, our hypothesis that AF^S has two eager labellings doesn't hold, which means that AF^S has a unique eager labelling.

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

Infinite SETAF: Consider the case that AF^S has no semi-stable extensions (which, as already proved, is possible for infinite SETAF). Then, $\mathbf{A} \subset \mathbf{Args}$ is an eager extension of AF^S iff it is a maximal (w.r.t. set inclusion) complete extension of AF^S (by Definition 3.9), and therefore $\mathbf{A}$ is an eager extension of AF^S iff it is a preferred extension of AF^S (by Definition 3.4). As already proved, an infinite SETAF has at least one preferred extension. Therefore, AF^S has at least one eager extension, and, by Theorem 5.10, at least one eager labelling. For the case that AF^S has at least one semi-stable extension, the proof is the same with the case of finite SETAF.

Any SETAF AF^S has zero or more stage extensions (labellings), and at least one if AF^S is finite.

Finite SETAF: There is at least one conflict-free subset of **Args**. Since **Args** is a finite set of arguments, it can only have a finite number of conflict-free subsets. Therefore, the set of sets $\mathbf{A} \cup \{b \in \mathbf{Args} \mid \mathbf{A} \blacktriangleright b\}$, where $\mathbf{A}$ is a conflict-free subset of AF^S, has at least one maximal element. Therefore, AF^S has at least one stage extension, and, by Theorem 5.10, at least one stage labelling.

Infinite SETAF: Example 6.14 shows an infinite SETAF with no stage extensions or labellings. □

References

[1] S.H. Nielsen, S. Parsons, A generalization of Dung's abstract framework for argumentation: arguing with sets of attacking arguments, in: Argumentation in Multi-Agent Systems, 2007, pp. 54–73.
[2] P.M. Dung, On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games, Artif. Intell. 77 (2) (1995) 321–357.
[3] H. Prakken, G. Sartor, Law and logic: a review from an argumentation perspective, Artif. Intell. 227 (Supplement C) (2015) 214–245.
[4] A. Hunter, M. Williams, Aggregating evidence about the positive and negative effects of treatments, Artif. Intell. Med. 56 (3) (2012) 173–190.
[5] J.a. Leite, J.a. Martins, Social abstract argumentation, in: Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence – Volume Three, IJCAI'11, AAAI Press, 2011, pp. 2287–2292.
[6] F. Bex, J. Lawrence, M. Snaith, C. Reed, Implementing the argument web, Commun. ACM 56 (10) (2013) 66–73.
[7] J. Schneider, T. Groza, A. Passant, A review of argumentation for the social semantic web, Semant. Web 4 (2) (2013) 159–218.
[8] P. Novák, C. Witteveen, Context-aware reconfiguration of large-scale surveillance systems: argumentative approach, Argument Comput. 6 (1) (2015) 3–23.
[9] P. Baroni, M. Romano, F. Toni, M. Aurisicchio, G. Bertanza, Automatic evaluation of design alternatives with quantitative argumentation, Argument Comput. 6 (1) (2015) 24–49.
[10] P. Baroni, M. Caminada, M. Giacomin, An introduction to argumentation semantics, Knowl. Eng. Rev. 26 (4) (2011) 365–410.
[11] H. Jakobovits, D. Vermeir, Robust semantics for argumentation frameworks, J. Log. Comput. 9 (2) (1999) 215–261.
[12] M. Caminada, D.M. Gabbay, A logical account of formal argumentation, Stud. Log. 93 (2/3) (2009) 109–145.
[13] D.M. Gabbay, Semantics for higher level attacks in extended argumentation frames part 1: overview, Stud. Log. 93 (2–3) (2009) 357–381.
[14] N. Oren, T.J. Norman, Semantics for evidence-based argumentation, in: Computational Models of Argument: Proceedings of COMMA 2008, Toulouse, France, May 28–30, 2008, 2008, pp. 276–284.
[15] R. Arisaka, K. Satoh, Coalition formability semantics with conflict-eliminable sets of arguments, in: Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems, AAMAS '17, International Foundation for Autonomous Agents and Multiagent Systems, 2017, pp. 1469–1471.
[16] W. Dvorák, J. Fandinno, S. Woltran, On the expressive power of collective attacks, in: Proceedings of the 7th Conference on Computational Models of Argument (COMMA-18), 2018, pp. 49–60.
[17] A. Bondarenko, P.M. Dung, R.A. Kowalski, F. Toni, An abstract, argumentation-theoretic approach to default reasoning, Artif. Intell. 93 (1–2) (1997) 63–101.
[18] M. Caminada, Semi-stable semantics, in: Proceedings of the 1st International Conference on Computational Models of Argument (COMMA-06), 2006, pp. 121–130.
[19] M. Caminada, Comparing two unique extension semantics for formal argumentation: ideal and eager, in: BNAIC 2007, 2007, pp. 81–87.
[20] M. Caminada, On the Issue of Reinstatement in Argumentation, Springer, Berlin, Heidelberg, 2006, pp. 111–123.
[21] P.M. Dung, P. Mancarella, F. Toni, Computing ideal sceptical argumentation, Artif. Intell. 171 (10–15) (2007) 642–674.
[22] M. Caminada, A labelling approach for ideal and stage semantics, Argument Comput. 2 (1) (2011) 1–21.
[23] B. Verheij, Two approaches to dialectical argumentation: admissible sets and argumentation stages, in: Proceedings of the Biannual International Conference on Formal and Applied Practical Reasoning (FAPR), Universiteit workshop, 1996, pp. 357–368.
[24] R. Baumann, C. Spanring, Infinite argumentation frameworks – on the existence and uniqueness of extensions, in: Advances in Knowledge Representation, Logic Programming, and Abstract Argumentation, in: Lecture Notes in Computer Science, vol. 9060, Springer, 2015, pp. 281–295.
[25] A. Bochman, Collective argumentation and disjunctive logic programming, J. Log. Comput. 13 (3) (2003) 405–428.
[26] C. Cayrol, M.-C. Lagasquie-Schiex, Coalitions of arguments: a tool for handling bipolar argumentation frameworks, Int. J. Intell. Syst. 25 (2010) 83–109.
[27] F. Nouioua, AFs with necessities: further semantics and labelling characterization, in: Proceedings of the International Conference on Scalable Uncertainty Management (SUM), 2013, pp. 120–133.
[28] D.M. Gabbay, M. Gabbay, Theory of disjunctive attacks, part I, Log. J. IGPL 24 (2) (2016) 186–218.
[29] J.L. Pollock, Cognitive Carpentry: A Blueprint for How to Build a Person, MIT Press, Cambridge, MA, USA, 1995.
[30] H. Prakken, A study of accrual of arguments, with applications to evidential reasoning, in: Proceedings of the 10th International Conference on Artificial Intelligence and Law, ICAIL '05, 2005, pp. 85–94.
[31] H. Verheij, Rules, Reasons, Arguments: Formal Studies of Argumentation and Defeat, Ph.D. thesis, Maastricht University, 1996.
[32] S. Modgil, T.J.M. Bench-Capon, Integrating dialectical and accrual modes of argumentation, in: COMMA, in: Frontiers in Artificial Intelligence and Applications, vol. 216, IOS Press, 2010, pp. 335–346.
[33] D. Grossi, S. Modgil, On the graded acceptability of arguments, in: IJCAI, AAAI Press, 2015, pp. 868–874.
[34] S. Modgil, Revisiting abstract argumentation frameworks, in: TAFA, in: Lecture Notes in Computer Science, vol. 8306, Springer, 2013, pp. 1–15.
[35] H. Prakken, An abstract framework for argumentation with structured arguments, Argument Comput. 1 (2) (2010) 93–124.
[36] S.H. Nielsen, S. Parsons, Computing preferred extensions for argumentation systems with sets of attacking arguments, in: Proceedings of the 2006 Conference on Computational Models of Argument: Proceedings of COMMA 2006, 2006, pp. 97–108.
[37] L. Amgoud, C. Cayrol, M.-C. Lagasquie-Schiex, P. Livet, On bipolarity in argumentation frameworks, in: Bipolar Representations of Information and Preference, Part 2: Reasoning and Learning, Int. J. Intell. Syst. 23 (10) (2008) 1062–1093.
[38] C. Cayrol, M.C. Lagasquie-Schiex, On the acceptability of arguments in bipolar argumentation frameworks, in: Symbolic and Quantitative Approaches to Reasoning with Uncertainty, 2005, pp. 378–389.

UNCORRECTED PROOF

Please cite this article in press as: G. Flouris, A. Bikakis, A comprehensive study of argumentation frameworks with sets of attacking arguments, Int. J. Approx. Reason. (2019), https://doi.org/10.1016/j.ijar.2019.03.006

[39] P.E. Dunne, A. Hunter, P. McBurney, S. Parsons, M. Wooldridge, Weighted argument systems: basic definitions, algorithms, and complexity results, Artif. Intell. 175 (2) (2011) 457–486.
[40] E. Bonzon, J. Delobelle, S. Konieczny, N. Maudet, A comparative study of ranking-based semantics for abstract argumentation, in: Proceedings of the 30th AAAI Conference on Artificial Intelligence (AAAI-16), 2016, pp. 914–920.
[41] T.J.M. Bench-Capon, Persuasion in practical argument using value-based argumentation frameworks, J. Log. Comput. 13 (3) (2003) 429–448.
[42] L. Amgoud, S. Vesic, Rich preference-based argumentation frameworks, Int. J. Approx. Reason. 55 (2) (2014) 585–606.
[43] M. Caminada, B. Verheij, On the existence of semi-stable extensions, in: Proceedings of the 22nd Benelux Conference on Artificial Intelligence, 2010.
[44] M. Zorn, A remark on method in transfinite algebra, Bull. Am. Math. Soc. 41 (1935) 667–670.

UNCORRECTED PROOF