Appendix B

Fractional Lower Order Moments of Products of SαS Random Variables

It is known that if \( X \) is a SαS random variable and \( p > 0 \), then \( E\{|X|^p\} < \infty \) if and only if \( p < \alpha \) [6]. Also, if \( X_1, \ldots, X_n \) are \( n \)-fold dependent SαS random variables and \( p_1, \ldots, p_n \) are positive numbers, then

\[
E\{|X_1|^{p_1} \cdots |X_n|^{p_n}\} < \infty \text{ if and only if } p_1 + \cdots + p_n < \alpha. \tag{B.1}
\]

Recently, it was proven in [45] that for a real SαS random variable \( X \) it holds

\[
E\{|X|^p\} < \infty \text{ for } -1 < p < \alpha. \tag{B.2}
\]

Similarly, for an isotropic complex SαS random variable \( X \) it holds

\[
E\{|X|^p\} < \infty \text{ for } -2 < p < \alpha. \tag{B.3}
\]

We now consider the problem of determining a range of values of the parameter \( p \) for which \( E\{|X||Y|^{p-1}\} < \infty \) and \( E\{|X|^2|Y|^{2p-2}\} < \infty \).

**Proof of Proposition 5.1** First, consider \( E\{|X|^2|Y|^{2p-2}\} \). It follows from (B.1) that \( E\{|X|^2|Y|^{2p-2}\} < \infty \) for \( p < \alpha/2 \), when \( X \) and \( Y \) are jointly SαS random variables. If \( \mu \) is the measure induced on \( \mathbb{R}^2 \) by \( X \) and \( Y \), then \( E\{|X|^2|Y|^{2p-2}\} \) can be written as

\[
E\{|X|^2|Y|^{2p-2}\} = \int_{\mathbb{R}^2} |x|^2|y|^{2p-2}d\mu(x, y) =: I_1. \tag{B.4}
\]
But $I_1 < \infty$ if and only if [94]

$$I'_1 := \int_{\mathbb{R}^2} |y|^{2p-2} d\mu(x, y) < \infty$$  \hspace{1cm} (B.5)

By using (B.2) and (B.3) we get that $I'_1 < \infty$ if $p > 1/2$ $(p > 0)$ when $X$ and $Y$ are real (complex isotropic) S.o.S random variables. Hence, it follows that $E\{||X||^2||Y||^{2p-2}\} < \infty$ if and only if $1/2 < p < \alpha/2$ $(0 < p < \alpha/2)$ when $X$ and $Y$ are real (complex isotropic).

Finally, since $E\{||X||^2||Y||^{2p-2}\} < \infty$ implies $E\{||X||||Y||^{p-1}\} < \infty$, it follows that for the aforementioned values of $p$, $E\{||X||||Y||^{p-1}\}$ is also finite. \hfill \blacksquare