APPENDIX A

Complex Variables

This Appendix presents a brief review of the functional theory of complex variables. In the context of the material considered in this book, a complex variable of interest is the variable z associated with the z-transform. We begin the review by defining analytic functions of a complex variable, and then derive the important theorems that make up the important subject of complex variables[1]. .

A.1 CAUCHY–REIMANN EQUATIONS

Consider a complex variable z defined by

$$z = x + jy$$

where $x = \mathrm{Re}[z]$, and $y = \mathrm{Im}[z]$. We speak of the plane in which the complex variable z is represented as the *z-plane*. Let $f(z)$ denote a *function of the complex variable* z, written as

$$w = f(z) = u + jv$$

The function $w = f(z)$ is *single-valued* if there is only one value of w for each z in a given region of the z-plane. If, on the other hand, more than one value of w corresponds to z, the function $w = f(z)$ is said to be *multiple-valued.*

[1]For a detailed treatment of the functional theory of complex variables, see Guillemin (1949), Levinson and Redheffer (1970), and Wylie and Barrett (1982).

We say that a point $z = x + jy$ in the z-plane approaches a fixed point $z_0 = x_0 + jy_0$ if $x \to x_0$ and $y \to y_0$. Let $f(z)$ denote a single-valued function of z that is defined in some neighborhood of the point $z = z_0$. The *neighborhood* of z_0 refers to the set of all points in a sufficiently small circular region centered at z_0. Let

$$\lim_{z \to z_0} f(z) = w_0$$

In particular, if $f(z_0) = w_0$, then the function $f(z)$ is said to be *continuous* at $z = z_0$.

Let $f(z)$ be written in terms of its real and imaginary parts as

$$f(z) = u(x, y) + jv(x, y)$$

Then, if $f(z)$ is continuous at $z_0 = x_0 + jy_0$, its real and imaginary parts $u(x, y)$ and $v(x, y)$ are continuous functions at (x_0, y_0), and vice versa.

Let $w = f(z)$ be continuous at each point of some region of interest in the z-plane. The complex quantities w and z may then be represented on separate planes of their own, referred to as the w- and z-planes, respectively. In particular, a point (x, y) in the z-plane corresponds to a point (u, v) in the w-plane by virtue of the relationship $w = f(z)$.

Consider an incremental change Δz such that the point $z_0 + \Delta z$ may lie anywhere in the neighborhood of z_0, and throughout which the function $f(z)$ is defined. We may then define the *derivative* of $f(z)$ with respect to z at $z = z_0$ as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{A.1}$$

Clearly, for the derivative $f'(z_0)$ to have a unique value, the limit in Eq. (A.1) must be independent of the way in which Δz approaches zero.

For a function $f(z)$ to have a unique derivative at some point $z = x + jy$, it is necessary that its real and imaginary parts satisfy certain conditions, as shown next. Let

$$w = f(z) = u(x, y) + jv(x, y)$$

With $\Delta w = \Delta u + j\Delta v$ and $\Delta z = \Delta x + j\Delta y$, we may write

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \\ &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u + j\Delta v}{\Delta x + j\Delta y} \end{aligned} \tag{A.2}$$

Suppose that we let $\Delta z \to 0$ by first letting $\Delta y \to 0$ and then $\Delta x \to 0$, in which case Δz is purely real. We then deduce from Eq. (A.2) that

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + j \frac{\Delta v}{\Delta x} \\ &= \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} \end{aligned} \tag{A.3}$$

Suppose next that we let $\Delta z \to 0$ by first letting $\Delta x \to 0$ and then $\Delta y \to 0$, in which case Δz is purely imaginary. This time we deduce from Eq. (A.2) that

$$\begin{aligned} f'(z) &= \lim_{\Delta y \to 0} \frac{\Delta v}{\Delta y} - j \frac{\Delta u}{\Delta y} \\ &= \frac{\partial v}{\partial y} - j \frac{\partial u}{\partial y} \end{aligned} \tag{A.4}$$

If the derivative $f'(z)$ is to exist, it is necessary that the two expressions in Eqs. (A.3) and (A.4) be one and the same. Hence, we require

$$\frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}$$

Accordingly, equating real and imaginary parts, we get the following pair of relations, respectively:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{A.5}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{A.6}$$

Equations (A.5) and (A.6), known as the *Cauchy–Riemann equations*, were derived from a consideration of merely two of the infinitely many ways in which Δz can approach zero. For $\Delta w/\Delta z$ evaluated along these other paths to also approach $f'(z)$, we need only make the additional requirement that the partial derivatives in Eqs. (A.5) and (A.6) are continuous at the point (x, y). In other words, provided that the real part $u(x, y)$ and the imaginary part $v(x, y)$ together with their first partial derivatives are continuous at the point (x, y), the Cauchy–Riemann equations are not only necessary but also sufficient for the existence of a derivative of the complex function $w = u(x, y) + jv(x, y)$ at the point (x, y).

A function $f(z)$ is said to be *analytic*, or *homomorphic*, at some point $z = z_0$ in the z-plane if it has a derivative at $z = z_0$ and at every point in the neighborhood of z_0; the point z_0 is called a *regular point* of the function $f(z)$. If the function $f(z)$ is *not* analytic at a point z_0, but if every neighborhood of z_0 contains points at which $f(z)$ is analytic, the point z_0 is referred to as a *singular point* of $f(z)$.

A.2 CAUCHY'S INTEGRAL FORMULA

Let $f(z)$ be any continuous function of the complex variable z, analytic or otherwise. Let $\mathscr{C}$ be a sectionally smooth path joining the points $A = z_0$ and $B = z_n$ in the z plane. Suppose that the path $\mathscr{C}$ is divided into n segments Δs_k by the points z_k, $k = 1, 2, \ldots, n-1$, as illustrated in Fig. A.1. This figure also shows an arbitrary point ζ_k on segment Δs_k, depicted as an elementary arc of length Δz_k. Consider then the summation $\sum_{k=1}^{n} f(\zeta_k)\, \Delta z_k$.

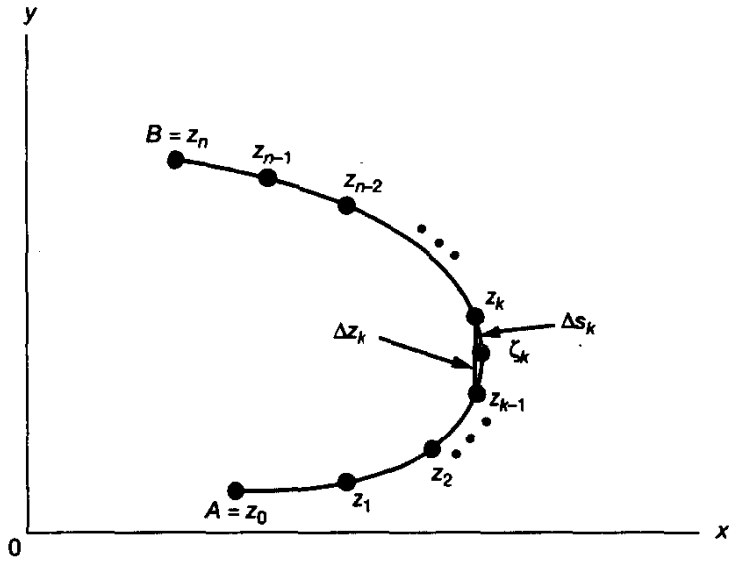

Figure A.1 Sectionally smooth path.

The *line integral* of $f(z)$ along the path $\mathscr{C}$ is defined by the limiting value of this summation as the number n of segments is allowed to increase indefinitely in such a way that Δz_k approaches zero. That is

$$\oint_{\mathscr{C}} f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\zeta_k)\,\Delta z_k \tag{A.7}$$

In the special case when the points A and B coincide and $\mathscr{C}$ is a closed curve, the integral in Eq. (A.7) is referred to as a *contour integral* that is written as $\oint_{\mathscr{C}} f(z)dz$. Note that, according to the notation described herein, the contour $\mathscr{C}$ is transversed in a *counterclockwise direction*.

Let $f(z)$ be an analytic function in a given region R, and let the derivative $f'(z)$ be continuous there. The line integral $\oint_{\mathscr{C}} f(z)dz$ is then independent of the path $\mathscr{C}$ that joins any pair of points in the region R. If the path $\mathscr{C}$ is a closed curve, the value of this integral is zero. We thus have *Cauchy's integral theorem*, stated as follows:

If a function $f(z)$ is analytic throughout a region R, then the contour integral of $f(z)$ along any closed path $\mathscr{C}$ lying inside the region R is zero, as shown by

$$\oint_{\mathscr{C}} f(z)\,dz = 0 \tag{A.8}$$

This theorem is of cardinal importance in the study of analytic functions.

An important consequence of Cauchy's theorem is known as *Cauchy's integral formula*. Let $f(z)$ be analytic within and on the boundary $\mathscr{C}$ of a simple connected region. Let z_0 be any point in the interior of $\mathscr{C}$. Then Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi j} \oint_{\mathscr{C}} \frac{f(z)}{z - z_0}\, dz \tag{A.9}$$

where the contour integration around $\mathscr{C}$ is taken in the counterclockwise direction.

Cauchy's integral formula expresses the value of the analytic function $f(z)$ at an interior point z_0 of $\mathscr{C}$ in terms of its values on the boundary of $\mathscr{C}$. Using this formula, it is a straightforward matter to express the derivative of $f(z)$ of all orders as follows:

$$f^{(n)}(z_0) = \frac{n!}{2\pi j} \oint_{\mathscr{C}} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \tag{A.10}$$

where $f^{(n)}(z_0)$ is the nth derivative of $f(z)$ evaluated at $z = z_0$. Equation (A.10) is obtained by repeated differentiation of Eq. (A.9) with respect to z_0.

Cauchy's Inequality

Let the contour $\mathscr{C}$ consist of a circle of radius r and center z_0. Then, using Eq. (A.10) to evaluate the magnitude of $f^{(n)}(z_0)$, we may write

$$\begin{aligned} |f^{(n)}(z_0)| &= \frac{n!}{2\pi} \left| \oint_{\mathscr{C}} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| \\ &\leq \frac{n!}{2\pi} \oint_{\mathscr{C}} \frac{|f(z)|}{|z - z_0|^{n+1}}\, |dz| \\ &\leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} \oint_{\mathscr{C}} |dz| \\ &= \frac{n!}{2\pi} \frac{M}{r^{n+1}}\, 2\pi r \\ &= n! \frac{M}{r^n} \end{aligned} \tag{A.11}$$

where M is the maximum value of $f(z)$ on $\mathscr{C}$. The inequality of (A.11) is known as *Cauchy's inequality*.

A.3 LAURENT'S SERIES

Let the function $f(z)$ be analytic in the annular region of Fig. A.2, including the boundary of the region. The annular region consists of two concentric circles $\mathscr{C}_1$ and $\mathscr{C}_2$, whose common center is z_0. Let the point $z = z_0 + h$ be located inside the annular region as depicted in Fig. A.2. According to *Lauren's series*, we have

$$f(z_0 + h) = \sum_{k=-\infty}^{\infty} a_k h^k \tag{A.12}$$

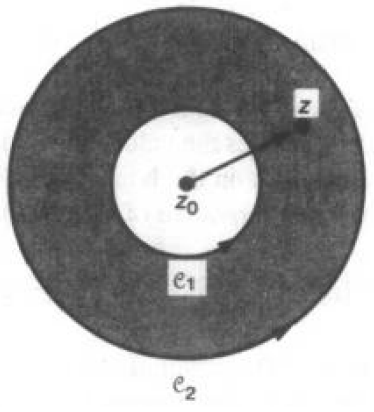

Figure A.2 Annular region.

where the coefficients a_k for varying k are given by

$$a_k = \begin{cases} \dfrac{1}{2\pi j} \oint_{c_2} \dfrac{f(z)\,dz}{(z - z_0)^{k+1}}, & k = 0, 1, 2, \ldots \\ \dfrac{1}{2\pi j} \oint_{c_1} \dfrac{f(z)\,dz}{(z - z_0)^{k+1}}, & k = -1, -2, \ldots \end{cases} \tag{A.13}$$

Note that we may also express the Laurent expansion of $f(z)$ around the point z as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{A.14}$$

When all the coefficients of negative index have the value zero, then Eq. (A.14) reduces to *Taylor's series*:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{A.15}$$

In light of Eq. (A.10) and the first line of Eq. (A.13), we may define the coefficient a_k as

$$a_k = \frac{f^{(k)}(z_0)}{k!}, \qquad k = 0, 1, 2, \ldots \tag{A.16}$$

Taylor's series provides the basis of Liouville's theorem, considered next.

Liouville's Theorem

Let a function $f(z)$ of the complex variable z be bounded and analytic for all values of z. Then, according to *Liouville's theorem*, $f(z)$ is simply a constant.

To prove this theorem, we first note that since $f(z)$ is analytic everywhere inside the z-plane, we may use Taylor's series to expand $f(z)$ about the origin:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \tag{A.17}$$

The power series of Eq. (A.17) is convergent, and therefore provides a valid representation of $f(z)$. Let contour $\mathscr{C}$ consist of a circle of radius r and origin as center. Then, invoking Cauchy's inequality of (A.11), we may write

$$|f^{(k)}(0)| \leq \frac{k!\, M_c}{r^k} \tag{A.18}$$

where M_c is the maximum value of $f(z)$ on $\mathscr{C}$. Correspondingly, the value of the kth coefficient in the power series expansion of Eq. (A.17) is bounded as

$$|a_k| = \frac{|f^{(k)}(0)|}{k!} \leq \frac{M_c}{r^k} \leq \frac{M}{r^k} \tag{A.19}$$

where M is the bound on $|f(z)|$ for all values of z. Since, by hypothesis, M does exist, it follows from (A.19) that for an arbitrarily large r:

$$a_k = \begin{cases} f(0), & k=0 \\ 0, & k = 1, 2, \ldots \end{cases} \tag{A.20}$$

Accordingly, Eq. (A.17) reduces to

$$f(z) = f(0) = \text{constant}$$

which proves Liouville's theorem.

A function $f(z)$ that is analytic for all values of z is said to be an *entire function*. Thus, Liouville's theorem may be restated as follows: *An entire function that is bounded for all values of* z *is a constant* (Wylie and Barrett, 1982).

A.4 SINGULARITIES AND RESIDUES

Let $z = z_0$ be a singular point of an analytic function $f(z)$. If the neighborhood of $z = z_0$ contains *no* other singular points of $f(z)$, the singularity at $z = z_0$ is said to be *isolated*. In the neighborhood of such a singularity, the function $f(z)$ may be represented by the Laurent series

$$\begin{aligned} f(z) &= \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \\ &= \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=-\infty}^{-1} a_k (z - z_0)^k \\ &= \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k} \end{aligned} \tag{A.21}$$

The particular coefficient a_{-1} in the Laurent expansion of $f(z)$ in the neighborhood of the isolated singularity at the point $z = z_0$ is called the *residue* of $f(z)$ at $z = a$. The residue plays an important role in the evaluation of integrals of analytic functions. In particular,

putting $k = -1$ in Eq. (A.13) we get the following connection between the residue a_{-1} and the integral of the function $f(z)$:

$$a_{-1} = \frac{1}{2\pi j} \oint_{\mathscr{C}} f(z)\, dz \tag{A.22}$$

There are two nontrivial cases to be considered:

1. The Laurent expansion of $f(z)$ contains *infinitely* many terms with negative powers of $z - z_0$, as in Eq. (A.21). The point $z - z_0$ is then called an *essential singular point* of $f(z)$.
2. The Laurent expansion of $f(z)$ contains at most *a finite* number of terms, m, with negative powers of $z - z_0$, as shown by

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots + \frac{a_{-m}}{(z - z_0)^m} \tag{A.23}$$

According to this latter representation, $f(z)$ is said to have a *pole of order m* at $z = z_0$. The *finite sum* of all the terms containing *negative powers* on the right-hand side of Eq. (A.22) is called the *principal part* of $f(z)$ at $z = z_0$.

Note that when the singularity at $z = z_0$ is a pole of order m, the residue of the pole may be determined by using the formula

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]_{z=z_0} \tag{A.24}$$

In effect, by using this formula we avoid the need for the deduction of the Laurent series. For the special case when the order $m = 1$, the pole is said to be *simple*. Correspondingly, the formula of Eq. (A.24) for the residue a_{-1} of a simple pole reduces to

$$a_{-1} = \lim_{z \to z_0} (z - z_0) f(z) \tag{A.25}$$

A.5 CAUCHY'S RESIDUE THEOREM

Consider a closed contour $\mathscr{C}$ in the z-plane containing within it a number of isolated singularities of some function $f(z)$. Let $z_1, z_2, \ldots, z_n$ define the locations of these isolated singularities. Around each singular point of the function $f(z)$, we draw a circle small enough to ensure that it does not enclose the other singular points of $f(z)$, as depicted in Fig. A.3. The original contour $\mathscr{C}$ together with these small circles constitute the boundary of a *multiply connected region* in which $f(z)$ is analytic everywhere and to which Cauchy's integral theorem may therefore be applied. Specifically, for the situation described in Fig. A.3 we may write

$$\frac{1}{2\pi j} \oint_{\mathscr{C}} f(z)\, dz + \frac{1}{2\pi j} \oint_{\mathscr{C}_1} f(z)\, dz + \cdots + \frac{1}{2\pi j} \oint_{\mathscr{C}_n} f(z)\, dz = 0 \tag{A.26}$$

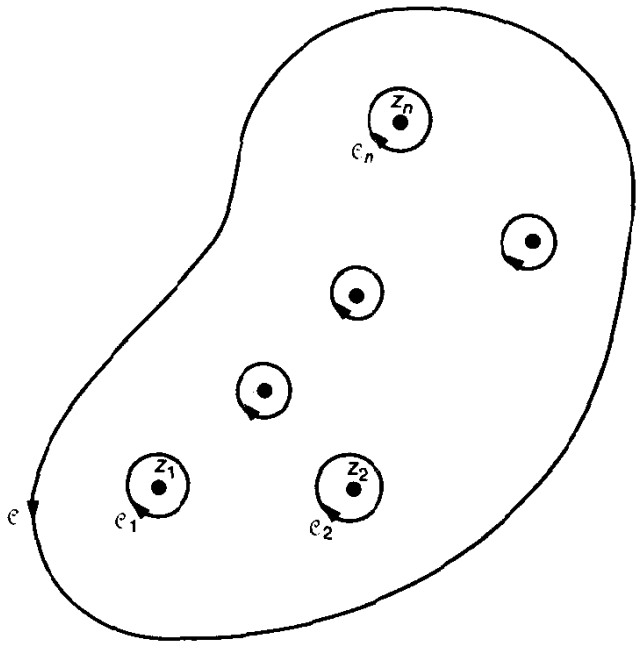

Figure A.3 Multiply connected region.

Note that in Fig. A.3 the contour $\mathscr{C}$ is traversed in the *positive* sense (i.e., counterclockwise direction), whereas the small circles are traversed in the *negative* sense (i.e., clockwise direction).

Suppose now we *reverse* the direction along which the integral around each small circle in Fig. A.3 is taken. This operation has the equivalent effect of applying a minus sign to each of the integrals in Eq. (A.26) that involve the small circles $\mathscr{C}_1, \ldots, \mathscr{C}_n$. Accordingly, for the case when *all* the integrals around the original contour $\mathscr{C}$ and the small circles $\mathscr{C}_1, \ldots, \mathscr{C}_n$ are taken in the counterclockwise direction, we may rewrite Eq. (A.26) as

$$\frac{1}{2\pi j}\oint_{\mathscr{C}} f(z)\,dz = \frac{1}{2\pi j}\oint_{\mathscr{C}_1} f(z)\,dz + \cdots + \frac{1}{2\pi j}\oint_{\mathscr{C}_n} f(z)\,dz \tag{A.27}$$

By definition, the integrals on the right-hand side of Eq. (A.27) are the residues of the function $f(z)$ evaluated at the various isolated singularities of $f(z)$ within the contour $\mathscr{C}$. We may thus express the integral of $f(z)$ around the contour $\mathscr{C}$ simply as

$$\oint_{\mathscr{C}} f(z)\,dz = 2\pi j \sum_{k=1}^{n} \text{Res}(f(z), z_k) \tag{A.28}$$

where $\text{Res}(f(z), z_k)$ stands for the residue of the function $f(z)$ evaluated at the isolated singular point $z = z_k$. Equation (A.28) is called *Cauchy's residue theorem*. This theorem is extremely important in the theory of functions in general and in evaluating definite integrals in particular.

A.6 PRINCIPLE OF THE ARGUMENT

Consider a complex function $f(z)$, characterized as follows:

1. The function $f(z)$ is analytic in the interior of a closed contour $\mathscr{C}$ in the z-plane, except at a finite number of poles.
2. The function $f(z)$ has neither poles nor zeros on the contour $\mathscr{C}$. By a "zero" we mean a point in the z-plane at which $f(z) = 0$. In contrast, at a "pole" as defined previously, we have $f(z) = \infty$. Let N be the *number of zeros* and P be the *number of poles* of the function $f(z)$ in the interior of contour $\mathscr{C}$, where each zero or pole is counted according to its *multiplicity*.

We may then state the following theorem (Levinson and Redheffer, 1970; Wylie and Barrett, 1982):

$$\frac{1}{2\pi j}\oint_{\mathscr{C}} \frac{f'(z)}{f(z)}\,dz = N - P \tag{A.29}$$

where $f'(z)$ is the derivative of $f(z)$. We note that

$$\frac{d}{dz}\ln f(z) = \frac{f'(z)}{f(z)}\,dz$$

where ln denotes the natural logarithm. Hence,

$$\begin{aligned}\oint_{\mathscr{C}} \frac{f'(z)}{f(z)}\,dz &= \ln f(z)|_{\mathscr{C}} \\ &= \ln |f(z)|_{\mathscr{C}} + j \arg f(z)|_{\mathscr{C}}\end{aligned} \tag{A.30}$$

where $|f(z)|$ denotes the magnitude of $f(z)$, and $\arg f(z)$ denotes its argument. The first term on the right-hand side of Eq. (A.30) is zero, since the logarithmic function $\ln f(z)$ is single-valued and the contour $\mathscr{C}$ is closed. Hence,

$$\oint_{\mathscr{C}} \frac{f'(z)}{f(z)}\,dz = j \arg f(z)|_{\mathscr{C}} \tag{A.31}$$

Thus, substituting Eq. (A.31) in (A.29), we get

$$N - P = \frac{1}{2\pi}\arg f(z)|_{\mathscr{C}} \tag{A.32}$$

This result, which is a reformulation of the theorem described in Eq. (A.29), is called the *principle of the argument.*

For a geometrical interpretation of this principle, let $\mathscr{C}$ be a closed contour in the z-plane as in Fig. A.4(a). As z traverses the contour $\mathscr{C}$ in a counterclockwise direction, we find that $w = f(z)$ traces out a contour $\mathscr{C}'$ of its own in the w-plane; for the purpose of illustration, $\mathscr{C}'$ is shown in Fig. A.4(b). Suppose now a line is drawn in the w-plane from the

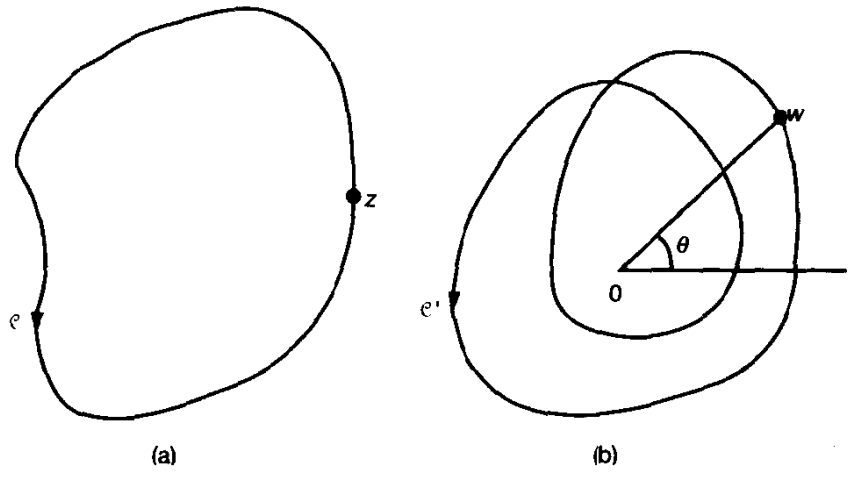

Figure A.4 (a) Contour $\mathscr{C}$ in the z-plane; (b) Contour $\mathscr{C}'$ in the w-plane, where $w = f(z)$.

origin to the point $w = f(z)$, as depicted in Fig. A.4(b). Then the angle θ which this line makes with a fixed direction (shown as the horizontal direction in Fig. A.4(b)) is $\arg f(z)$. The principle of the argument thus provides a description of the number of times the point $w = f(z)$ winds around the origin of the w-plane (i.e., the point $w = 0$) as the complex variable z traverses the contour $\mathscr{C}$ in a counterclockwise direction.

Rouché's Theorem

Let the function $f(z)$ be analytic on a closed contour $\mathscr{C}$ and in the interior of $\mathscr{C}$. Let $g(z)$ be a second function which, in addition to satisfying the same condition for analyticity as $f(z)$, also fulfills the following condition on the contour $\mathscr{C}$:

$$|f(z)| > |g(z)|$$

In other words, on the contour $\mathscr{C}$ we have

$$\left|\frac{g(z)}{f(z)}\right| < 1 \tag{A.33}$$

Define the function

$$F(z) = 1 + \frac{g(z)}{f(z)} \tag{A.34}$$

which has no poles or zeros on $\mathscr{C}$. By the principle of the argument applied to $F(z)$, we have

$$N - P = \frac{1}{2\pi} \arg F(z)|_{\mathscr{C}} \tag{A.35}$$

However, the implication of the condition (A.33) is that when z is on the contour $\mathscr{C}$, then

$$|F(z) - 1| < 1 \tag{A.36}$$

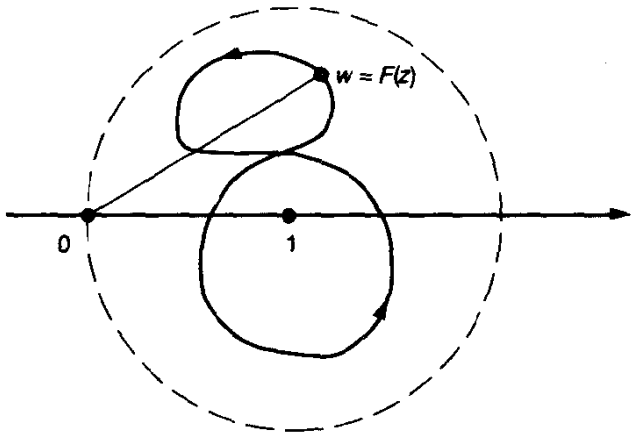

Figure A.5 Point $w = F(z)$ on a closed contour inside the unit circle.

In other words, the point $w = F(z)$ lies inside a circle with center at $w = 1$ and unit radius, as illustrated in Fig. A.5. It follows therefore that

$$|\arg F(z)| < \frac{\pi}{2} \qquad \text{for } z \text{ on } \mathscr{C} \tag{A.37}$$

Equivalently, we may write

$$\arg F(z)|_{\mathscr{C}} = 0 \tag{A.38}$$

Hence, from Eq. (A.38) we deduce that $N = P$, where both N and P refer to $f(z)$. From the definition of the function $F(z)$ given in Eq. (A.34) we note that the poles of $F(z)$ are the zeros of $f(z)$, and the zeros of $F(z)$ are the zeros of the sum $f(z) + g(z)$. Accordingly, the fact that $N = P$ means that $f(z) + g(z)$ and $f(z)$ have the same numbers of zeros. The result that we have just established is known as *Rouché's theorem*, which may be formally stated as follows:

Let $f(z)$ and $g(z)$ be analytic on a closed contour $\mathscr{C}$ and in the interior of $\mathscr{C}$. Let $|f(z)| > |g(z)|$ on $\mathscr{C}$. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside contour $\mathscr{C}$.

Example

Consider the contour depicted in Fig. A.6(a) that constitutes the boundary of a multiply connected region in the z-plane. Let $F(z)$ and $G(z)$ be two polynomials in z^{-1}, both of which are analytic on this contour and in the interior of it. Moreover, Let $|F(z)| > |G(z)|$. Then, according to Rouché's theorem, both $F(z)$ and $F(z) + G(z)$ have the same number of zeros inside the contour described in Fig. A.6(a).

Suppose now that we let the radius R of the outside circle $\mathscr{C}$ in Fig. A.6(a) approach infinity. Also, let the separation l between the two straight-line portions of the contour approach zero. Then, in the limit, the region enclosed by the contour described in Fig. A.6(a)

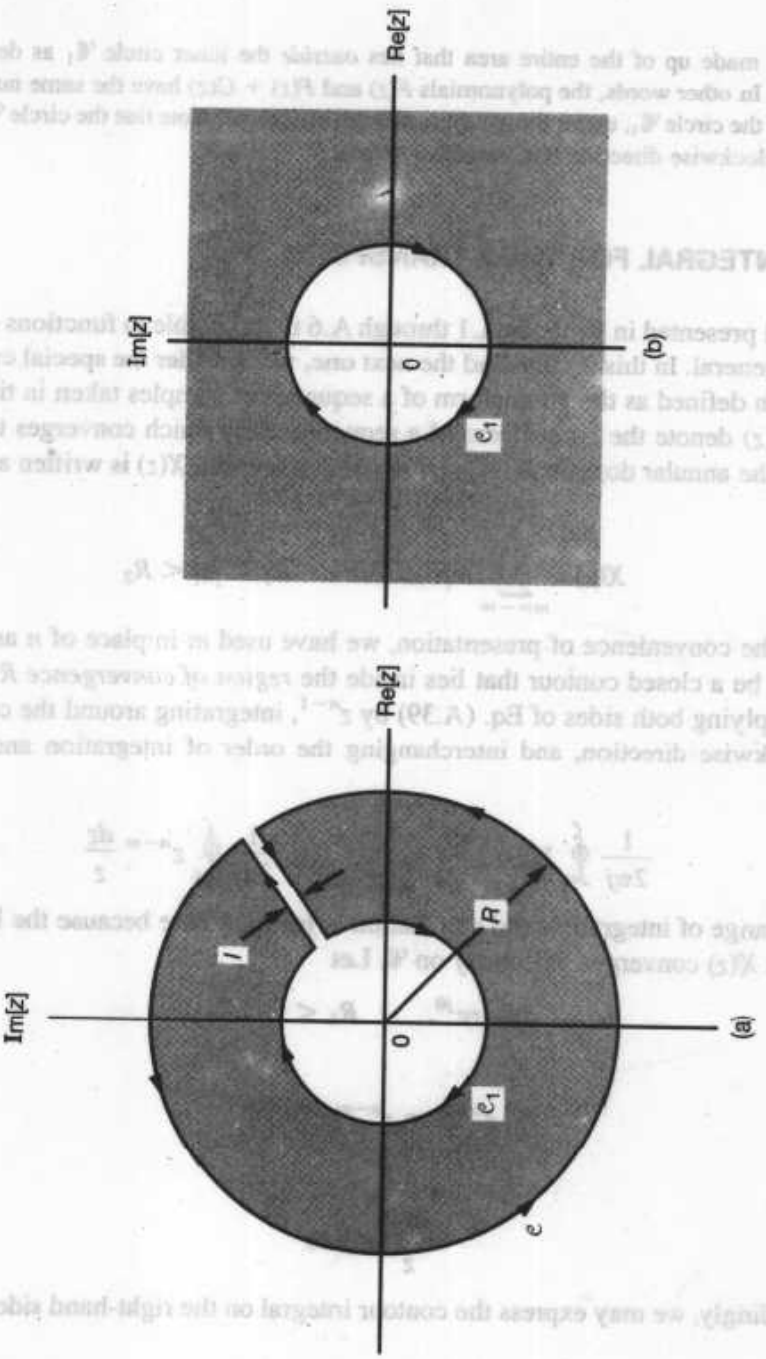

Figure A.6 (a) Multiply connected region; (b) limiting form of the region in (a) as $R \to \infty$ and $l \to 0$.

will be made up of the entire area that lies *outside* the inner circle $\mathscr{C}_i$ as depicted in Fig. A.6(b). In other words, the polynomials $F(z)$ and $F(z) + G(z)$ have the same number of zeros outside the circle $\mathscr{C}_1$, under the conditions described above. Note that the circle $\mathscr{C}_1$ is traversed in the clockwise direction (i.e., negative sense).

A.7 INVERSION INTEGRAL FOR THE *z*-TRANSFORM

The material presented in Sections A.1 through A.6 is applicable to functions of a complex variable in general. In this section and the next one, we consider the special case of a complex function defined as the z-transform of a sequence of samples taken in time.

Let $X(z)$ denote the z-transform of a sequence $x(n)$, which converges to an analytic function in the annular domain $R_1 < |z| < R_2$. By definition, $X(z)$ is written as the Laurent series

$$X(z) = \sum_{m=-\infty}^{\infty} x(m)z^{-m}, \qquad R_1 < |z| < R_2 \tag{A.39}$$

where, for the convenience of presentation, we have used m in place of n as the index of time. Let $\mathscr{C}$ be a closed contour that lies inside the *region of convergence* $R_1 < |z| < R_2$. Then, multiplying both sides of Eq. (A.39) by z^{n-1}, integrating around the contour $\mathscr{C}$ in a counterclockwise direction, and interchanging the order of integration and summation, we get

$$\frac{1}{2\pi j}\oint_{\mathscr{C}} X(z)z^n \frac{dz}{z} = \sum_{m=-\infty}^{\infty} x(n)\frac{1}{2\pi j}\oint_{\mathscr{C}} z^{n-m}\frac{dz}{z} \tag{A.40}$$

The interchange of integration and summation is justified here because the Laurent series that defines $X(z)$ converges uniformly on $\mathscr{C}$. Let

$$z = re^{j\theta}, \qquad R_1 < r < R_2 \tag{A.41}$$

Hence,

$$z^{n-m} = r^{n-m}e^{j(n-m)\theta}$$

and

$$\frac{dz}{z} = j\,d\theta$$

Correspondingly, we may express the contour integral on the right-hand side of Eq. (A.40) as

$$\frac{1}{2\pi j}\oint_{\mathscr{C}} z^{n-m}\frac{dz}{z} = \frac{1}{2\pi}\int_0^{2\pi} r^{n-m}e^{j(n-m)\theta}\,d\theta \tag{A.42}$$

$$= \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

Inserting Eq. (A.42) in (A.40), we get

$$x(n) = \frac{1}{2\pi j} \oint_{\mathscr{C}} X(z) z^n \frac{dz}{z} \tag{A.43}$$

Equation (A.43) is called the *inversion integral formula* for the z-transform.

A.8 PARSEVAL'S THEOREM

Let $X(z)$ denote the z-transform of the sequence $x(n)$ with the region of convergence $R_{1x} < |z| < R_{2x}$. Let $Y(z)$ denote the z-transform of a second sequence $y(n)$ with the region of convergence $R_{1y} < |z| < R_{2y}$. Then *Parseval's theorem* states that

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi j} \oint_{\mathscr{C}} X(z) Y^*\left(\frac{1}{z^*}\right) \frac{dz}{z} \tag{A.44}$$

where $\mathscr{C}$ is a closed contour defined in the overlap of the regions of convergence of $X(z)$ and $Y(z)$, both of which are analytic. The function $Y^*(1/z^*)$ is obtained from the z-transform $Y(z)$ by using $1/z^*$ in place of z, and then complex-conjugating the resulting function. Note that the function $Y^*(1/z^*)$ obtained in this way is analytic too.

To prove Parseval's theorem, we use the inversion integral of Eq. (A.43) to write

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi j} \sum_{n=-\infty}^{\infty} y^*(n) \oint_{\mathscr{C}} X(z) z^n \frac{dz}{z} \tag{A.45}$$

$$= \frac{1}{2\pi j} \oint_{\mathscr{C}} X(z) \sum_{n=-\infty}^{\infty} y^*(n) z^n \frac{dz}{z}$$

From the definition of the z-transform of $y(n)$, namely,

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

we note that

$$Y^*\left(\frac{1}{z^*}\right) = \sum_{n=-\infty}^{\infty} y^*(n) z^n \tag{A.46}$$

Hence, using Eq. (A.46) in (A.45), we get the result given in Eq. (A.44), and the proof of Parseval's theorem is completed.

APPENDIX B

Differentiation with Respect to a Vector

An issue commonly encountered in the study of optimization theory is that of differentiating a cost function with respect to a parameter vector of interest. In the text we used an ordinary gradient operation. The purpose of Appendix B is to address the more difficult issue of differentiating a cost function with respect to a complex-valued parameter vector. We begin by introducing some basic definitions.

B.1 BASIC DEFINITIONS

Consider a complex function $f(\mathbf{w})$ that is dependent on a parameter vector $\mathbf{w}$. When $\mathbf{w}$ is complex valued, there are two different mathematical concepts that require individual attention: (1) the vector nature of $\mathbf{w}$, and (2) the fact that each element of $\mathbf{w}$ is a complex number.

Dealing with the issue of complex numbers first, let x_k and y_k denote the real and imaginary parts of the kth element w_k of the vector $\mathbf{w}$; that is,

$$w_k = x_k + jy_k \tag{B.1}$$

We thus have a function of the real quantities x_k and y_k. Hence, we may use Eq. (B.1) to express the real part x_k in terms of the pair of *complex conjugate coordinates* w_k and w_k^* as

$$x_k = \tfrac{1}{2}(w_k + w_k^*) \tag{B.2}$$

and express the imaginary part y_k as

$$y_k = \frac{1}{2j}(w_k - w_k^*) \tag{B.3}$$

where the asterisk denotes complex conjugation. The real quantities x_k and y_k are functions of both w_k and w_k^*. It is only when we deal with analytic functions f that we are permitted to abandon the complex-conjugated term w_k^* by virtue of the Cauchy–Riemann equations. However, most functions encountered in physical sciences and engineering are *not* analytic.

The notion of a derivative must tie in with the concept of a differential. In particular, the chain rule of changes of variables must be obeyed. With these important points in mind, we may define certain complex derivatives in terms of real derivatives, as shown by (Schwartz, 1967)

$$\frac{\partial}{\partial w_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - j\frac{\partial}{\partial y_k}\right) \tag{B.4}$$

and

$$\frac{\partial}{\partial w_k^*} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + j\frac{\partial}{\partial y_k}\right) \tag{B.5}$$

The derivatives defined here satisfy the following two basic requirements:

$$\frac{\partial w_k}{\partial w_k} = 1$$

$$\frac{\partial w_k}{\partial w_k^*} = \frac{\partial w_k^*}{\partial w_k} = 0$$

(An analytic function f satisfies $\partial f/\partial z^* = 0$ everywhere.)

The next issue to be considered is that of differentiation with respect to a vector. Let $w_0, w_1, \ldots, w_{M-1}$ denote the elements of an M-by-1 complex vector $\mathbf{w}$. We may extend the use of Eqs. (B.4) and (B.5) to deal with this new situation by writing (Miller, 1974)

$$\frac{\partial}{\partial \mathbf{w}} = \frac{1}{2}\begin{bmatrix} \dfrac{\partial}{\partial x_0} - j\dfrac{\partial}{\partial y_0} \\ \dfrac{\partial}{\partial x_1} - j\dfrac{\partial}{\partial y_1} \\ \vdots \\ \dfrac{\partial}{\partial x_{M-1}} - j\dfrac{\partial}{\partial y_{M-1}} \end{bmatrix} \tag{B.6}$$

and

$$\frac{\partial}{\partial \mathbf{w}^*} = \frac{1}{2}\begin{bmatrix} \dfrac{\partial}{\partial x_0} + j\dfrac{\partial}{\partial y_0} \\ \dfrac{\partial}{\partial x_1} + j\dfrac{\partial}{\partial y_1} \\ \vdots \\ \dfrac{\partial}{\partial x_{M-1}} + j\dfrac{\partial}{\partial y_{M-1}} \end{bmatrix} \tag{B.7}$$

where we have $w_k = x_k + jy_k$ for $k = 0, 1, \ldots, M-1$. We refer to $\partial/\partial \mathbf{w}$ as a *derivative* with respect to the vector $\mathbf{w}$, and to $\partial/\partial \mathbf{w}^*$ as a *conjugate derivative* also with respect to the vector $\mathbf{w}$. These two derivatives must be considered together. They obey the following relations:

$$\frac{\partial \mathbf{w}}{\partial \mathbf{w}} = \mathbf{I}$$

and

$$\frac{\partial \mathbf{w}}{\partial \mathbf{w}^*} = \frac{\partial \mathbf{w}^*}{\partial \mathbf{w}} = \mathbf{O}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{O}$ is the null matrix.

For subsequent use, we will adopt the definition of (B.7) as the derivative with respect to a complex-valued vector.

B.2 EXAMPLES

In this section, we illustrate some applications of the derivative defined in Eq. (B.7). The examples are taken from Chapter 5 dealing with optimum linear filtering, and Chapter 11 dealing with the method of least squares.

Example 1

Let $\mathbf{p}$ and $\mathbf{w}$ denote two complex-valued M-by-1 vectors. There are two inner products, $\mathbf{p}^H\mathbf{w}$ and $\mathbf{w}^H\mathbf{p}$, to be considered

Let $c_1 = \mathbf{p}^H\mathbf{w}$. The conjugate derivative of c_1 with respect to the vector $\mathbf{w}$ is

$$\frac{\partial c_1}{\partial \mathbf{w}^*} = \frac{\partial}{\partial \mathbf{w}^*}(\mathbf{p}^H\mathbf{w}) = \mathbf{0} \tag{B.8}$$

where $\mathbf{0}$ is the null vector. Here we note that $\mathbf{p}^H\mathbf{w}$ is an analytic function; see Problem 1 of Chapter 5. We therefore find that the derivative of $\mathbf{p}^H\mathbf{w}$ with respect to $\mathbf{w}$ is zero, in agreement with Eq. (B.8).

Consider next $c_2 = \mathbf{w}^H\mathbf{p}$. The conjugate derivative of c_2 with respect to $\mathbf{w}$ is

$$\frac{\partial c_2}{\partial \mathbf{w}^*} = \frac{\partial}{\partial \mathbf{w}^*}(\mathbf{w}^H\mathbf{p}) = \frac{\partial}{\partial \mathbf{w}^*}(\mathbf{p}^T\mathbf{w}^*) = \mathbf{p} \tag{B.9}$$

Here we note that $\mathbf{w}^H\mathbf{p}$ is not an analytic function; see Problem 1 of Chapter 5. Hence, the derivative of $\mathbf{w}^H\mathbf{p}$ with respect to $\mathbf{w}^*$ is nonzero, as in Eq. (B.9).

Example 2

Consider next the quadratic form

$$c = \mathbf{w}^H\mathbf{R}\mathbf{w}$$

where $\mathbf{R}$ is a Hermitian matrix. The conjugate derivative of c (which is real) with respect to $\mathbf{w}$ is

$$\begin{aligned}\frac{\partial c}{\partial \mathbf{w}^*} &= \frac{\partial}{\partial \mathbf{w}^*}(\mathbf{w}^H\mathbf{R}\mathbf{w}) \\ &= \mathbf{R}\mathbf{w}\end{aligned} \tag{B.10}$$

Example 3

Consider the real-valued cost function (see Chapter 5)

$$J(\mathbf{w}) = \sigma_d^2 - \mathbf{w}^H\mathbf{p} - \mathbf{p}^H\mathbf{w} + \mathbf{w}^H\mathbf{R}\mathbf{w}$$

Using the results of Examples 1 and 2, we find that the conjugate derivative of J with respect to the tap-weight vector $\mathbf{w}$ is

$$\frac{\partial J}{\partial \mathbf{w}^*} = -\mathbf{p} + \mathbf{R}\mathbf{w} \tag{B.11}$$

Let $\mathbf{w}_o$ be the optimum value of the tap-weight vector $\mathbf{w}$ for which the cost function J is minimum or, equivalently, the derivative $(\partial J/\partial \mathbf{w}^*) = \mathbf{0}$. Hence, from Eq. (B.11) we deduce that

$$\mathbf{R}\mathbf{w}_o = \mathbf{p} \tag{B.12}$$

This is the matrix form of the Wiener–Hopf equations for a transversal filter operating in a stationary environment.

Example 4

Consider the real log-likelihood function (see Chapter 11)

$$l(\tilde{\mathbf{w}}) = F - \frac{1}{\sigma^2}\boldsymbol{\epsilon}^H\boldsymbol{\epsilon} \tag{B.13}$$

where F is a constant and

$$\boldsymbol{\epsilon} = \mathbf{b} - \mathbf{A}\tilde{\mathbf{w}} \tag{B.14}$$

Substituting Eq. (B.14) in (B.13), we get

$$l(\tilde{\mathbf{w}}) = F - \frac{1}{\sigma^2}\mathbf{b}^H\mathbf{b} + \frac{1}{\sigma^2}\mathbf{b}^H\mathbf{A}\tilde{\mathbf{w}} + \frac{1}{\sigma^2}\tilde{\mathbf{w}}^H\mathbf{A}^H\mathbf{b} - \frac{1}{\sigma^2}\tilde{\mathbf{w}}^H\mathbf{A}^H\mathbf{A}\tilde{\mathbf{w}} \tag{B.15}$$

Evaluating the conjugate derivative of l with respect to $\tilde{\mathbf{w}}$, and adapting the results of Examples 1 and 2 to fit our present situation, we get

$$\frac{\partial l}{\partial \tilde{\mathbf{w}}^*} = \frac{1}{\sigma^2}\mathbf{A}^H\mathbf{b} - \frac{1}{\sigma^2}\mathbf{A}^H\mathbf{A}\tilde{\mathbf{w}}$$

Setting $(\partial l/\partial \tilde{\mathbf{w}}^*) = \mathbf{0}$, and then simplifying, we thus get

$$\mathbf{A}^H\mathbf{b} - \mathbf{A}^H\mathbf{A}\mathbf{w}_o = \mathbf{0}$$

where $\mathbf{w}_o$ is the special value of $\tilde{\mathbf{w}}$ for which the log-likelihood function is maximum. Hence,

$$\mathbf{A}^H\mathbf{A}\mathbf{w}_o = \mathbf{A}^H\mathbf{b} \tag{B.16}$$

This is the matrix form of the normal equations for the method of least squares.

B.3 RELATION BETWEEN THE DERIVATIVE WITH RESPECT TO A VECTOR AND THE GRADIENT VECTOR

Consider the real cost function $J(\mathbf{w})$ that defines the error-performance surface of a linear transversal filter whose tap-weight vector is $\mathbf{w}$. In Chapter 5, we defined the *gradient vector* of the error-performance surface as

$$\nabla J = \begin{bmatrix} \dfrac{\partial J}{\partial x_0} + j\dfrac{\partial J}{\partial y_0} \\ \dfrac{\partial J}{\partial x_1} + j\dfrac{\partial J}{\partial y_1} \\ \vdots \\ \dfrac{\partial J}{\partial x_{M-1}} + j\dfrac{\partial J}{\partial y_{M-1}} \end{bmatrix} \tag{B.17}$$

where $x_k + jy_k$ is the kth element of the tap-weight vector $\mathbf{w}$, and $k = 0, 1, \ldots, M-1$. The gradient vector is *normal* to the error-performance surface. Comparing Eqs. (B.7) and (B.17), we see that the conjugate derivative $\partial J/\partial \mathbf{w}^*$ and the gradient vector ∇J are related by

$$\nabla J = 2\frac{\partial J}{\partial \mathbf{w}^*} \tag{B.18}$$

Thus, except for a scaling factor, the definition of the gradient vector introduced in Chapter 5 is the same as the conjugate derivative defined in Eq. (B.7).

APPENDIX

C

Method of Lagrange Multipliers

Optimization consists of determining the values of some specified variables that minimize or maximize an *index of performance* or *cost function*, which combines important properties of a system into a single real-valued number. The optimization may be *constrained* or *unconstrained*, depending on whether the variables are also required to satisfy side equations or not. Needless to say, the additional requirement to satisfy one or more side equations complicates the issue of constrained optimization. In this appendix, we derive the classical *method of Lagrange multipliers* for solving the *complex* version of a constrained optimization problem. The notation used in the derivation is influenced by the nature of applications that are of interest to us. We consider first the case when the problem involves a single side equation, followed by the more general case of multiple side equations.

C.1 OPTIMIZATION INVOLVING A SINGLE EQUALITY CONSTRAINT

Consider the minimization of a real-valued function $f(\mathbf{w})$ that is a quadratic function of a vector $\mathbf{w}$, subject to the *constraint*

$$\mathbf{w}^H\mathbf{s} = g \tag{C.1}$$

where $\mathbf{s}$ is a prescribed vector and g is a complex constant. We may redefine the constraint by introducing a new function $c(\mathbf{w})$ that is linear in $\mathbf{w}$, as shown by

$$\begin{aligned} c(\mathbf{w}) &= \mathbf{w}^H\mathbf{s} - g \\ &= 0 + j0 \end{aligned} \tag{C.2}$$

In general, the vectors $\mathbf{w}$ and $\mathbf{s}$ and the function $c(\mathbf{w})$ are all *complex*. For example, in a beamforming application the vector $\mathbf{w}$ represents a set of complex weights applied to the individual sensor outputs, and $\mathbf{s}$ represents a steering vector whose elements are defined by a prescribed "look" direction; the function $f(\mathbf{w})$ to be minimized represents the mean-square value of the overall beamformer output. In a harmonic retrieval application, $\mathbf{w}$ represents the tap-weight vector of a transversal filter, and $\mathbf{s}$ represents a sinusoidal vector whose elements are determined by the angular frequency of a complex sinusoid contained in the filter input; the function $f(\mathbf{w})$ represents the mean-square value of the filter output. In any event, assuming that the issue is one of minimization, we may state the constrained optimization problem as follows:

$$\begin{aligned} &\text{Minimize a real-valued function } f(\mathbf{w}), \\ &\text{subject to the constraint } c(\mathbf{w}) = 0 + j0 \end{aligned} \tag{C.3}$$

The *method of Lagrange multipliers* converts the problem of constrained minimization described above into one of unconstrained minimization by the introduction of *Lagrange multipliers*. First we use the real function $f(\mathbf{w})$ and the complex constraint function $c(\mathbf{w})$ to define a new real-valued function

$$h(\mathbf{w}) = f(\mathbf{w}) + \lambda_1 \operatorname{Re}[c(\mathbf{w})] + \lambda_2 \operatorname{Im}[c(\mathbf{w})] \tag{C.4}$$

where λ_1 and λ_2 are *real Lagrange* multipliers, and

$$c(\mathbf{w}) = \operatorname{Re}[c(\mathbf{w})] + j \operatorname{Im}[c(\mathbf{w})] \tag{C.5}$$

Define a *complex Lagrange multiplier*:

$$\lambda = \lambda_1 + j\lambda_2 \tag{C.6}$$

We may then rewrite Eq. (C.4) in the form

$$h(\mathbf{w}) = f(\mathbf{w}) + \operatorname{Re}[\lambda^* c(\mathbf{w})] \tag{C.7}$$

where the asterisk denotes complex conjugation.

Next, we minimize the function $h(\mathbf{w})$ with respect to the vector $\mathbf{w}$. To do this, we set the conjugate derivative $\partial h/\partial \mathbf{w}^*$ equal to the null vector, as shown by

$$\frac{\partial f}{\partial \mathbf{w}^*} + \frac{\partial}{\partial \mathbf{w}^*}(\operatorname{Re}[\lambda^* c(\mathbf{w})]) = \mathbf{0} \tag{C.8}$$

The system of simultaneous equations, consisting of Eq. (C.8) and the original constraint given in Eq. (C.2), define the optimum solutions for the vector $\mathbf{w}$ and the Lagrange multiplier λ. We call Eq. (C.8) the *adjoint equation* and Eq. (C.2) the *primal equation* (Dorny, 1975).

C.2 OPTIMIZATION INVOLVING MULTIPLE EQUALITY CONSTRAINTS

Consider next the minimization of a real function $f(\mathbf{w})$ that is a quadratic function of the vector $\mathbf{w}$, subject to a set of *multiple linear constraints*

$$\mathbf{w}^H\mathbf{s}_k = g_k, \qquad k = 1, 2, \ldots, K \tag{C.9}$$

where the number of constraints, K, is less than the dimension of the vector $\mathbf{w}$, and the g_k are complex constants. We may state the multiple-constrained optimization problem as follows:

> Minimize a real function $f(\mathbf{w})$, subject to the constraints $c_k(\mathbf{w}) = 0 + j0$ for $k = 1, 2, \ldots, K$ (C.10)

The solution to this optimization problem is readily obtained by generalizing the previous results of Section C.1. Specifically, we formulate a system of simultaneous equations, consisting of the adjoint equation

$$\frac{\partial f}{\partial \mathbf{w}^*} + \sum_{k=1}^{K} \frac{\partial}{\partial \mathbf{w}^*}(\mathrm{Re}[\lambda_k^* c_k(\mathbf{w})]) = \mathbf{0} \tag{C.11}$$

and the primal equation

$$c_k(\mathbf{w}) = 0^* + \mathrm{j}0, \qquad k = 1, 2, \ldots, K \tag{C.12}$$

This system of equations defines the optimum solutions for the vector $\mathbf{w}$ and the set of complex Lagrange multipliers $\lambda_1, \lambda_2, \ldots, \lambda_K$.

C.3 Example

By way of an example, consider the problem of finding the vector $\mathbf{w}$ that minimizes the function

$$f(\mathbf{w}) = \mathbf{w}^H\mathbf{w} \tag{C.13}$$

and which satisfies the constraint

$$c(\mathbf{w}) = \mathbf{w}^H\mathbf{s} - g = 0 + j0 \tag{C.14}$$

The adjoint equation for this problem is

$$\frac{\partial}{\partial \mathbf{w}^*}(\mathbf{w}^H\mathbf{w}) + \frac{\partial}{\partial \mathbf{w}^*}(\mathrm{Re}[\lambda^*(\mathbf{w}^H\mathbf{s} - g)]) = \mathbf{0} \tag{C.15}$$

Using the rules for differentiation developed in Appendix B, we have

$$\frac{\partial}{\partial \mathbf{w}^*}(\mathbf{w}^H\mathbf{w}) = \mathbf{w}$$

and

$$\frac{\partial}{\partial \mathbf{w}^*}(\mathrm{Re}[\lambda^*(\mathbf{w}^H\mathbf{s} - g)]) = \lambda^*\mathbf{s}$$

Substituting these results in Eq. (C.15), we get

$$\mathbf{w} + \lambda^* \mathbf{s} = \mathbf{0} \tag{C.16}$$

or, equivalently,

$$\mathbf{w}^H + \lambda \mathbf{s}^H = \mathbf{0}^T \tag{C.17}$$

Next, postmultiplying both sides of Eq. (C.17) by s and then solving for the unknown λ, we obtain

$$\begin{aligned} \lambda &= -\frac{\mathbf{w}^H \mathbf{s}}{\mathbf{s}^H \mathbf{s}} \\ &= -\frac{g}{\mathbf{s}^H \mathbf{s}} \end{aligned} \tag{C.18}$$

Finally, substituting Eq. (C.18) in (C.16) and solving for the optimum value $\mathbf{w}_o$ of the weight vector w, we get

$$\mathbf{w}_o = \left(\frac{g^*}{\mathbf{s}^H \mathbf{s}} \right) \mathbf{s} \tag{C.19}$$

This solution is optimum in the sense that $\mathbf{w}_o$ satisfies the constraint of Eq. (C.14) and has minimum length.

APPENDIX D

Estimation Theory

Estimation theory is a branch of probability and statistics that deals with the problem of deriving information about properties of random variables and stochastic processes, given a set of observed samples. This problem arises frequently in the study of communications and control systems. *Maximum likelihood* is by far the most general and powerful method of estimation. It was first used by the famous statistician R. A. Fisher in 1906. In principle, the method of maximum likelihood may be applied to any estimation problem with the proviso that we formulate the joint probability density function of the available set of observed data. As such, the method yields almost all the well-known estimates as special cases.

D.1 LIKELIHOOD FUNCTION

The method of maximum likelihood is based on a relatively simple idea: Different populations generate different data samples and any given data sample is more *likely* to have come from some population than from others (Kmenta, 1971).

Let $f_{\mathbf{U}}(\mathbf{u}|\boldsymbol{\theta})$ denote the *conditional joint probability density function* of the *random vector* $\mathbf{U}$ represented by the observed *sample* vector $\mathbf{u}$, where the sample vector $\mathbf{u}$ has u_1, u_2, . . ., u_M for its elements, and $\boldsymbol{\theta}$ is a *parameter vector* with θ_1, θ_2, . . ., θ_K as elements.

The method of maximum likelihood is based on the principle that we should estimate the parameter vector $\boldsymbol{\theta}$ by its most *plausible values*, given the observed sample vector $\mathbf{u}$. In other words, the maximum-likelihood estimators of $\theta_1, \theta_2, \ldots, \theta_K$ are those values of the parameter vector for which the conditional joint probability density function $f_{\mathbf{U}}(\mathbf{u}|\boldsymbol{\theta})$ is at maximum.

The name *likelihood function*, denoted by $l(\boldsymbol{\theta})$, is given to the conditional joint probability density function $f_{\mathbf{U}}(\mathbf{u}|\boldsymbol{\theta})$, viewed as a function of the parameter vector $\boldsymbol{\theta}$. We thus write

$$l(\boldsymbol{\theta}) = f_{\mathbf{U}}(\mathbf{u}|\boldsymbol{\theta}) \tag{D.1}$$

Although the conditional joint probability density function and the likelihood function have exactly the same formula, nevertheless, it is vital that we appreciate the physical distinction between them. In the case of the conditional joint probability density function, the parameter vector $\boldsymbol{\theta}$ is fixed and the observation vector $\mathbf{u}$ is variable. On the other hand, in the case of the likelihood function, the parameter vector $\boldsymbol{\theta}$ is variable and the observation vector $\mathbf{u}$ is fixed.

In many cases, it turns out to be more convenient to work with the natural logarithm of the likelihood function rather than with the likelihood itself. Thus, using $L(\boldsymbol{\theta})$ to denote the *log-likelihood function*, we write

$$\begin{aligned} L(\boldsymbol{\theta}) &= \ln[l(\boldsymbol{\theta})] \\ &= \ln[f_{\mathbf{U}}(\mathbf{u}|\boldsymbol{\theta})] \end{aligned} \tag{D.2}$$

The logarithm of $l(\boldsymbol{\theta})$ is a *monotonic transformation* of $l(\boldsymbol{\theta})$. This means that whenever $l(\boldsymbol{\theta})$ decreases, its logarithm $L(\boldsymbol{\theta})$ also decreases. Since $l(\boldsymbol{\theta})$, being a formula for conditional joint probability density function, can never become negative, it follows that there is no problem in evaluating its logarithm $L(\boldsymbol{\theta})$. We conclude therefore that the parameter vector for which the likelihood function $l(\boldsymbol{\theta})$ is at maximum is exactly the same as the parameter vector for which the log-likelihood function $L(\boldsymbol{\theta})$ is at its maximum.

To obtain the ith element of the maximum-likelihood estimate of the parameter vector $\boldsymbol{\theta}$, we differentiate the log-likelihood function with respect to θ_i and set the result equal to zero. We thus get a set of first-order conditions:

$$\frac{\partial L}{\partial \theta_i} = 0, \qquad i = 1, 2, \ldots, K \tag{D.3}$$

The first derivative of the log-likelihood function with respect to parameter θ_i is called the *score* for that parameter. The vector of such parameters is known as the *scores vector* (i.e., the gradient vector). The scores vector is identically zero at the maximum-likelihood estimates of the parameters, that is, at the values of $\boldsymbol{\theta}$ that result from the solutions of Eq. (D.3).

To find how effective the method of maximum likelihood is, we can compute the *bias* and *variance* for the estimate of each parameter. However, this is frequently difficult to do. Rather than approach the computation directly, we may derive a *lower bound* on the

variance of any *unbiased* estimate. We say an estimate is unbiased if the average value of the estimate equals the parameter we are trying to estimate. Later we show how the variance of the maximum-likelihood estimate compares with this lower bound.

D.2 CRAMER–RAO INEQUALITY

Let U be a random vector with conditional joint probability density function $f_{\mathbf{U}}(\mathbf{u}|\boldsymbol{\theta})$, where $\mathbf{u}$ is the observed sample vector with elements $u_1, u_2, \ldots, u_M$ and $\boldsymbol{\theta}$ is the parameter vector with elements $\theta_1, \theta_2, \ldots, \theta_K$. Using the definition of Eq. (D.2) for the log-likelihood function $L(\hat{\mathbf{u}})$ in terms of the conditional joint probability density function $f_{\mathbf{U}}(\mathbf{u}|\boldsymbol{\theta})$, we form the K-by-K matrix:

$$\mathbf{J} = -\begin{bmatrix} E\left[\frac{\partial^2 L}{\partial\theta_1^2}\right] & E\left[\frac{\partial^2 L}{\partial\theta_1\partial\theta_2}\right] & \cdots & E\left[\frac{\partial^2 L}{\partial\theta_1\partial\theta_K}\right] \\ E\left[\frac{\partial^2 L}{\partial\theta_2\partial\theta_1}\right] & E\left[\frac{\partial^2 L}{\partial\theta_2^2}\right] & \cdots & E\left[\frac{\partial^2 L}{\partial\theta_2\partial\theta_K}\right] \\ \vdots & \vdots & & \vdots \\ E\left[\frac{\partial^2 L}{\partial\theta_K\partial\theta_1}\right] & E\left[\frac{\partial^2 L}{\partial\theta_K\partial\theta_2}\right] & \cdots & E\left[\frac{\partial^2 L}{\partial\theta_K^2}\right] \end{bmatrix} \tag{D.4}$$

The matrix $\mathbf{J}$ is called *Fisher's information matrix.*

Let $\mathbf{I}$ denote the inverse of Fisher's information matrix $\mathbf{J}$. Let I_{ii} denote the ith diagonal element (i.e., the element in the ith row and ith column) of the inverse matrix $\mathbf{I}$. Let $\hat{\theta}_i$ be *any* unbiased estimate of the parameter θ_i, based on the observed sample vector $\mathbf{u}$. We may then write (Van Trees, 1968; Nahi, 1969)

$$\text{var}[\hat{\theta}_i] \geq I_{ii}, \qquad i = 1, 2, \ldots, K \tag{D.5}$$

Equation (D.5) is called the *Cramér–Rao inequality*. This theorem enables us to construct a lower limit (greater than zero) for the variance of any unbiased estimator, provided, of course, that we know the functional form of the log-likelihood function. The lower limit specified in the theorem is called the *Cramér–Rao lower bound* (CRLB).

If we can find an unbiased estimator whose variance equals the Cramér–Rao lower bound, then according to the theorem of Eq. (D.5) there is no other unbiased estimator with a smaller variance. Such an estimator is said to be *efficient.*

D.3 PROPERTIES OF MAXIMUM-LIKELIHOOD ESTIMATORS

Not only is the method of maximum likelihood based on an intuitively appealing idea (that of choosing those parameters from which the actually observed sample vector is most likely to have come), but also the resulting estimates have some desirable properties.

Indeed, under quite general conditions, the following *asymptotic* properties may be proved (Kmenta, 1971):

1. Maximum-likelihood estimators are *consistent*. That is, the value of θ_i for which the score $\partial L/\partial\theta_i$ is identically zero *converges in probability* to the true value of the parameter θ_i, $i = 1, 2, \ldots, K$, as the *sample size* M approaches infinity.
2. Maximum-likelihood estimators are *asymptotically efficient*; that is,

$$\lim_{M\to\infty}\left\{\frac{\text{var}[\theta_{i,\text{ml}} - \theta_i]}{I_{ii}}\right\} = 1, \qquad i = 1, 2, \ldots, K$$

   where $\theta_{i,\text{ml}}$ is the maximum-likelihood estimate of parameter θ_i, and I_{ii} is the ith diagonal element of the inverse of Fisher's information matrix.
3. Maximum-likelihood estimators are *asymptotically Gaussian*.

In practice, we find that the large-sample (asymptotic) properties of maximum-likelihood estimators hold rather well for sample size $M \geq 50$.

D.4 CONDITIONAL MEAN ESTIMATOR

Another classic problem in estimation theory is that of the *Bayes estimation of a random parameter*. There are different answers to this problem, depending on how the *cost function* in the Bayes estimation is formulated (Van Trees, 1968). A particular type of the Bayes estimator of interest to us in this book is the so-called *conditional mean estimator*. We now wish to do two things: (1) derive the formula for the conditional mean estimator from first principles, and (2) show that such an estimator is the same as a minimum mean-squared-error estimator.

Consider a *random parameter* x. We are given an *observation* y that depends on x, and the requirement is to estimate x. Let $\hat{x}(y)$ denote an *estimate* of the parameter x; the symbol $\hat{x}(y)$ emphasizes the fact that the estimate is a function of the observation y. Let $C(x, \hat{x}(y))$ denote a *cost function*. Then, according to Bayes estimation theory, we may write an expression for the *risk* as follows (Van Trees, 1968):

$$\begin{aligned}\mathcal{R} &= E[C(x, \hat{x}(y))] \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} C(x, \hat{x}(y)) f_{X,Y}(x, y)\, dy\end{aligned} \tag{D.6}$$

where $f_{X,Y}(x, y)$ is the joint probability density function of x and y. For a specified cost function $C(x, \hat{x}(y))$, the *Bayes estimate* is defined as the estimate $\hat{x}(y)$ that *minimizes* the risk $\mathcal{R}$.

A cost function of particular interest (and which is very much in the spirit of the material covered in this book) is the *mean-squared error*. In this case, the cost function is

specified as the square of the *estimation error*. The estimation error is itself defined as the difference between the actual parameter value x and the estimate $\hat{x}(y)$, as shown by

$$\epsilon = x - \hat{x}(y) \tag{D.7}$$

Correspondingly, the cost function is defined by

$$C(x, \hat{x}(y)) = C(x - \hat{x}(y)) \tag{D.8}$$

or, more simply,

$$C(\epsilon) = \epsilon^2 \tag{D.9}$$

Thus, the cost function varies with the estimation error ϵ in the manner indicated in Fig. D.1. It is assumed here that x and y are both real. Accordingly, for the situation at hand, we may rewrite Eq. (D.6) as follows:

$$\mathcal{R}_{\text{ms}} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} [x - \hat{x}(y)]^2 f_{X,Y}(x, y)\, dy \tag{D.10}$$

where the subscripts in the risk $\mathcal{R}_{\text{ms}}$ indicate the use of mean-squared error as its basis.

Using *Bayes' rule*, we have

$$f_{X,Y}(x, y) = f_X(x|y)\, f_Y(y) \tag{D.11}$$

where $f_X(x|y)$ is the conditional probability density function of x, given y, and $f_Y(y)$ is the (marginal) probability density function of y. Hence, using Eq. (D.11) in (D.10), we have

$$\mathcal{R}_{\text{ms}} = \int_{-\infty}^{\infty} dy f_Y(y) \int_{-\infty}^{\infty} [x - \hat{x}(y)]^2 f_X(x|y)\, dx \tag{D.12}$$

We now recognize that the inner integral and $f_Y(y)$ in Eq. (D.12) are both nonnegative. We may therefore minimize the risk $\mathcal{R}_{\text{ms}}$ by simply minimizing the inner integral. Let the estimate so obtained be denoted by $\hat{x}_{\text{ms}}(y)$. We find $\hat{x}_{\text{ms}}(y)$ by differentiating the inner integral with respect to $\hat{x}(y)$ and then setting the result equal to zero.

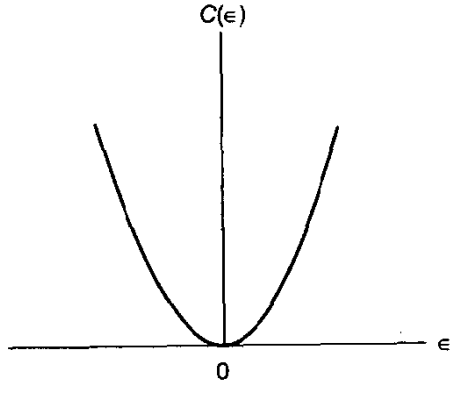

Figure D.1 Mean-squared error as the cost function.

To simplify the presentation, let I denote the inner integral in Eq. (D.12). Then differentiating I with respect to $\hat{x}(y)$ yields

$$\frac{dI}{d\hat{x}} = -2\int_{-\infty}^{\infty} x f_X(x|y)\,dx + 2\hat{x}(y)\int_{-\infty}^{\infty} f_X(x|y)\,dx \tag{D.13}$$

The second integral on the right-hand side of Eq. (D.13) represents the total area under a probability density function and therefore equals 1. Hence, setting the derivative $dI/d\hat{x}$ equal to zero, we obtain

$$\hat{x}_{\text{ms}}(y) = \int_{-\infty}^{\infty} x f_X(x|y)\,dx \tag{D.14}$$

The solution defined by Eq. (D.14) is a unique minimum.

The estimator $\hat{x}_{ms}(y)$ defined in Eq. (D.14) is naturally a *minimum mean-squared-error estimator*, hence the use of the subscripts "ms." For another interpretation of this estimator, we recognize that the integral on the right-hand side of Eq. (D.14) is just the *conditional mean* of the parameter x, given the observation y.

We therefore conclude that the minimum mean-squared error estimator and the conditional mean estimator are indeed one and the same. In other words, we have

$$\hat{x}_{\text{ms}}(y) = E[x|y] \tag{D.15}$$

Substituting Eq. (D.15) for the estimate $\hat{x}(y)$ in Eq. (D.12), we find that the inner integral is just the *conditional variance* of the parameter x, given y. Accordingly, the minimum value of the risk $\mathscr{R}_{\text{ms}}$ is just the average of this conditional variance over all observations y.

APPENDIX E

Maximum-Entropy Method

The *maximum-entropy method (MEM)* was originally devised by Burg (1967, 1975) to overcome fundamental limitations of Fourier-based methods for estimating the power spectrum of a stationary stochastic process. The basic idea of MEM is to choose the particular spectrum that corresponds to the most *random* or the most *unpredictable* time series whose autocorrelation function agrees with a set of known values. This condition is equivalent to an extrapolation of the autocorrelation function of the available time series by *maximizing* the *entropy* of the process, hence the name of the method. Entropy is a measure of the average information content of the process (Shannon, 1948). Thus, MEM bypasses the problems that arise from the use of window functions, a feature that is common to all Fourier-based methods of spectrum analysis. In particular, MEM avoids the use of a periodic extension of the data (as in the method based on smoothing the periodogram and its computation using the fast Fourier transform algorithm) or of the assumption that data outside the available record length are zero (as in the Blackman–Tukey method based on the sample autocorrelation function). An important feature of the MEM spectrum is that it is *nonnegative at all frequencies*, which is precisely the way it should be.

E.1 MAXIMUM-ENTROPY SPECTRUM

Suppose that we are given $2M + 1$ values of the autocorrelation function of a stationary stochastic process $u(n)$ of zero mean. We wish to obtain the special value of the power

spectrum of the process that corresponds to the most random time series whose autocorrelation function is consistent with the set of $2M + 1$ known values. In terms of information theory, this statement corresponds to the *principle of maximum entropy* (Jaynes, 1982).

In the case of a set of Gaussian-distributed random variables of zero mean, the entropy is given by (Middleton, 1960)

$$H = \tfrac{1}{2}\ln[\det(\mathbf{R})] \tag{E.1}$$

where $\mathbf{R}$ is the correlation matrix of the process. When the process is of infinite duration, however, we find that the entropy H diverges, and so we cannot use it as a measure of information content. To overcome this divergence problem, we may use the *entropy rate* defined by

$$\begin{aligned} h &= \lim_{M\to\infty} \frac{H}{M+1} \\ &= \lim_{M\to\infty} \tfrac{1}{2}\ln[\det(\mathbf{R})]^{1/(M+1)} \end{aligned} \tag{E.2}$$

Let $S(\omega)$ denote the power spectrum of the process $u(n)$. The limiting form of the determinant of the correlation matrix $\mathbf{R}$ is related to the power spectrum $S(\omega)$ as follows (see Problem 14 of Chapter 4):

$$\lim_{M\to\infty} [\det(\mathbf{R})]^{1/(M+1)} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S(\omega)\, d\omega\right\} \tag{E.3}$$

Hence, substituting Eq. (E.3) in (E.2), we get

$$h = \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln[S(\omega)]d\omega \tag{E.4}$$

Although this relation was derived on the assumption that the process $u(n)$ is Gaussian, nevertheless, the form of the relation is valid for any stationary process.

We may now restate the MEM problem in terms of the entropy rate. We wish to find a real positive-valued power spectrum characterized by entropy rate h, satisfying two simultaneous requirements:

1. The entropy rate h is *stationary* with respect to the *unknown* values of the autocorrelation function of the process.
2. The power spectrum is *consistent* with respect to the *known* values of the autocorrelation function of the process.

We will address these two requirements in turn.

Since the autocorrelation sequence $r(m)$ and power spectrum $S(\omega)$ of a stationary process $u(n)$ form a discrete-time Fourier-transform pair, we write

$$S(\omega) = \sum_{m=-\infty}^{\infty} r(m)\exp(-jm\omega) \tag{E.5}$$

Equation (E.5) assumes that the sampling period of the process $u(n)$ is normalized to unity. Substituting Eq. (E.5) in (E.4), we get

$$h = \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left[\sum_{m=-\infty}^{\infty} r(m)\exp(-jm\omega)\right] d\omega \tag{E.6}$$

We extrapolate the autocorrelation sequence $r(m)$ outside the range of known values, $-M \le m \le M$, by choosing the unknown values of the autocorrelation function in such a way that no information or entropy is added to the process. That is, we impose the condition

$$\frac{\partial h}{\partial r(m)} = 0, \qquad |m| \ge M + 1 \tag{E.7}$$

Hence, differentiating Eq. (E.6) with respect to $r(m)$ and setting the result equal to zero, we find that the conditions for *maximum entropy* are as follows:

$$\int_{-\pi}^{\pi} \frac{\exp(-jm\omega)}{S_{\text{MEM}}(\omega)}\, d\omega = 0, \qquad |m| \ge M + 1 \tag{E.8}$$

where $S_{\text{MEM}}(\omega)$ is the special value of the power spectrum resulting from the imposition of the condition in Eq. (E.7). Equation (E.8) implies that the power spectrum $S_{\text{MEM}}(\omega)$ is expressible in the form of a truncated Fourier series:

$$\frac{1}{S_{\text{MEM}}(\omega)} = \sum_{k=-M}^{M} c_k \exp(-jk\omega) \tag{E.9}$$

The complex Fourier coefficient c_k of the expansion satisfies the Hermitian condition

$$c_k^* = c_{-k} \tag{E.10}$$

so as to ensure that $S_{\text{MEM}}(\omega)$ is real for all ω.

The next requirement is to make the power spectrum $S_{\text{MEM}}(\omega)$ consistent with the set of known values of the autocorrelation function $r(m)$ for the interval $-M \le m \le M$. Since $r(m)$ is a Hermitian function, we need only concern ourselves with $0 \le m \le M$. Accordingly, $r(m)$ must equal the inverse discrete-time Fourier transform of $S_{\text{MEM}}(\omega)$ for $0 \le m \le M$, as shown by

$$r(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{\text{MEM}}(\omega)\exp(jm\omega)\, d\omega, \qquad 0 \le m \le M \tag{E.11}$$

Therefore, substituting Eq. (E.9) in (E.11), we get

$$r(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp(jm\omega)}{\sum_{k=-M}^{M} c_k \exp(-jk\omega)}\, d\omega, \qquad 0 \le m \le M \tag{E.12}$$

Clearly, in the set of complex Fourier coefficients $\{c_k\}$, we have the available degrees of freedom needed to satisfy the conditions of Eq. (E.12).

To proceed with the analysis, however, we find it convenient to use z-transform notation by changing from the variable ω to z. Define

$$z = \exp(j\omega) \tag{E.13}$$

Hence,

$$d\omega = \frac{1}{j}\frac{dz}{z}$$

and so we rewrite Eq. (E.12) in terms of the variable z as the contour integral

$$r(m) = \frac{1}{j2\pi}\oint \frac{z^{m-1}}{\sum_{k=-M}^{M} c_k z^{-k}}\, dz, \qquad 0 \le m \le M \tag{E.14}$$

The contour integration in Eq. (E.14) is performed on the unit circle in the z-plane in a counterclockwise direction. Since the complex Fourier coefficient c_k satisfies the Hermitian condition of Eq. (E.10), we may express the summation in the denominator of the integral in Eq. (E.14) as the product of two polynomials, as follows:

$$\sum_{k=-M}^{M} c_k z^{-k} = G(z)G^*\left(\frac{1}{z^*}\right) \tag{E.15}$$

where

$$G(z) = \sum_{k=0}^{M} g_k z^{-k} \tag{E.16}$$

and

$$G^*\left(\frac{1}{z^*}\right) = \sum_{k=0}^{M} g_k^* z^k \tag{E.17}$$

We choose the first polynomial $G(z)$ to be minimum phase, in that its zeros are all located inside the unit circle in the z-plane. Correspondingly, we choose the second polynomial $G^*(1/z^*)$ to be maximum phase, in that its zeros are all located outside the unit circle in the z-plane. Moreover, the zeros of these two polynomials are the inverse of each other with respect to the unit circle. Thus, substituting Eq. (E.15) in (E.14), we get

$$r(m) = \frac{1}{j2\pi}\oint \frac{z^{m-1}}{G(z)G^*(1/z^*)}\, dz, \qquad 0 \le m \le M \tag{E.18}$$

We next form the summation

$$\sum_{k=0}^{M} g_k r(m-k) = \frac{1}{j2\pi} \oint \frac{z^{m-1} \sum_{k=0}^{M} g_k z^{-k}}{G(z)G^*(1/z^*)}\, dz$$
$$= \frac{1}{j2\pi} \oint \frac{z^{m-1}}{G^*(1/z^*)}\, dz, \qquad 0 \le m \le M \tag{E.19}$$

where in the first line we have used Eq. (E.18), and in the second line we have used Eq. (E.16).

To evaluate the contour integral of Eq. (E.19), we use *Cauchy's residue theorem* of complex variable theory (see Appendix A). According to this theorem, the contour integral equals $2\pi j$ times the sum of *residues* of the poles of the integral $z^{m-1}/G^*(1/z^*)$ that lie inside the unit circle used as the contour of integration. Since the polynomial $G^*(1/z^*)$ is chosen to have no zeros inside the unit circle, it follows that the integral in Eq. (E.19) is analytic on and inside the unit circle for $m \ge 1$. For $m = 0$ the integral has a simple pole at $z = 0$ with a *residue* equal to $1/g_0^*$. Hence, application of Cauchy's residue theorem yields

$$\oint \frac{z^{m-1}}{G^*(1/z^*)}\, dz = \begin{cases} \dfrac{2\pi j}{g_0^*}, & m = 0 \\ 0, & m = 1, 2, \ldots, M \end{cases} \tag{E.20}$$

Thus, substituting Eq. (E.20) in (E.19), we get

$$\sum_{k=0}^{M} g_k r(m-k) = \begin{cases} \dfrac{1}{g_0^*}, & m = 0 \\ 0, & m = 1, 2, \ldots, M \end{cases} \tag{E.21}$$

We recognize that the set of $(M + 1)$ equations in (E.21) has a mathematical form similar to that of the augmented Wiener–Hopf equations for forward prediction of order M (see Chapter 6). In particular, by comparing Eqs. (E.21) and (6.16), we deduce that

$$g_k^* = \frac{1}{g_0 P_M} a_{M,k}, \qquad 0 \le k \le M \tag{E.22}$$

where the $a_{M,k}$ are coefficients of a prediction-error filter of order M, and P_M is the average output power of the filter. Since $a_{M,0} = 1$ for all M, by definition, we find from Eq. (E.22) that for $k = 0$:

$$|g_0|^2 = \frac{1}{P_M} \tag{E.23}$$

Finally, substituting Eqs. (E.15), (E.22), and (E.23) in (E.9) with $z = \exp(j\omega)$, we get

$$S_{\text{MEM}}(\omega) = \frac{P_M}{\left|1 + \sum_{k=1}^{M} a_{M,k} e^{-jk\omega}\right|^2} \tag{E.24}$$

We refer to the formula of Eq. (E.24) as the *MEM spectrum*.

E.2 COMPUTATION OF THE MEM SPECTRUM

The formula for the MEM spectrum given in Eq. (E.24) may be recast in the alternative form

$$S_{\text{MEM}}(\omega) = \frac{1}{\sum_{k=-M}^{M} \psi(k) e^{-j\omega k}} \tag{E.25}$$

where $\psi(k)$ is defined in terms of the prediction-error filter coefficients as follows:

$$\psi(k) = \begin{cases} \dfrac{1}{P_M} \displaystyle\sum_{i=0}^{M-k} a_{M,i} a^*_{M,i+k} & \text{for } k = 0, 1, \ldots, M \\ \psi^*(-k) & \text{for } k = -M, \ldots, -1 \end{cases} \tag{E.26}$$

The parameter $\psi(k)$ may be viewed as some form of a *correlation coefficient for prediction-error filter coefficients*.

Examination of the denominator polynomial in Eq. (E.25) reveals that it represents the *discrete Fourier transform* of the sequence $\psi(k)$. Accordingly, we may use the *fast Fourier transform (FFT)* algorithm (Oppenheim and Schafer, 1989) for the efficient computation of the denominator polynomial and therefore the MEM spectrum. Given the autocorrelation sequence $r(0), r(1), \ldots, r(M)$, pertaining to a wide-sense stationary stochastic process $u(n)$, we may now summarize an efficient procedure for computing the MEM spectrum:

Step 1: Levinson–Durbin Recursion.

Initialize the algorithm by setting

$$a_{0,0} = 1$$
$$P_0 = r(0)$$

For $m = 1, 2, \ldots, M$, compute

$$\kappa_m = -\frac{1}{P_{m-1}} \sum_{i=0}^{M-1} r(i-m) a_{m-1,i}$$

$$a_{m,i} = \begin{cases} 1 & \text{for } i = 0 \\ a_{m-1,i} + \kappa_m a^*_{m-1,m-i} & \text{for } i = 1, 2, \ldots, m-1 \\ \kappa_m & \text{for } i = m \end{cases}$$

$$P_m = P_{m-1}(1 - |\kappa_m|^2)$$

Step 2: Correlation for Prediction-Error Filter Coefficients.

Compute the correlation coefficient

$$\psi(k) = \begin{cases} \dfrac{1}{P_M} \displaystyle\sum_{i=0}^{M-k} a_{M,i} a^*_{M,i+k} & \text{for } k = 0, 1, \ldots, M \\ \psi^*(-k) & \text{for } k = -M, \ldots, -1 \end{cases} \tag{E.26}$$

Step 3: MEM Spectrum.

Use the fast Fourier transform algorithm to compute the MEM spectrum for varying angular frequency:

$$S_{\text{MEM}}(\omega) = \frac{1}{\displaystyle\sum_{k=-M}^{M} \psi(k) e^{-j\omega k}}$$

APPENDIX F

Minimum-Variance Distortionless Response Spectrum

In Section 5.8, we derived the formula for the *minimum-variance distortionless response (MVDR)* spectrum for a wide-sense stationary stochastic process. In this appendix we do two things. First, we develop a fast algorithm for computing the MVDR spectrum, given the ensemble-averaged correlation matrix of the process (Musicus, 1985); the algorithm exploits the Toeplitz property of the correlation matrix. Second, in deriving the algorithm, we develop an insightful relationship between the MVDR and MEM spectra.

F.1 FAST MVDR SPECTRUM COMPUTATION

Consider a zero-mean wide-sense stationary stochastic process $u(n)$ characterized by an $(M + 1)$-by-$(M + 1)$ ensemble-averaged correlation matrix $\mathbf{R}$. The *minimum-variance distortionless response (MVDR) spectrum* for such a process is defined in terms of the inverse matrix $\mathbf{R}^{-1}$ by

$$S_{\text{MVDR}}(\omega) = \frac{1}{\mathbf{s}^H(\omega)\mathbf{R}^{-1}\mathbf{s}(\omega)} \tag{F.1}$$

where

$$\mathbf{s}(\omega) = [1, e^{-j\omega}, e^{-j2\omega}, \ldots, e^{-jM\omega}]^T$$

Let $R^{-1}_{l,k}$ denote the (l, k)th element of $\mathbf{R}^{-1}$. Then, we may rewrite Eq. (F.1) in the form

$$S_{\text{MVDR}}(\omega) = \frac{1}{\sum_{k=-M}^{M} \mu(k) e^{-j\omega k}} \tag{F.2}$$

where

$$\mu(k) = \sum_{l=\max(0,k)}^{\min(M-k,M)} R^{-1}_{l,l+k} \tag{F.3}$$

We recognize that the correlation matrix $\mathbf{R}$ is Toeplitz. We may therefore use the *Gohberg–Semencul formula* (Kailath et al., 1979) to express the (l,k)th element of the inverse matrix $\mathbf{R}^{-1}$ as follows:

$$R^{-1}_{l,k} = \frac{1}{P_M} \sum_{i=0}^{l} (a_{M,i} a^*_{M,i+k-l} - a^*_{M,M+1-i} a_{M,M+1-i-k+l}), \qquad k \geq l \tag{F.4}$$

where $1, a_{M,1}, \ldots, a_{M,M}$ are the coefficients of a prediction-error filter of order M, and P_M is the average prediction-error power. Substituting Eq. (F.4) in (F.3) and confining attention to $k \geq 0$, we get

$$\mu(k) = \frac{1}{P_M} \sum_{l=0}^{M-k} \sum_{i=0}^{l} a_{M,i} a^*_{M,i+k} - \frac{1}{P_M} \sum_{l=0}^{M-k} \sum_{i=0}^{l} a^*_{M,M+1-i} a_{M,M+1-i-k} \tag{F.5}$$

Interchanging the order of summations and setting $j = M + 1 - i - k$, we may rewrite $\mu(k)$ as

$$\mu(k) = \frac{1}{P_M} \sum_{i=0}^{M-k} \sum_{l=i}^{M-k} a_{M,i} a^*_{M,i+k} - \frac{1}{P_M} \sum_{j=1}^{M+1-k} \sum_{l=M+1-j-k}^{M-k} a^*_{M,j+k} a_{M,j} \tag{F.6}$$

The terms that do not involve the index l permit us to collapse the summation over l into a multiplicative integer constant. We may thus combine the two summations in Eq. (F.6). Moreover, we may use the Levinson–Durbin recursion for computing the prediction-error filter coefficients. Given the autocorrelation sequence $r(0), r(1), \ldots, r(M)$, we may now formulate a fast algorithm for computing the MVDR spectrum as follows (Musicus, 1985):

Step 1: Levinson–Durbin Recursion.

Initialize the algorithm by setting

$$a_{0,0} = 1$$

$$P_0 = r(0)$$

Hence, compute for $m = 1, 2, \ldots, M$:

$$\kappa_m = -\frac{1}{P_{m-1}} \sum_{i=0}^{m-1} r(i-m) a_{m-1,i}$$

$$a_{m,i} = \begin{cases} 1 & \text{for } i = 0 \\ a_{m-1,i} + \kappa_m a^*_{m-1,m-i} & \text{for } i = 1, 2, \ldots, m-1 \\ \kappa_m & \text{for } i = m \end{cases}$$

$$P_m = P_{m-1}(1 - |\kappa_m|^2)$$

Step 2: Correlation of the Predictor Coefficients.

Compute the parameter $\mu(k)$ for varying k:

$$\mu(k) = \begin{cases} \dfrac{1}{P_M} \displaystyle\sum_{i=0}^{M-k} (M + 1 - k - 2i) a_{M,i} a^*_{M,i+k} & \text{for } k = 0, \ldots, M \\ \mu^*(-k) & \text{for } k = -M, \ldots, -1 \end{cases} \tag{F.7}$$

Step 3: MVDR Spectrum Computation.

Use the fast Fourier transform algorithm to compute the MVDR spectrum for varying angular frequency:

$$S_{\text{MVDR}}(\omega) = \frac{1}{\displaystyle\sum_{k=-M}^{M} \mu(k) e^{-j\omega k}} \tag{F.8}$$

F.2 COMPARISON OF MVDR AND MEM SPECTRA

Comparing the formula for computing the MVDR spectrum with that for computing the MEM spectrum, we see that the only difference between the MVDR formula in Eq. (F.8) and the MEM formula in Eq. (E.25) lies in the definitions of their respective correlations of predictor coefficients. In particular, a *linear taper* is used in the definition of $\mu(k)$ given in Eq. (F.7) for the MVDR formula. On the other hand, the definition of the corresponding parameter $\psi(k)$ given in Eq. (E.26) for the MEM formula does *not* involve a taper. This means that for a large-enough model order M, such that $a_{M,i} = 0$ for $i > M/2$, the linear taper involved in the computation of $\mu(k)$ acts like a triangular window on the product terms $a_{M,i} a^*_{M,i+k}$. This has the effect of deemphasizing higher-order terms with large i for large values of lag k (Musicus, 1985). Accordingly, for a given process, an MVDR spectrum is *smoother* in appearance than the corresponding MEM spectrum.

APPENDIX G

Gradient Adaptive Lattice Algorithm

The adaptive lattice filtering algorithms considered in Chapter 15 are all *exact* manifestations of recursive least-squares estimation, exact in the sense that no approximations are made in their derivations. In this appendix we derive another adaptive lattice filtering algorithm known as the *gradient adaptive lattice (GAL) algorithm* (Griffiths, 1977, 1978), which is a natural extension of the least-mean-square (LMS) algorithm.

Consider a single-stage lattice structure the input–output relation of which is characterized by a single parameter, namely, the *reflection coefficient* κ_m. We assume that the input data are wide-sense stationary and that κ_m is complex valued. Define a cost function for this stage as

$$J_m = E[|f_m(n)|^2 + |b_m(n)|^2] \tag{G.1}$$

where $f_m(n)$ is the forward prediction error and $b_m(n)$ is the backward prediction error, both measured at the output of the stage; E is the statistical expectation operator. The input–output relations of the lattice stage under consideration are described by

$$f_m(n) = f_{m-1}(n) + \kappa_m^* b_{m-1}(n-1)$$

$$b_m(n) = b_{m-1}(n-1) + \kappa_m f_{m-1}(n)$$

The gradient of the cost function J_m with respect to the real and imaginary parts of the reflection coefficient κ_m is given by

$$\nabla J_m = 2E[f_m^*(n)b_{m-1}(n-1) + b_m(n)f_{m-1}^*(n)] \tag{G.2}$$

where $f_{m-1}(n)$ is the forward prediction error and $b_{m-1}(n-1)$ is the delayed backward prediction error, both measured at the input of the lattice stage; the other two prediction errors in Eq. (G.2) refer to the output of the stage. Following the development of the LMS algorithm as presented in Chapter 9, we may use instantaneous estimates of the expectations in Eq. (G.2) and thus write

$$E[f_m^*(n)b_{m-1}(n-1)] \simeq f_m^*(n)b_{m-1}(n-1)$$
$$E[b_m(n)f_{m-1}^*(n)] \simeq b_m(n)f_{m-1}^*(n)$$

Correspondingly, we may express the *instantaneous estimate* of the gradient $\nabla_m J$ as

$$\hat{\nabla}_m J(n) = 2[f_m^*(n)b_{m-1}(n-1) + b_m(n)f_{m-1}^*(n)] \tag{G.3}$$

Let $\hat{\kappa}_m(n-1)$ denote the *old estimate* of the reflection coefficient κ_m of the mth lattice stage. Let $\hat{\kappa}_m(n)$ denote the *updated estimate* of this reflection coefficient. We may compute this updated estimate by adding to the old estimate $\kappa_m(n-1)$ a *correction* term proportional to the gradient estimate $\hat{\nabla}_m J(n)$, as shown by

$$\hat{\kappa}_m(n) = \hat{\kappa}_m(n-1) - \frac{1}{2}\mu_m(n)\hat{\nabla}_m J(n) \tag{G.4}$$

where μ_m denotes a *time-varying step-size parameter* associated with the mth lattice stage. Substituting Eq. (G.3) in (G.4), we thus get

$$\hat{\kappa}_m(n) = \hat{\kappa}_m(n-1) - \mu_m(n)[f_m^*(n)b_{m-1}(n-1) + b_m(n)f_{m-1}^*(n)] \tag{G.5}$$

The adaptation parameter $\mu_m(n)$ is chosen as

$$\mu_m(n) = \frac{\tilde{\mu}}{\mathscr{E}_{m-1}(n)} \tag{G.6}$$

where

$$\begin{aligned}\mathscr{E}_{m-1}(n) &= \sum_{i=1}^{n} [|f_{m-1}(i)|^2 + |b_{m-1}(i-1)|^2] \\ &= \mathscr{E}_{m-1}(n-1) + |f_{m-1}(n)|^2 + |b_{m-1}(n-1)|^2]\end{aligned} \tag{G.7}$$

For a well-behaved convergence of the algorithm, we usually set $\tilde{\mu} < 0.1$. The parameter $\mathscr{E}_{m-1}(n)$ represents the total energy of both the forward and backward prediction errors at the input of the mth stage, measured up to and including time n.

In practice, a minor modification is made to the *energy estimator* of Eq. (G.7) by writing it in the form of a *single-pole average* of squared data, as shown by (Griffiths, 1977, 1978)

$$\mathscr{E}_{m-1}(n) = \beta\mathscr{E}_{m-1}(n-1) + (1-\beta)[|f_{m-1}(n)|^2 + |b_{m-1}(n-1)|^2] \tag{G.8}$$

where $0 < \beta < 1$. The introduction of the parameter β in Eq. (G.8) provides the GAL algorithm with a finite *memory*, which helps it deal better with statistical variations when operating in a nonstationary environment.

TABLE G.1 SUMMARY OF THE GAL ALGORITHM

Parameters: M = final prediction order
β = constant, lying in the range $0 < \beta < 1$
$\tilde{\mu} < 0.1$

Initialization: For prediction order $m = 1, 2, \ldots, M$, put

$$f_m(0) = b_m(0) = 0$$

$$\mathscr{E}_{m-1}(0) = \delta, \qquad \delta = \text{small constant}$$

$$\hat{\kappa}_m(0) = 0$$

For time $n = 1, 2, \ldots$, put

$$f_0(n) = b_0(n) = u(n), \qquad u(n) = \text{lattice predictor input}$$

Prediction: For prediction order $m = 1, 2, \ldots, M$ and time $n = 1, 2, \ldots$, compute

$$f_m(n) = f_{m-1}(n) + \hat{\kappa}_m^*(n) b_{m-1}(n-1)$$

$$b_m(n) = b_{m-1}(n-1) + \hat{\kappa}_m(n) f_{m-1}(n)$$

$$\mathscr{E}_{m-1}(n) = \beta \mathscr{E}_{m-1}(n-1) + (1-\beta)(|f_{m-1}(n)|^2 + |b_{m-1}(n-1)|^2)$$

$$\hat{\kappa}_m(n) = \hat{\kappa}_m(n-1) - \frac{\tilde{\mu}}{\mathscr{E}_{m-1}(n)} [f_{m-1}^*(n) b_m(n) + b_{m-1}(n-1) f_m^*(n)]$$

A summary of the GAL algorithm is presented in Table G.1.

Properties of the GAL Algorithm

The use of time-varying step-size parameter $\mu_m(n) = \tilde{\mu}/\mathscr{E}_{m-1}(n)$ in the update equation for the reflection coefficient $\hat{\kappa}_m(n)$ introduces a form of *normalization* similar to that in the normalized LMS algorithm. From Eq. (G.8) we see that for small magnitudes of the prediction errors $f_{m-1}(n)$ and $b_{m-1}(n)$ the value of the parameter $\mathscr{E}_{m-1}(n)$ is correspondingly small or, equivalently, the step-size parameter $\mu_m(n)$ has a correspondingly large value. Such a behavior is desirable from a practical point of view. Basically, a small value for the prediction errors means that the adaptive lattice predictor is providing an accurate model of the external environment in which it is operating. Hence, if there is any increase in the prediction errors, it should be due to variations in the external environment, in which case it is highly desirable for the adaptive lattice predictor to respond rapidly to such variations. This objective is indeed realized by having the step-size parameter $\mu_m(n)$ assume a large value, which makes it possible for the GAL algorithm to provide an initially rapid convergence to the new environmental conditions. If, on the other hand, the input data applied to the adaptive lattice predictor are too noisy (i.e., they contain a strong white-noise component in addition to the signal of interest), we find that the prediction errors produced by the

adaptive lattice predictor are correspondingly large. In such a situation, the parameter $\mathscr{E}_{m-1}(n)$ has a large value or, equivalently, the step-size parameter $\mu_m(n)$ has a small value. Accordingly, the GAL algorithm does *not* respond rapidly to variations in the external environment, which is precisely the way we would like the algorithm to behave (Alexander, 1986a).

Another point of interest is that the convergence behavior of the GAL algorithm is somewhat more rapid than that of the LMS algorithm, but inferior to that of exact recursive LSL algorithms.

APPENDIX H

Solution of the Difference Equation (9.75)

In this appendix we fill in the mathematical details concerning the mean-squared error analysis of the LMS algorithm. We begin by reproducing Eq. (9.75):

$$\mathbf{x}(n+1) = \mathbf{B}\mathbf{x}(n) + \mu^2 J_{\min}\boldsymbol{\lambda} \tag{H.1}$$

where $\mathbf{B}$ is a real, positive, and symmetric matrix; $\boldsymbol{\lambda}$ is a vector of eigenvalues pertaining to an ensemble-averaged correlation matrix $\mathbf{R}$ of size M-by-M.

Equation (H.1) is a difference equation of order 1 in the vector $\mathbf{x}(n)$. Therefore, assuming an initial value $\mathbf{x}(0)$, the solution to this equation is[1]

$$\mathbf{x}(n) = \mathbf{B}^n\mathbf{x}(0) + \mu^2 J_{\min}\sum_{i=0}^{n-1}\mathbf{B}^i\boldsymbol{\lambda} \tag{H.2}$$

By analogy with the formula for the sum of a geometric series, we may express the finite sum $\sum_{i=0}^{n-1}\mathbf{B}^i$ as follows:

$$\sum_{i=0}^{n-1}\mathbf{B}^i = (\mathbf{I} - \mathbf{B}^n)(\mathbf{I} - \mathbf{B})^{-1} \tag{H.3}$$

where $\mathbf{I}$ is the identity matrix. Substituting Eq. (H.3) in (H.2), we thus get

$$\mathbf{x}(n) = \mathbf{B}^n[\mathbf{x}(0) - \mu^2 J_{\min}(\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\lambda}] + \mu^2 J_{\min}(\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\lambda} \tag{H.4}$$

[1] The approach we follow here is adapted from Mazo (1979). However, we differ from Mazo in that our analysis is for complex data, whereas that of Mazo is for real data.

The first term on the right-hand side of Eq. (H.4) is the *transient* component of the vector $\mathbf{x}(n)$, and the second term is the *steady-state* component. Since the matrix $\mathbf{B}$ is symmetric, we may apply to it an orthogonal similarity transformation. We may thus write

$$\mathbf{G}^T\mathbf{B}\mathbf{G} = \mathbf{C} \tag{H.5}$$

The matrix $\mathbf{C}$ is a diagonal matrix with elements $c_i = 1, 2, \ldots, M$, which are the eigenvalues of $\mathbf{B}$. The matrix $\mathbf{G}$ is an *orthonormal matrix* whose ith column is the eigenvector $\mathbf{g}_i$ of $\mathbf{B}$, associated with eigenvalue c_i. Because of the property

$$\mathbf{G}\mathbf{G}^T = \mathbf{I} \tag{H.6}$$

we find that

$$\mathbf{B}^n = \mathbf{G}\mathbf{C}^n\mathbf{G}^T \tag{H.7}$$

Hence, we may rewrite Eq. (H.4) in the form

$$\mathbf{x}(n) = \mathbf{G}\mathbf{C}^n\mathbf{G}^T[\mathbf{x}(0) - \mu^2 J_{\min}(\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\lambda}] + \mu^2 J_{\min}(\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\lambda} \tag{H.8}$$

Since $\mathbf{C}$ is a diagonal matrix, we have

$$\mathbf{C}^n = \text{diag}[c_1^n, c_2^n, \ldots, c_M^n] \tag{H.9}$$

It follows therefore that the solution defined by Eq. (H.8) is stable if and only if the eigenvalues of matrix $\mathbf{B}$ all have a magnitude less than 1. The eigenvalues of matrix $\mathbf{B}$ are all positive, since the matrix $\mathbf{B}$ is positive definite. For stability, we therefore require the condition

$$0 < c_i < 1 \qquad \text{for all } i \tag{H.10}$$

When this condition is satisfied, the transient component in Eq. (H.8) decays to zero as the number of iterations, n, approaches infinity. This would then leave the steady-state component as the only component. We may thus write

$$\mathbf{x}(\infty) = \mu^2 J_{\min}(\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\lambda} \tag{H.11}$$

Substituting Eq. (H.11) in (H.8), we may rewrite the solution as

$$\mathbf{x}(n) = \mathbf{G}\mathbf{C}^n\mathbf{G}^T[\mathbf{x}(0) - \mathbf{x}(\infty)] + \mathbf{x}(\infty) \tag{H.12}$$

In view of the diagonal nature of matrix $\mathbf{C}^n$, and since the orthonormal matrix $\mathbf{G}$ consists of the eigenvectors of $\mathbf{B}$ as its columns, we may express the matrix product $\mathbf{G}\mathbf{C}^n\mathbf{G}^T$ as follows:

$$\mathbf{G}\mathbf{C}^n\mathbf{G}^T = \sum_{i=1}^{M} c_i^n \mathbf{g}_i\mathbf{g}_i^T \tag{H.13}$$

Accordingly, we may rewrite Eq. (H.12) one more time in the equivalent form

$$\mathbf{x}(n) = \sum_{i=1}^{M} c_i^n \mathbf{g}_i\mathbf{g}_i^T[\mathbf{x}(0) - \mathbf{x}(\infty)] + \mathbf{x}(\infty) \tag{H.14}$$

This is the desired solution to the difference equation (H.1).

APPENDIX I

Steady-State Analysis of the LMS Algorithm Without Invoking the Independence Assumption

In this Appendix, we revisit the steady-state analysis of the LMS algorithm by taking an iterative approach that avoids the independence assumption (Butterweck, 1995a). The theory applies to small values of the step-size parameter. It proceeds in two stages. First, a power series solution is derived for the weight-error vector in terms of the step-size parameter. The result so obtained is next used to derive a corresponding expansion for the weight-error correlation matrix.

I.1 ITERATIVE SOLUTION FOR THE WEIGHT-ERROR VECTOR

The weight-error vector $\boldsymbol{\epsilon}(n)$ computed by the LMS algorithm is defined by the stochastic difference equation (9.55), reproduced here for convenience of presentation:

$$\boldsymbol{\epsilon}(n+1) = [\mathbf{I}-\mu\mathbf{u}(n)\mathbf{u}^H(n)]\boldsymbol{\epsilon}(n) + \mu\mathbf{u}(n)e_o^*(n) \tag{I.1}$$

where $\mathbf{u}(n)$ is the tap-input vector, μ is the step-size parameter, and $e_o(n)$ is the estimation error produced by the Wiener solution. Under the condition that μ is small, the direct-averaging method leads us to say that the solution of this equation is approximately the same as that of Eq. (9.56), reproduced here in the form

$$\boldsymbol{\epsilon}_0(n+1) = (\mathbf{I}-\mu\mathbf{R})\boldsymbol{\epsilon}_0(n) + \mu\mathbf{u}(n)e_o^*(n) \tag{I.2}$$

where $\mathbf{R} = E[\mathbf{u}(n)\mathbf{u}^H(n)]$. For reasons that will become apparent presently, we have used a different symbol for the weight-error vector in Eq. (I.2). Note that the solutions of Eqs. (I.1) and (I.2) become equal for the limiting case of a vanishing step-size parameter μ.

In the iterative procedure described by Butterweck (1995a), the solution of Eq. (I.2) is used as a starting point for generating a whole set of solutions of the original stochastic difference equation (I.1). The accuracy of the solution so obtained improves with increasing iteration order. Thus, starting with the solution $\boldsymbol{\epsilon}_0(n)$, the solution of Eq. (I.1) is expressed as a sum of partial functions, as shown by

$$\boldsymbol{\epsilon}(n) = \boldsymbol{\epsilon}_0(n) + \boldsymbol{\epsilon}_1(n) + \boldsymbol{\epsilon}_2(n) + \cdots \tag{I.3}$$

Define the zero-mean difference matrix:

$$\mathbf{P}(n) = \mathbf{u}(n)\mathbf{u}^H(n) - \mathbf{R} \tag{I.4}$$

Then, substituting Eq. (I.4) in (I.1) yields

$$\begin{aligned}\boldsymbol{\epsilon}_0(n+1) + \boldsymbol{\epsilon}_1(n+1) + \boldsymbol{\epsilon}_2(n+1) + \cdots \\ = (\mathbf{I} - \mu\mathbf{R})[\boldsymbol{\epsilon}_0(n) + \boldsymbol{\epsilon}_1(n) + \boldsymbol{\epsilon}_2(n) + \cdots \\ - \mu\mathbf{P}(n)[\boldsymbol{\epsilon}_1(n) + \boldsymbol{\epsilon}_2(n) + \cdots] + \mu\mathbf{u}(n)e_o^*(n)\end{aligned}$$

from which we readily deduce that

$$\boldsymbol{\epsilon}_i(n+1) = (\mathbf{I} - \mu\mathbf{R})\boldsymbol{\epsilon}_i(n) + \mathbf{f}_i(n), \qquad i = 0, 1, 2, \cdots \tag{I.5}$$

where the subscript i refers to the iteration order. The "driving force" $\mathbf{f}_i(n)$ for the difference equation (I.5) is defined by

$$\mathbf{f}_i(n) = \begin{cases} \mu\mathbf{u}(n)e_o^*(n), & i = 0 \\ -\mu\mathbf{P}(n)\boldsymbol{\epsilon}_{i-1}(n), & i = 1, 2, \cdots \end{cases} \tag{I.6}$$

Thus, a time-varying system characterized by the stochastic difference equation (I.1) is transformed into a set of equations having the same basic format as described in (I.5), such that the solution to the ith equation in the set (i.e., step i in the iterative procedure) follows from the $(i-1)$th equation. In particular, the problem is reduced to a study of the transmission of a stationary stochastic process through a low-pass filter with an extremely low cutoff frequency.

I.2 SERIES EXPANSION OF THE WEIGHT-ERROR CORRELATION MATRIX

On the basis of Eq. (I.3), we may express the weight-error correlation matrix in the form of a corresponding series as follows:

$$\begin{aligned}\mathbf{K}(n) &= E[\boldsymbol{\epsilon}(n)\boldsymbol{\epsilon}^H(n)] \\ &= \sum_i \sum_k E[\boldsymbol{\epsilon}_i(n)\boldsymbol{\epsilon}_k^H(n)], \qquad (i,k) = 0, 1, 2, \cdots\end{aligned} \tag{I.7}$$

Expanding this series in light of the definitions given in Eqs. (I.5) and (I.6), and then grouping equal-order terms in the step-size parameter μ, we get the following series expansion:

$$\mathbf{K}(n) = \mathbf{K}_0(n) + \mu\mathbf{K}_1(n) + \mu^2\mathbf{K}_2(n) + \cdots \tag{I.8}$$

where the various matrix coefficients are themselves defined as follows:

$$\mathbf{K}_j(n) = \begin{cases} E[\boldsymbol{\epsilon}_0(n)\boldsymbol{\epsilon}_0^H(n)] & \text{for } j = 0 \\ \displaystyle\sum_i \sum_k E[\boldsymbol{\epsilon}_i(n)\boldsymbol{\epsilon}_k^H(n)] & \text{for all } (i,k) \geq 0 \\ & \text{such that } i + k = 2j-1, 2j \end{cases} \tag{I.9}$$

These matrix coefficients are defined, albeit in a rather complex fashion, by the spectral and probability distribution of the environment in which the LMS algorithm operates. In a general setting with arbitrarily colored signals, the calculation of $\mathbf{K}_j(n)$ for $j \geq 1$ can be rather tedious, except in some special cases (Butterweck, 1995a).

The zero-order term $\mathbf{K}_0(n)$ in Eq. (I.8) is of special interest for two reasons. First, for a small μ it may be used as an approximation to the actual $\mathbf{K}(n)$, as discussed in Section 9.4. Second, it lends itself to examination without any statistical assumptions concerning the environment in which the LMS algorithm operates. In particular, we find that under steady-state conditions (i.e., large n), $\mathbf{K}_0(n)$ is determined as the solution to the equation (Butterweck, 1995b):

$$\mathbf{R}\mathbf{K}_0(n) + \mathbf{K}_0(n)\mathbf{R} = \mu \sum_l J_{\min}^{(l)} \mathbf{R}^{(l)}, \qquad \text{large } n \tag{I.10}$$

where

$$J_{\min}^{(l)} = E[e_o(n)\, e_o^*(n-l)], \qquad l = 0, 1, 2, \ldots \tag{I.11}$$

$$\mathbf{R}^{(l)} = E[\mathbf{u}(n)\mathbf{u}^H(n-l)], \qquad l = 0, 1, 2, \ldots \tag{I.12}$$

Note that for $l = 0$, we have $J_{\min}^{(0)} = J_{\min}$ and $\mathbf{R}^{(0)} = \mathbf{R}$.

The steady-state value of the misadjustment $\mathcal{M}$ derived in Chapter 9 under the independence assumption corresponds to setting $l = 0$ in Eq. (I.10) and ignoring all higher-order terms. This special case corresponds to the assumption that the estimation error $e(n)$ produced by the LMS algorithm is drawn from a white noise process. Thus, Eq. (I.10) is approximated by

$$\mathbf{R}\mathbf{K}_0(n) + \mathbf{K}_0(n)\mathbf{R} \simeq \mu J_{\min}\mathbf{R}, \qquad \text{large } n$$

from which we readily find that the misadjustment is

$$\begin{aligned} \mathcal{M} &= \frac{\text{tr}[\mathbf{R}\mathbf{K}_0(n)]}{J_{\min}} \\ &\simeq \frac{\mu}{2}\,\text{tr}[\mathbf{R}] \\ &= \frac{\mu}{2}\sum_{i=1}^{M} \lambda_i \end{aligned}$$

This is indeed the result derived in Eq. (9.95).

APPENDIX J

The Complex Wishart Distribution

The Wishart distribution plays an important role in statistical signal processing. In this appendix we present a summary of some important properties of the Wishart distribution for complex-valued data. In particular, we derive a result that is pivotal to a rigorous analysis of the convergence behavior of the standard RLS algorithm, presented in Chapter 13. We begin the discussion with a definition of the complex Wishart distribution.

J.1 DEFINITION

Consider an M-by-M time-averaged (sample) correlation matrix $\mathbf{\Phi}(n)$, defined by

$$\mathbf{\Phi}(n) = \sum_{i=1}^{n} \mathbf{u}(i)\mathbf{u}^H(i) \tag{J.1}$$

where

$$\mathbf{u}(i) = [u_1(i), u_2(i), \ldots, u_M(i)]^T$$

In what follows, we assume that $\mathbf{u}(1), \mathbf{u}(2), \ldots, \mathbf{u}(n)$ $(n > M)$ are *independently and identically distributed.* We may then formally define the *complex Wishart distribution* as follows (Muirhead, 1982):

If $\{u_1(i), u_2(i), \ldots, u_M(i) \mid i = 1, 2, \ldots, n\}$, $n \geq M$, is a sample from the M-dimensional Gaussian distribution $\mathcal{N}(\mathbf{0},\mathbf{R})$, and if $\mathbf{\Phi}(n)$ is the time-averaged correlation matrix defined in Eq. (J.1), then the elements of $\mathbf{\Phi}(n)$ have the complex Wishart distribution $\mathcal{W}_M(n,\mathbf{R})$, which is characterized by the parameters M, n, and $\mathbf{R}$.

In specific terms, we may say that if matrix $\mathbf{\Phi}$ is $\mathcal{W}_M(n,\mathbf{R})$, then the probability density function of $\mathbf{\Phi}$ is

$$f(\mathbf{\Phi}) = \frac{1}{2^{Mn/2}\Gamma_M\left(\frac{1}{2}n\right)(\det(\mathbf{R}))^{n/2}} \operatorname{etr}\left(-\frac{1}{2}\mathbf{R}^{-1}\mathbf{\Phi}\right)(\det(\mathbf{\Phi}))^{(n-M-1)/2} \tag{J.2}$$

where $\det(\bullet)$ denotes the determinant of the enclosed matrix, $\operatorname{etr}(\bullet)$ denotes the exponential raised to the trace of the enclosed matrix, and $\Gamma_M(a)$ is the *multivariate gamma function* defined by

$$\Gamma_M(a) = \int_{\mathbf{A}} \operatorname{etr}(-\mathbf{A})(\det(\mathbf{A}))^{a-(M+1)/2} d\mathbf{A} \tag{J.3}$$

where $\mathbf{A}$ is a positive definite matrix.

J.2 THE CHI-SQUARE DISTRIBUTION AS A SPECIAL CASE

For the special case of a univariate distribution, that is, $M = 1$, Eq. (J.1) reduces to the scalar form:

$$\varphi(n) = \sum_{i=1}^{n} |u(i)|^2 \tag{J.4}$$

Correspondingly, the correlation matrix $\mathbf{R}$ reduces to the variance σ^2. Let

$$\chi^2(n) = \frac{\varphi(n)}{\sigma^2} \tag{J.5}$$

Then, using Eq. (J.2) we may define the normalized probability density function of the normalized random variable $\chi^2(n)$ as

$$f(\chi^2) = \frac{\left(\frac{\chi^2}{2}\right)^{n/2-1} e^{-\chi^2/2}}{2^{n/2}\Gamma\left(\frac{1}{2}n\right)} \tag{J.6}$$

where $\Gamma(1/2n)$ is the (scalar) *gamma function*.[1] The variable $\chi^2(n)$, defined above, is said to have a *chi-square distribution with n degrees of freedom*. We may thus view the complex Wishart distribution as a generalization of the univariate chi-square distribution.

A useful property of a chi-square distribution with n degrees of freedom is the fact that it is *reproductive with respect to* $1/2n$ (Wilks, 1962). That is, the rth moment of $\chi^2(n)$ is

$$E[\chi^{2r}(n)] = \frac{2^r \Gamma\left(\frac{n}{2} + r\right)}{\Gamma\left(\frac{n}{2}\right)} \tag{J.7}$$

Thus, the mean, mean-square, and variance of $\chi^2(n)$ are as follows, respectively:

$$E[\chi^2(n)] = n \tag{J.8}$$

$$E[\chi^4(n)] = n(n+2) \tag{J.9}$$

$$\text{var}[\chi^2(n)] = n(n+2) - n^2 = 2n \tag{J.10}$$

Moreover, from Eq. (J.7) we find that the mean of the reciprocal of $\chi^2(n)$ is

$$\begin{aligned} E\left[\frac{1}{\chi^2(n)}\right] &= \frac{1}{2} \frac{\Gamma\left(\frac{n}{2} - 1\right)}{\Gamma\left(\frac{n}{2}\right)} \\ &= \frac{1}{2} \frac{\Gamma\left(\frac{n}{2} - 1\right)}{\left(\frac{n}{2} - 1\right)\Gamma\left(\frac{n}{2} - 1\right)} = \frac{1}{n-2} \end{aligned} \tag{J.11}$$

[1]For the general case of a complex number g whose real part is positive, the *gamma function* $\Gamma(g)$ is defined by the definite integral (Wilks, 1962)

$$\Gamma(g) = \int_0^\infty x^{g-1} e^{-x} \, dx$$

Integrating it by parts, we readily find that

$$\Gamma(g) = (g-1)\Gamma(g-1)$$

For the case when g is a positive integer, we may thus express the gamma function $\Gamma(g)$ as the factorial

$$\Gamma(g) = (g-1)!$$

When $g > 0$, but not an integer, we have

$$\Gamma(g) = (g-1)\,\Gamma(\delta)$$

where $0 < \delta < 1$. For the particular case of $\delta = 1/2$, we have $\Gamma(\delta) = \sqrt{\pi}$.

J.3 PROPERTIES OF THE COMPLEX WISHART DISTRIBUTION

Returning to the main theme of this appendix, the complex Wishart distribution has some important properties of its own, which are summarized as follows (Muirhead, 1982; Anderson, 1984):

1. If Φ is $\mathcal{W}_M(n, \mathbf{R})$ and $\mathbf{a}$ is any M-by-1 random vector distributed independently of Φ with $P(\mathbf{a} = \mathbf{0}) = 0$ (i.e., the probability that $\mathbf{a} = \mathbf{0}$ is zero), then $\mathbf{a}^H\Phi\mathbf{a}/\mathbf{a}^H\mathbf{R}\mathbf{a}$ is chi-square distributed with n degrees of freedom, and is independent of $\mathbf{a}$.
2. If Φ is $\mathcal{W}_M(n,\mathbf{R})$ and $\mathbf{Q}$ is a matrix of dimensions M-by-k and rank k, then $\mathbf{Q}^H\Phi\mathbf{Q}$ is $\mathcal{W}_k(n,\mathbf{Q}^H\mathbf{R}\mathbf{Q})$.
3. If Φ is $\mathcal{W}_M(n,\mathbf{R})$ and $\mathbf{Q}$ is a matrix of dimensions M-by-k and rank k, then $(\mathbf{Q}^H\Phi^{-1}\mathbf{Q})^{-1}$ is $\mathcal{W}_k(n - M + k,(\mathbf{Q}^H\mathbf{R}^{-1}\mathbf{Q})^{-1})$.
4. If Φ is $\mathcal{W}_M(n,\mathbf{R})$ and $\mathbf{a}$ is any M-by-1 random vector distributed independently of Φ with $P(\mathbf{a} = \mathbf{0}) = 0$, then $\mathbf{a}^H\mathbf{R}^{-1}\mathbf{a}/\mathbf{a}^H\Phi^{-1}\mathbf{a}$ is chi-square distributed with $n - M + 1$ degrees of freedom.
5. Let Φ and $\mathbf{R}$ be partitioned into p and $M - p$ rows and columns, as shown by

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix}$$

If Φ is distributed according to $\mathcal{W}_M(n,\mathbf{R})$, then Φ_{11} is distributed according to $\mathcal{W}_p(n,\mathbf{R}_{11})$.

J.4 EXPECTATION OF THE INVERSE CORRELATION MATRIX $\Phi^{-1}(n)$

Property 4 of the complex Wishart distribution may be used to find the expectation of the inverse correlation matrix $\Phi^{-1}(n)$, which is associated with the convergence of the RLS algorithm in the mean square. Specifically, for any fixed and nonzero α in $\mathbb{R}^M$, we know from Property 4 described above that $\alpha^H\mathbf{R}^{-1}\alpha/\alpha^H\Phi^{-1}\alpha$ is chi-square distributed with $n - M + 1$ degrees of freedom. Let $\chi^2(n - M + 1)$ denote this ratio. Then, using the result described in Eq. (J.11), we may write

$$E[\alpha^H\Phi^{-1}(n)\alpha] = \alpha^H\mathbf{R}^{-1}\alpha E\left[\frac{1}{\chi^2(n - M + 1)}\right]$$

$$= \frac{1}{n - M - 1}\alpha^H\mathbf{R}^{-1}\alpha, \qquad n > M + 1$$

which, in turn, implies that

$$E[\Phi^{-1}(n)] = \frac{1}{n - M - 1}\mathbf{R}^{-1}, \qquad n > M + 1 \tag{J.12}$$