Non-Linear ZMP based State Estimation for Humanoid Robot Locomotion*

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Abstract—This article presents a novel state estimation scheme for humanoid robot locomotion using an Extended Kalman Filter (EKF) for fusing encoder, inertial and Foot Sensitive Resistor (FSR) measurements. The filter’s model is based on the non-linear Zero Moment Point (ZMP) dynamics and thus, coupling the dynamic behavior in the frontal and the lateral plane. Furthermore, it provides state estimates for variables that are commonly used by walking pattern generators and posture balance controllers, such as the Center of Mass (CoM) and the linear time-varying Divergent Component of Motion (DCM) position and velocity, in the 3-D space. Modeling errors are taken into account as external forces acting on the robot in the acceleration level. In addition, an observability analysis for the non-linear system dynamics and the linearized discrete-time EKF dynamics is presented. Subsequently, by utilizing ground-truth data obtained from a vicon motion capture system with a NAO humanoid robot, we demonstrate the effectiveness and robustness of the proposed scheme contrasted to the linear filters, even in the case where disturbances are introduced to the system. Finally, the proposed approach is implemented and employed for feedback to a real-time posture controller, rendering a NAO robot able to walk on an outdoors inclined pavement.

I. INTRODUCTION

Humanoid robot locomotion is a challenging task with many difficulties. Mainly, due to the non-linear multi-body dynamics along with the many Degrees of Freedoms (DoFs) the humanoid robots have, the under-actuation which occurs during the gait, and the unilateral type of contact the robots experience with the ground. The non-linear multi-body dynamics prohibit exact solutions to be obtained in real-time. Therefore, many researchers approximated those dynamics with simplified models that could describe the dynamic behavior of a humanoid while walking. However, those models are based on the assumption that the robot’s dynamics are decoupled in the frontal and the lateral plane, which is not true, especially when the robot exhibits highly dynamic motions. In addition, since the robot does not have a fixed base, it is under-actuated. Nevertheless, when the assumptions that a rigid type contact between the support leg and the ground along with sufficient friction exist, all the under-actuated DOFs vanish. Unfortunately, this is not the case in a realistic environment. Vertical displacement with respect to the ground can cause acceleration in the same direction which must be taken into consideration while planning or controlling the robot in order to avoid undesired ground reaction forces.

In this paper, we propose a novel estimation scheme with an Extended Kalman Filter (EKF) which has its dynamics based on the non-linear Zero Moment Point (ZMP) formulation, thus, effectively coupling the dynamic behavior in the frontal and lateral plane and fusing information from sensors that are widely available on humanoids, namely, encoders, Inertial Measurement Units (IMU), and Foot Sensitive Resistors (FSRs). This filter provides accurate estimates for variables that are commonly used by walking pattern generators and posture stabilization controllers, such as the Center of Mass (CoM) and the linear time-varying Divergent Component of Motion (DCM) position and velocity, in the x, y, and z axes, as experimentally validated with a NAO humanoid robot under real world conditions.

II. RELATED WORK

Biped state-estimation plays an important role in realizing stable walking motions and in posture balance control [3]. Xinjilefu et al. [4] solved a Quadratic Program (QP) utilizing the robot’s full-body dynamics. The proposed approach was advantageous, in the sense that it did not require a state-space model as in the Kalman Filter (KF) case, could naturally handle equalities and inequalities as constraints, and consider modeling error in the state vector. However, due to the imposed constraints and the high-dimensionality of the framework was computationally expensive for real-time execution, did not generalize since it was based on the robot’s dynamics and required force/torque sensors on robot’s joints. Stephens [5] used simplified models based on the the Linear Inverted Pendulum Model (LIPM) dynamics [6] for state-estimation in order to control the posture of the force-controlled Sarcos Primus humanoid. He was able to estimate modeling errors as incoming external forces, and possible CoM biases by fusing CoM and Center of Pressure (CoP) measurements from the joint encoders and the FSRs respectively. Nevertheless, he observed that there was a trade-off between disturbance estimation and state estimation, since time-varying disturbances demanded a carefully tuning of the noise covariances. Based on that approach, Xinjilefu and Atkeson [7] compared two KF schemes; one based on the LIPM dynamics and one based on robot’s planar dynamics. They observed that LIPM KF was simple to design and implement, easy to tune, robust to modeling errors, and can generalize to other robots, while, as expected, the Planar KF yielded more accurate estimates since it is based on a more

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accurate representation of the robot’s dynamics. Another approach based on the LIPM dynamics was presented by Kwon and Oh [8], where the current CoP measurement was the input of a KF and the output was the CoM position. The filter’s state was augmented with a CoM bias and a state for external forces. A similar approach but without the CoM bias in the state vector was proposed by Wittmann et al. [9], where a state estimator for biped robots fusing encoders, IMU, and force/torque measurements with a KF based on the LIPM dynamics was presented. Nevertheless, by using the LIPM dynamics, one assumes that the CoM is constrained to lie on a constant horizontal plane and furthermore that the motion in the $x$ and $y$ axes are decoupled. Unfortunately, neither holds in real world conditions and especially when a robot locomotes on uneven and/or rough terrain.

Other approaches, treat the robot as a floating rigid body, based on Newton-Euler dynamics. Bloesch et al. [10] proposed a state estimation scheme for quadruped robots, fusing leg kinematics and the IMU measurements to estimate not only the position and the velocity of the floating base but also the body’s orientation, expressed as a quaternion, and the foothold positions. This approach was extended in the humanoid robot case by Rotella et al. [11]. However, since those approaches are based on generic dynamics, they are very sensitive to noise and thus, difficult to tune. To this end, the authors carried out an Alan variance analysis [12] to carefully identify the noise characteristics of the IMU. A similar approach was proposed by Bry et al. [13] for Micro-Aerial-Vehicle (MAV) navigation, where the orientation uncertainty was expressed as a screw in exponential coordinates around the body frame. This approach was extended to the Atlas robot by Kuindersma et al. [14], where it yielded drift-free estimates [15] at a cost of incorporating exteroceptive LIDAR measurements with a Gaussian Particle Filter (GPF).

In our work, the proposed state estimator fuses effectively three different kind of sensors, generalizes with little to no effort to other humanoids, can be easily tuned, and yields accurate 3-D estimates for important quantities in humanoid planning and control, even in the $z$-axis contrasted to the LIPM approaches.

This paper is organized as follows, section III, presents the underlying EKF dynamics. In addition, section IV demonstrates an observability analysis for both the non-linear dynamics and the EKF linearized dynamics. Next, in section V experimental results with a real NAO humanoid robot are presented. Section VI concludes the paper and discusses possible future work.

### III. Extended Kalman Filter Based State Estimation

In this section, we will present the EKF’s process and measurement model which will be used for the state estimation task. The dynamics are based on the non-linear ZMP equation, where we treat the ZMP location on the plane and the vertical ground reaction force (GRF) as the input to the system and the output are the position and the acceleration of the CoM in the 3-D space.

The ZMP is defined as the point on the ground at which the moments generated by the reaction forces vanish. By also considering external forces acting on the robot’s body, the equations of motion are formulated as:

\[
\begin{align*}
\dot{c}_x &= \frac{z_x - z_c(\ddot{c}_y + g)}{c_z} + \frac{1}{m}f_x \\
\dot{c}_y &= \frac{y_y - y_c(\ddot{c}_y + g)}{c_z} + \frac{1}{m}f_y \\
\dot{c}_z &= \frac{1}{m}(f_N + f_z) - g
\end{align*}
\]

where $z_x$, $z_y$, $f_x$, $f_y$ are the ZMP coordinates and external forces/modeling errors in the $x$ and $y$ axes respectively, $c_x$, $c_y$, $c_z$ is the position of the CoM with respect to an inertial frame of reference, $\ddot{c}_x$, $\ddot{c}_y$, $\ddot{c}_z$ is the corresponding acceleration, $g$ is the gravitational acceleration and $m$ is the robot’s mass. Furthermore, for the $z$-axis the dynamics are:

\[
\dot{c}_z = \frac{1}{m}f_N - g + \frac{1}{m}f_z
\]

where $f_N$ is the vertical GRF and $f_z$ the external force/modeling error in the $z$ direction.

Replacing (3) in (1), (2), yields the following 3-D non-linear dynamics:

\[
\begin{align*}
\dot{c}_x &= \frac{z_x - z_c(\ddot{c}_y + g)}{mc_z} + \frac{1}{m}f_x \\
\dot{c}_y &= \frac{y_y - y_c(\ddot{c}_y + g)}{mc_z} + \frac{1}{m}f_y \\
\dot{c}_z &= \frac{1}{m}(f_N + f_z) - g
\end{align*}
\]

#### A. Process Model

Assume the following state vector for the process dynamics:

\[
x_t = \begin{bmatrix} c_x & c_y & c_z & \dot{c}_x & \dot{c}_y & \dot{c}_z & f_x & f_y & f_z \end{bmatrix}^T
\]

with $\dot{c}_x$, $\dot{c}_y$, $\dot{c}_z$ the CoM velocity. Furthermore, assume the input $u$ to the filter is the ZMP in the $x$ and $y$ axes along with the vertical GRF as measured by the FSRs:

\[
u_t = \begin{bmatrix} \dot{c}_x & \dot{c}_y & \dot{c}_z & f_x & f_y & f_z \end{bmatrix}^T
\]

Consequently, the process model takes the standard non-linear form:

\[
x_{t+1} = f(x_t, u_t, \epsilon_t)
\]

where

\[
\begin{bmatrix} c_x \\
\dot{c}_x \\
\dot{c}_y \\
\dot{c}_z \\
f_x \\
f_y \\
f_z \end{bmatrix} =
\begin{bmatrix}
\dot{c}_x \\
\dot{c}_y \\
\dot{c}_z \\
0 \\
0 \\
0
\end{bmatrix}
\]

and $\epsilon_t$ is a Gaussian zero-mean additive noise with covariance $Q_t$, $\epsilon_t \sim \mathcal{N}(0, Q_t)$.
Taking the appropriate continuous dynamic linearization, yields the following Jacobian matrix of the state vector $x$:

$$G_t = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & I & 0 \\ C_t & 0 & D_t \end{bmatrix}$$

with

$$C_t = \begin{bmatrix} \frac{u_3 + f_x}{mc_x} & 0 & -\frac{(u_3 + f_x)(c_y - u_1)}{mc_x} \\ 0 & \frac{u_3 + f_x}{mc_y} & -\frac{(u_3 + f_x)(c_y - u_2)}{mc_x} \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_t = \begin{bmatrix} \frac{1}{m} & 0 & \frac{c_y - u_1}{mc_x} \\ 0 & \frac{1}{m} & \frac{c_y - u_2}{mc_x} \\ 0 & 0 & \frac{1}{m} \end{bmatrix}$$

Although, a more accurate approximation could be used to compute the discretized matrix $G_k$, Euler integration is used for simplicity:

$$G_k = I + G_t \Delta t$$

where $\Delta t$ is the sampling time.

To this end, the prediction step of the EKF is readily formulated as:

$$\dot{x}_{k|k-1} = f(\dot{x}_{k-1|k-1}, u_k, 0) \Delta t + \ddot{x}_{k-1|k-1}$$

$$P_{k|k-1} = G_t P_{k-1|k-1} G_t^T + Q_k$$

with $P$ being the estimate error covariance matrix.

### B. Measurement Model

For the output dynamics we employ sensors that are commonly available on humanoid robots nowadays. We assume that the robot is equipped with encoders on every joint and thus we are able to compute the CoM position with respect to the torso local frame. Moreover, with the IMU we compute the corresponding CoM accelerations, again in the torso’s local frame. Notice that all measurements, denoted as $y_i$, need to be transformed to the inertial frame of reference.

$$y_1 = c_{ENC}, \quad y_2 = c_{y}, \quad y_3 = c_{z}, \quad y_4 = c_{z}^{MU}, \quad y_5 = \dot{c}_{y}^{MU}, \quad y_6 = \dot{c}_{z}^{MU},$$

Subsequently, since the CoM acceleration is not part of the state vector, the output equation is non-linear:

$$y_t = h(x_t, u_t) + \delta_t$$

with

$$h(x_t, u_t) = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} \frac{c_y - u_1}{mc_x}(u_3 + f_x) + \frac{1}{m} f_y \\ \frac{c_y - u_2}{mc_x}(u_3 + f_x) + \frac{1}{m} f_y \\ \frac{1}{m}(u_3 + f_x) - g \end{bmatrix}$$

and $\delta_t$ be the Gaussian zero-mean measurement noise with covariance $R_t$, $\delta_t \sim \mathcal{N}(0, R_t)$. After discretizing, the Jacobian matrix $H_k = \frac{\partial h}{\partial x}$ can be readily computed, following the derivation of $G_t$. Then, the EKF update step is realized as:

$$K_k = H_k P_{k|k-1} H_k^T + R_k$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - h(\hat{x}_{k|k-1}, u_k))$$

$$P_{k|k} = (I - K_k H_k) P_{k|k-1}$$

where $K_k$ is the Kalman gain.

### IV. Observability Analysis

#### A. Non-linear Observability Analysis

Non-linear observability analysis is far from trivial, as it is in the linear time-invariant case, where the observability properties are excessively studied over the years and thus, well-understood. This is mainly due to the strong dependence the analysis has on the underlying non-linear dynamics and the neighborhood of the current system’s state and input. Therefore, we have included a brief Appendix shortly describing some important results from the non-linear geometric control theory that are used in our analysis.

Following the notation introduced in the Appendix, the dimension of the state-space and the measurement’s model is $n = 9$ and $m = 6$ respectively, therefore, by choosing the following coordinates $(h_1, \varphi_1, h_2, \varphi_2, h_3, \varphi_3, h_4, h_5, h_6)$, defined on the current operating point $(x_t^*, u_t^*)$, we obtain the following map:

$$\Phi(x_t^*, u_t^*) = \begin{bmatrix} \frac{c_y^*}{m} \\ \frac{c_z^*}{m} \\ \frac{u_3 + f_z^*}{m} + \frac{1}{m} f_y \end{bmatrix}$$

By re-ordering the quantities to obtain a mathematically convenient form $\Phi$ and then taking the Jacobian with respect to $x_t^*$, we get the local non-linear observability matrix:

$$\mathcal{O} = \frac{\partial \Phi(x_t^*, u_t^*)}{\partial x_t^*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ C_t & 0 & D_t \end{bmatrix}$$

and since $\det \mathcal{O} = -\frac{1}{m}$ we have that

$$\text{rank} \mathcal{O} = 9$$

rendering the non-linear dynamics in (8), (16) to be locally observable in all cases.

#### B. Linear Time-Varying Observability Analysis

Since, we are using a discrete EKF for the state estimation task, we must explore the observability of the filter which is based on the following linear time-varying dynamics:

$$x_{k+1} = G_k x_k + \epsilon_k$$

$$y_k = H_k x_k + \delta_k$$

where $G_k$ is the process noise.
We will give the local observability analysis based on the linear time-varying observability matrix $\mathcal{M}$, as proposed by Chen et al. [16]. Notice, in the general case, the observability properties of a discrete linear time-varying system, can differ from the observability properties of the true underlying nonlinear continuous system. This is due to the errors that arise by the linearization and/or discretization procedure.

For the state-space and output model in (24), (25), this matrix is defined as:

$$
\mathcal{M} = \begin{bmatrix}
H_k \\
H_{k+1} G_k \\
\vdots \\
H_{k+8} G_{k+7} \cdots G_k
\end{bmatrix}
$$

(26)

and the sufficient condition for the local observability is:

$$\text{rank}(\mathcal{M}) = 9$$

(27)

By examining the first $9 \times 9$ submatrix of $\mathcal{M}$:

$$\mathcal{M}^* = \begin{bmatrix}
I & 0 \\
C_k & 0 & D_k \\
0 & \Delta t I & 0
\end{bmatrix}
$$

(28)

where $C_k$ and $D_k$ are the matrices in (11), (12) evaluated at the $k$-th discrete time instant, it is straightforward to derive:

$$\det(\mathcal{M}^*) = -\left(\frac{\Delta t}{m}\right)^3$$

(29)

Therefore, the linear time-varying observability matrix $\mathcal{M}$ is full rank and cannot drop rank under any circumstances.

V. EXPERIMENTAL RESULTS

In the current section we outline representative results that demonstrate the effectiveness and robustness of the proposed scheme, contrasted to others well-established approaches. All conducted experiments were performed with a real NAO humanoid robot. First, the EKF is compared to a KF based on the LIPM dynamics, on ground-truth data sets obtained with a vicon motion capture system. Next, the proposed approach is employed to feedback a posture stabilization controller based on the DCM, enabling a NAO robot to walk outdoors on an inclined pavement. The filter’s response is contrasted to an EKF based on Newton-Euler dynamics. In all our experiments the covariance matrices were determined experimentally and set to:

$$Q = \text{diag}(1e - 6I, 1e - 5I, I)$$

(30)

$$R = \text{diag}(5e - 6I, 5e - 4I)$$

(31)

A. Evaluation on Motion Captured Data Sets

A first result regards an estimation accuracy study, in terms of the Root Mean Square Error (RMSE), contrasted to a KF based on the LIPM, as proposed by Wittman et al. [9]. The LIPM KF can estimate the following state vector:

$$x_t = [c_x \ c_y \ \dot{c}_x \ \dot{c}_y \ f_x \ f_y]^{T}$$

(32)

utilizing the CoM position and velocity, as measured by the encoders, and transformed to the inertial frame with the IMU. The employed covariance matrices are given by (30), (31), where this time the matrices-dimensions are obtained by neglecting the $z$-axis dynamics.

We’ve selected 10 ground-truth data sets, collected with a vicon motion capture system consisting of 15 infra-red cameras, and a NAO v3.3 humanoid robot [17]. To begin with, we calibrated the IMU, to remove the biases and cut off unwanted high frequencies with a low-pass filter. Then, we computed the CoM with respect to the local torso frame using kinematics and the CoP using the FSR in the feet which was then transformed to the torso local frame with kinematics. Subsequently, all the acquired data were transformed to the inertial frame of reference using the estimated rotation matrix obtained by the IMU. Notice, since the NAO robot is not equipped with a 3-axis gyroscope, namely no gyro rate is available about the $z$-axis, the vicon yaw angle was used instead. This is why no drifting is observed in all our data, although the robot drifts in many cases. Nevertheless, this does not pose any limitation, since the same data are used as input and measurements signals for both approaches.

Since we cannot plot the response for every data set, we selected one where the motion is unstable and thus more dynamic [17]. In Figure 1, the CoM trajectories are shown where the EKF estimated trajectory is overlapped by the actual signal, notice in the start the robot rises from a sitting position, the EKF can accurately capture that motion. In addition, Figure 2 shows that the EKF estimated more accurately the corresponding CoM velocities in all axes.

Figure 3 illustrates the external forces/modeling error for the corresponding motion; notice that there is no delay in the force estimation as observed in the KF’s case, also reported in [9]. This is due to the fusion of the acceleration measurement which yields a lag free estimation. Furthermore, notice in the $z$-axis, at time $0$–$2s$ where the robot is practically
still, the estimation is almost 15N, this is reasonable since the NAO’s FSR have a reliable working range up to 25N and when the robot is still they measure approximately 35N, therefore since the robot weights 4.789kg the modeling error needs to be approximately 1.5kg.

Furthermore, we computed the position and velocity of the DCM. In the KF’s case, we are forced to compute the Linear Time-Invariant (LTI) DCM since the assumption that the DCM. In the KF’s case, we are forced to compute the

\[ \dot{\xi}_{\text{LTI}} = c + \frac{1}{\omega_0} \dot{c} \]  

where \( \omega_0 = \sqrt{\frac{\rho}{\ell}} \) and \( \ell \) is the constant CoM height. The corresponding LTI-DCM velocity is thus:
corresponding LTV-DCM velocity is given by:

\[ \dot{\xi}_{LT} = \omega_0 (\xi_{LT} - c) + \frac{1}{\omega_{0l}} \ddot{c} \]  

(34)

On the other hand, in the EKF’s case we can compute the Linear Time-Varying (LTV) DCM to approximate the true non-linear DCM more effectively. The LTV-DCM is formulated as:

\[ \xi_{LTV} = c + \frac{1}{\omega_{lt}} \dot{c} \]  

(35)

where \( \omega_{lt} = \sqrt{\omega_{l}^2 + \omega_{l}^2}. \) Since \( \omega_{lt} \) is now time-dependent, the corresponding LTV-DCM velocity is given by:

\[ \dot{\xi}_{LTV} = \left( \omega_{lt} - \frac{\dot{\omega}_{lt}}{\omega_{lt}} \right) (\xi_{LTV} - c) + \frac{1}{\omega_{lt}} \ddot{c} \]  

(36)

with \( \dot{\omega}_{lt} = -\frac{1}{2} \omega_{lt}^{1/2} \dot{\omega}_{lt}. \)

In both DCM velocity cases, we used the calibrated low-pass filtered acceleration by the IMU, and the corresponding CoM position and velocity estimate by each filter respectively.

Figure 4 shows the corresponding DCM trajectories for this case of study, while Figure 5 demonstrates the DCM velocities. Finally, in Figure 6 the average RMSE for all quantities of interest for the 10 data sets used in our study is presented. Notice that the EKF not only yields more accurate estimates in the RMSE sense, but also more certain ones, especially when the estimated quantities are the velocities.

\[ x_t = [c \ \dot{c} \ q \ b_f \ b_w]^{T} \]  

(37)

where \( q, b_f, \) and \( b_w, \) are the torso’s attitude quaternion, the acceleration and the gyroscope biases respectively, utilizing the inertial CoM position as measured by the encoders. Notice, we’ve collected raw IMU data for 48 hours in order to perform an Allan variance analysis [12] and carefully tune the process noise covariance, as also suggested by the authors, to maximize in such a way the filter’s efficiency.

Figure 8 and Figure 9 illustrate the CoM position and velocity in the 3-D space as estimated by the two filters, for a diagonally forward gait on a 7° inclined pave-
ment for approximately 25s. Figure 10 shows the external force/modeling errors during the gait, notice only the ZMP based EKF can estimate those quantities. In addition, note that the magnitude of the external forces can be justified by considering that when the robot locomotes on an uneven and rough terrain, early ground contact can commonly occur, giving rise to larger external forces. Moreover, Figure 11 and Figure 12 demonstrate the LTV-DCM position and velocity as estimated by the two filters for the corresponding gait.

Furthermore, we’ve conducted a variety of indoors and outdoors experiments, namely, walking indoors on a hallway, walking outdoors on an even pavement, walking in place on grass while heavily disturbing the robot and, as also illustrated in the supplementary material (for a higher quality video, please check https://goo.gl/by3cB5), all the ZMP based EKF estimates were pretty similar, within noise margins, to the IMU based EKF ones, validating in such a way the proposed estimation scheme.

Notice that in all experiments reported above, the estimated z-axis components contain higher noise compared to the x and y-axis ones. This is due to the fact that the noisy FSR measurements are employed in the z-axis dynamics.

VI. CONCLUSION & FUTURE WORK

In this paper, a novel state estimation scheme for humanoid robot locomotion was presented, fusing effectively three different sensor sources, namely the joint encoders, the IMU, and the FSRs. We utilized the nonlinear ZMP equation with an EKF to surpass the limitation of the constant CoM height and the planar dynamic decouple, as assumed by the LIPM, and readily estimate control variables commonly used by walking pattern generators and posture stabilization controllers. In addition, modeling errors were considered as external forces acting on the CoM in the acceleration level.

Someone would assume that the observability would be lost when the robot experience accelerations in the z-axis equal to g, e.g. the robot is in free fall. As proved by our observability analysis for both the non-linear dynamics and the EKF this is not the case, since the local-observability matrix is full rank under all circumstances. Nevertheless, the ZMP is not well-defined when the robot is in flight.

Our experimental result demonstrated that the filter showed robustness to perturbations, quick convergence properties and provided more accurate estimates contrasted to a KF based on the LIPM. In addition, when incorporated with a real-time stabilization controller, a NAO robot was able to walk on an outdoors inclined pavement and on grass, effectively sensing and negotiating the incoming disturbances. Moreover, the filter’s estimates were pretty similar to the ones obtained by an EKF based on generic rigid body dynamics and the IMU, validating the proposed approach.

In future work, we aim in including the torque that acts about the robot’s CoM in the design. Thus, we will be able
to estimate rotational variables of interest (as it is the case in contemporary IMU-based estimation schemes), such as the angular momentum and the rate of angular momentum.

REFERENCES


APPENDIX

NON-LINEAR OBSERVABILITY

Consider the following non-linear dynamical system:

\[
\dot{x} = f(x, u) \quad (38)
\]

\[
y = h(x, u) \quad (39)
\]

with \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^l \) is the input vector and \( y \in \mathbb{R}^m \) is the measured output. In addition, without the loss of generality we assume that \( f \) and \( h \) are smooth functions. The general question is under which conditions we are able to reconstruct the state vector \( x \) by observing the system’s output \( y \). There are many results on non-linear observability i.e \([18],[19],[20],[21]\), nevertheless, we will follow the results presented by Del Vecchio and Murray [22], and effectively used in [23], since our output dynamics are dependent on the input \( u \).

Let \( h(x, u) = [h_1(x, u), \ldots, h_m(x, u)]^\top \), in addition, let \( \bar{u} = [u_1, \ldots, u_1^{(n_1-1)}, \ldots, u_l, \ldots, u_l^{(n_l-1)}]^\top \) with \( \sum_{i=1}^l n_i = n_u \), and define the functions:

\[
\varphi_0^i = h_i \quad (40)
\]

\[
\varphi_i^j = L_{f}\varphi_i^{j-1} = \frac{\partial \varphi_i^{j-1}}{\partial x}f + \sum_{k=0}^{j-1} \frac{\partial \varphi_i^{j-1}}{\partial u(k)}u(k+1) = y_i^j \quad (41)
\]

where \( L_{f}\varphi_i^{j-1} \) is the Lie derivative of \( \varphi_i^{j-1} \) in the direction of the vector field \( f \) and coincides with the \( j \)-th derivative of the \( i \)-th output, \( y_i^j \).

Next, define the map \( \Phi(x, \hat{u}) : \mathbb{R}^n \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^n \) to be:

\[
\Phi(x, \hat{u}) = [h_1, \varphi_1^1, \varphi_1^2, \ldots, h_m, \varphi_m^1, \ldots, \varphi_m^{k_m-1} ]^\top
\]

\( \forall k_i \mid \sum_{i=1}^l k_i = n \).

Then, the system in (38), (39) is locally observable if there exists a non-empty set \( X \times U \subseteq \mathbb{R}^n \times \mathbb{R}^{n_u} \), such that the map \( \Phi(x, \hat{u}) \), for some \( k_i \), is invertible with respect to \( x \) and its inverse is smooth \( \forall (x, \hat{u}) \in X \times U \), in other words:

\[
\text{rank} \mathcal{O} = \text{rank} \left( \frac{\partial \Phi(x, \hat{u})}{\partial x} \right) = n \quad (42)
\]

where \( \mathcal{O} \) is the local non-linear observability matrix. Notice, the choice of coordinates needed to define the map \( \Phi \) depend on the dynamics and are not unique, since there are many combination of \( k_i \)'s that sum up to \( n \). To this end, it suffices to find a map that satisfies the condition in (42).