CHAPTER

3

Spectrum Analysis

The autocorrelation function is a *time-domain description* of the second-order statistics of a stochastic process. The *frequency-domain description* of the second-order statistics of such a process is the *power spectral density*, which is also commonly referred to as the *power spectrum* or simply *spectrum*. Indeed, the power spectral density of a stochastic process is firmly established as the most useful description of the time series commonly encountered in engineering and physical sciences.

This chapter is devoted in part to the definition of the power spectral density of a wide-sense stationary discrete-time stochastic process, the properties of power spectral density, and methods for its estimation. We begin the discussion by establishing a mathematical definition of the power spectral density of a stationary process in terms of the Fourier transform of a single realization of the process.

3.1 POWER SPECTRAL DENSITY

Consider an infinitely long time series $u(n)$, $n = 0, \pm 1, \pm 2, \ldots$, that represents a *single realization* of a wide-sense stationary discrete-time stochastic process. For convenience of presentation, we assume that the process has zero mean. Initially, we focus our attention on a *windowed* portion of this time series, written as

$$u_N(n) = \begin{cases} u(n), & n = 0, 1, \ldots, N-1 \\ 0, & n > N, n < 0 \end{cases} \tag{3.1}$$

where N is the *total length (duration) of the window*. By definition, the *discrete-time Fourier transform* of the windowed time series $u_N(n)$ is given by

$$U_N(\omega) = \sum_{n=0}^{N-1} u_N(n) e^{-j\omega n} \tag{3.2}$$

where ω is the *angular frequency*, lying in the interval $(-\pi, \pi]$. In general, $U_N(\omega)$ is complex valued; specifically, its *complex conjugate* is given by

$$U_N^*(\omega) = \sum_{k=0}^{N-1} u_N^*(k) e^{j\omega k} \tag{3.3}$$

where the asterisk denotes complex conjugation. In Eq. (3.3) we have used the variable k to denote discrete time for reasons that will become apparent immediately. In particular, we may multiply Eq. (3.2) by (3.3) to express the squared magnitude of $U_N(n)$ as follows:

$$|U_N(\omega)|^2 = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} u_N(n) u_N^*(k) e^{-j\omega(n-k)} \tag{3.4}$$

Each realization $U_N(n)$ produces such a result. The *expected* result is obtained by taking the statistical expectation of both sides of Eq. (3.4), and interchanging the order of expectation and double summation:

$$E[|U_N(\omega)|^2] = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} E[u_N(n) u_N^*(k)] e^{-j\omega(n-k)} \tag{3.5}$$

We now recognize that for the wide-sense stationary discrete-time stochastic process under discussion, the autocorrelation function of $u_N(n)$ for lag $n - k$ is

$$r_N(n-k) = E[u_N(n) u_N^*(k)] \tag{3.6}$$

which may be rewritten as follows, in light of the defining equation (3.1):

$$r_N(n-k) = \begin{cases} E[u(n)u^*(k)] = r(n-k) & \text{for } 0 \le (n,k) \le N-1 \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

Accordingly, Eq. (3.6) takes on the form

$$E[|U_N(\omega)|^2] = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} r(n-k) e^{-j\omega(n-k)} \tag{3.8}$$

Let $l = n - k$, and so rewrite Eq. (3.8) as follows:

$$\frac{1}{N} E[|U_N(\omega)|^2] = \sum_{l=-N+1}^{N-1} \left(1 - \frac{|l|}{N}\right) r(l) e^{-j\omega l} \tag{3.9}$$

Equation (3.9) may be interpreted as the discrete-time Fourier transform of the product of two time functions: the autocorrelation function $r_N(l)$ for lag l, and a triangular window

$w_B(l)$ known as the *Barlett window*. The latter function is defined by

$$w_B(l) = \begin{cases} 1 - \dfrac{|l|}{N}, & |l| \leq N - 1 \\ 0, & |l| \geq N \end{cases} \tag{3.10}$$

As N approaches infinity, the Barlett window $w_B(l)$ approaches unity for all l. Correspondingly, we may write

$$\lim_{N\to\infty} \frac{1}{N} E[|U_N(\omega)|^2] = \sum_{l=-\infty}^{\infty} r(l) e^{-j\omega l} \tag{3.11}$$

where $r(l)$ is the autocorrelation function of the original time series $u(n)$, assumed to have infinite length. The quantity $U_N(\omega)$ is the discrete-time Fourier transform of a rectangular *windowed* portion of this time series that has length N.

Equation (3.11) leads us to define the quantity

$$S(\omega) = \lim_{N\to\infty} \frac{1}{N} E[|U_N(\omega)|^2] \tag{3.12}$$

where the quantity $|U_N(\omega)|^2/N$ is called the *periodogram* of the windowed time series $u_N(n)$. Note that the order of expectation and limiting operations indicated in Eq. (3.12) cannot be changed. Note also that the periodogram converges to $S(\omega)$ only in the mean value, but *not* in mean square or any other meaningful way.

When the limit in Eq. (3.12) exists, the quantity $S(\omega)$ has the following interpretation (Priestley, 1981):

$$\begin{aligned} S(\omega)\, d\omega = {} & \text{average of the contribution to the total power from components} \\ & \text{of a wide-sense stationary stochastic process with angular} \\ & \text{frequencies located between } \omega \text{ and } \omega + d\omega\text{; the average is} \\ & \text{taken over all possible realizations of the process} \end{aligned} \tag{3.13}$$

Accordingly, the quantity $S(\omega)$ is the "spectral density of expected power," which is abbreviated as the *power spectral density* of the process. Thus, equipped with the definition of power spectral density given in Eq. (3.12), we may now rewrite Eq. (3.11) as

$$S(\omega) = \sum_{l=-\infty}^{\infty} r(l) e^{-j\omega l} \tag{3.14}$$

In summary, Eq. (3.12) gives a basic definition of the power spectral density of a wide-sense stationary stochastic process, and Eq. (3.14) defines the mathematical relationship between the autocorrelation function and the power spectral density of such a process.

3.2 PROPERTIES OF POWER SPECTRAL DENSITY

Property 1. *The autocorrelation function and power spectral density of a wide-sense stationary stochastic process form a Fourier transform pair.*

Consider a wide-sense stationary stochastic process represented by the time series $u(n)$, assumed to be of infinite length. Let $r(l)$ denote the autocorrelation function of such a process for lag l, and let $S(\omega)$ denote its power spectral density. According to Property 1, these two quantities are related by the pair of relations:

$$S(\omega) = \sum_{l=-\infty}^{\infty} r(l)e^{-j\omega l}, \qquad -\pi < \omega \leq \pi \tag{3.15}$$

and

$$r(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)e^{j\omega l}d\omega, \qquad l = 0, \pm 1, \pm 2, \ldots \tag{3.16}$$

Equation (3.15) states that *the power spectral density is the discrete-time Fourier transform of the autocorrelation function.* On the other hand, Eq. (3.16) states that *the autocorrelation function is the inverse discrete-time Fourier transform of the power spectral density.* This fundamental pair of equations constitutes the *Einstein–Wiener–Khintchine* relations.

In a way, we already have a proof of this property. Specifically, Eq. (3.15) is merely a restatement of Eq. (3.14), previously established in Section 3.1. Equation (3.16) follows directly from this result by invoking the formula for the inverse discrete-time Fourier transform.

Property 2. *The frequency support of the power spectral density $S(\omega)$ is the Nyquist interval $-\pi < \omega \leq \pi$.*

Outside this interval, $S(\omega)$ is periodic as shown by

$$S(\omega + 2k\pi) = S(\omega) \quad \text{for integer } k \tag{3.17}$$

Property 3. *The power spectral density of a stationary discrete-time stochastic process is real.*

To prove this property, we rewrite Eq. (3.15) as

$$S(\omega) = r(0) + \sum_{k=1}^{\infty} r(k)e^{-j\omega k} + \sum_{k=-\infty}^{-1} r(k)e^{-j\omega k}$$

Replacing k with $-k$ in the third term on the right-hand side of this equation, and recognizing that $r(-k) = r^*(k)$, we get

$$\begin{aligned} S(\omega) &= r(0) + \sum_{k=1}^{\infty} [r(k)e^{-j\omega k} + r^*(k)e^{j\omega k}] \\ &= r(0) + 2\sum_{k=1}^{\infty} \mathrm{Re}[r(k)e^{-j\omega k}] \end{aligned} \tag{3.18}$$

where Re denotes the *real part operator.* Equation (3.18) shows that the power spectral density $S(\omega)$ is a real-valued function of ω. It is because of this property that we have used the notation $S(\omega)$ rather than $S(e^{j\omega})$ for the power spectral density.

Property 4. *The power spectral density of a real-valued stationary discrete-time stochastic process is even (i.e., symmetric); if the process is complex-valued, its power spectral density is not necessarily even.*

For a real-valued stochastic process, we find that $S(-\omega) = S(\omega)$, indicating that $S(\omega)$ is an even function of ω; that is, it is symmetric about the origin. If, however, the process is complex-valued, then $r(-k) = r^*(k)$, in which case we find that $S(-\omega) \neq S(\omega)$, and $S(\omega)$ is *not* an even function of ω.

Property 5. *The mean-square value of a stationary discrete-time stochastic process equals, except for the scaling factor* $1/2\pi$, *the area under the power spectral density curve for* $-\pi < \omega \leq \pi$.

This property follows directly from Eq. (3.16), evaluated for $l = 0$. For this condition, we may thus write

$$r(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\ d\omega \tag{3.19}$$

Since $r(0)$ equals the mean-square value of the process, we see that Eq. (3.19) is a mathematical description of Property 5. The mean-square value of a process is equal to the *expected power* of the process developed across a load resistor of 1 ohm. On this basis, the terms "expected power" and "mean-square value" are used interchangeably in what follows.

Property 6. *The power spectral density of a stationary discrete-time stochastic process is nonnegative.*

That is,

$$S(\omega) \geq 0 \qquad \text{for all } \omega \tag{3.20}$$

This property follows directly from the basic formula of Eq. (3.12), reproduced here for convenience of presentation:

$$S(\omega) = \lim_{N\to\infty} \frac{1}{N} E[|U_N(\omega)|^2]$$

We first note that $|U_N(\omega)|^2$, representing the squared magnitude of the discrete-time Fourier transform of a windowed portion of the time series $u(n)$, is nonnegative for all ω. The expectation $E[|U_N(\omega)|^2]$ is also nonnegative for all ω. Thus, using the basic definition of $S(\omega)$ in terms of $U_N(\omega)$, the property described by Eq. (3.20) follows immediately.

3.3 TRANSMISSION OF A STATIONARY PROCESS THROUGH A LINEAR FILTER

Consider a discrete-time filter that is *linear*, *time invariant*, and *stable*. Let the filter be characterized by the *discrete transfer function* $H(z)$, defined as the *ratio of the z-transform of the filter output to the z-transform of the filter input*. Suppose that we feed the filter with

Figure 3.1 Transmission of stationary process through a discrete-time linear filter.

a stationary discrete-time stochastic process of power spectral density $S(\omega)$, as in Fig. 3.1. Let $S_o(\omega)$ denote the power spectral density of the filter output. We may then write

$$S_o(\omega) = |H(e^{j\omega})|^2 S(\omega) \tag{3.21}$$

where $H(e^{j\omega})$ is the *frequency response* of the filter. The frequency response $H(e^{j\omega})$ equals the discrete transfer function $H(z)$ evaluated on the unit circle in the z-plane. The important feature of this result is that the value of the output spectral density at angular frequency ω depends purely on the squared *amplitude response* of the filter and the input power spectral density at the same angular frequency ω.

Equation (3.21) is a fundamental relation in stochastic process theory. To prove it, we may proceed as follows. Let the time series $y(n)$ denote the filter output in Fig. 3.1, produced in response to the time series $u(n)$ applied to the filter input. Assuming that $u(n)$ represents a single realization of a wide-sense stationary discrete-time stochastic process, we find that $y(n)$ also represents a single realization of a wide-sense stationary discrete-time stochastic process modified by the filtering operation. Thus, given that the autocorrelation function of the filter input $u(n)$ is written as

$$r_u(l) = E[u(n)u^*(n-l)]$$

we may express the autocorrelation function of the filter output $y(n)$ in a corresponding way as

$$r_y(l) = E[y(n)y^*(n-l)] \tag{3.22}$$

where $y(n)$ is related to $u(n)$ by the convolution sum

$$y(n) = \sum_{i=-\infty}^{\infty} h(i)u(n-i) \tag{3.23}$$

Similarly, we may write

$$y^*(n-l) = \sum_{k=-\infty}^{\infty} h^*(k)u^*(n-l-k) \tag{3.24}$$

Substituting Eqs. (3.23) and (3.24) in (3.22), and interchanging the orders of expectation and summations, we find that the autocorrelation functions $r_y(l)$ and $r_u(l)$, for lag l, are related as follows:

$$r_y(l) = \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h(i)h^*(k)r_u(k-i+l) \tag{3.25}$$

Finally, taking the discrete-time Fourier transforms of both sides of Eq. (3.25), and invoking Property 1 of the power spectral density and the fact that the transfer function of a linear filter is equal to the Fourier transform of its impulse response, we get the result described in Eq. (3.21).

Power Spectrum Analyzer

Suppose that the discrete-time linear filter in Fig. 3.1 is designed to have a bandpass characteristic. That is, the amplitude response of the filter is defined by

$$|H(e^{j\omega})| = \begin{cases} 1, & |\omega - \omega_c| \le \Delta\omega \\ 0, & \text{remainder of the interval } -\pi < \omega \le \pi \end{cases} \tag{3.26}$$

This amplitude response is depicted in Fig. 3.2. We assume that the *angular bandwidth* of the filter, $2\Delta\omega$, is small enough for the spectrum inside this bandwidth to be essentially constant. Then using Eq. (3.21) we may write

$$S_o(\omega) = \begin{cases} S(\omega_c), & |\omega - \omega_c| \le \Delta\omega \\ 0, & \text{remainder of the interval } -\pi < \omega \le \pi \end{cases} \tag{3.27}$$

Next, using Properties 4 and 5 of the power spectral density, we may express the mean-square value of the filter output resulting from a real-valued stochastic input as

$$\begin{aligned} P_o &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_o(\omega)\,d\omega \\ &= \frac{2\Delta\omega}{2\pi}S(\omega_c) + \frac{2\Delta\omega}{2\pi}S(-\omega_c) \\ &= 2\frac{\Delta\omega}{\pi}S(\omega_c) \qquad \text{for real data} \end{aligned}$$

Equivalently, we may write

$$S(\omega_c) = \frac{\pi P_o}{2\Delta\omega} \tag{3.28}$$

where $\Delta\omega/\pi$ is that fraction of the Nyquist interval that corresponds to the passband of the filter. Equation (3.28) states that the value of the power spectral density of the filter input $u(n)$, measured at the center frequency ω_c of the filter, is equal to the mean-square value P_o of the filter output, scaled by a constant factor. We may thus use Eq. (3.28) as the mathematical basis for building a *power spectrum analyzer*, as depicted in Fig. 3.3. Ideally, the discrete-time bandpass filter employed here should satisfy two requirements: *fixed bandwidth* and *adjustable center frequency*. Clearly, in a practical filter design, we can only approximate these two ideal requirements. Note also that the reading of the *average power meter* at the output end of Fig. 3.3 approximates (for finite averaging time) the expected power of an ergodic process $y(n)$.

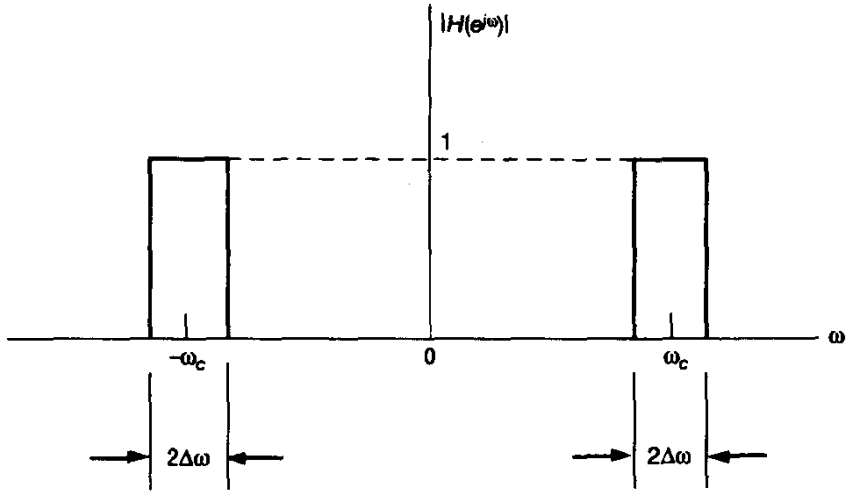

Figure 3.2 Ideal bandpass characteristic.

Example 1: White Noise

A stochastic process of zero mean is said to be *white* if its power spectral density $S(\omega)$ is constant for all frequencies, as shown by

$$S(\omega) = \sigma^2 \quad \text{for } -\pi < \omega \leq \pi$$

where σ^2 is the variance of a sample taken from the process. Suppose that this process is passed through a discrete-time bandpass filter, characterized as in Fig. 3.2. Hence, from Eq. (3.28), we find that the mean-square value of the filter output is

$$P_o = \frac{2\sigma^2\Delta\omega}{\pi}$$

White noise has the property that any two of its samples are uncorrelated, as shown by the autocorrelation function

$$r(\tau) = \sigma^2\delta_{\tau,0}$$

where $\delta_{\tau,0}$ is the Kronecker delta:

$$\delta_{\tau,0} = \begin{cases} 1, & \tau = 0 \\ 0, & \text{otherwise} \end{cases}$$

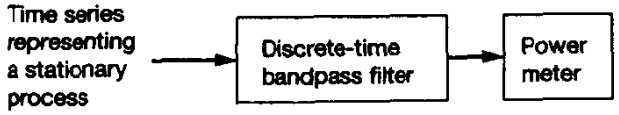

Figure 3.3 Power spectrum analyzer.

3.4 CRAMÉR SPECTRAL REPRESENTATION FOR A STATIONARY PROCESS

Equation (3.12) provides one way of defining the power spectral density of a wide-sense stationary process. Another way of defining the power spectral density is to use the *Cramér spectral representation for a stationary process*. According to this representation, a sample $u(n)$ of a discrete-time stochastic process is written as an inverse Fourier transform (Thomson, 1982, 1988):

$$u(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega n} dZ(\omega) \tag{3.29}$$

If the process represented by the time series $u(n)$ is wide-sense stationary with no periodic components, then the *increment process* $dZ(\omega)$ has the following three properties:

1. The mean of the increment process $dZ(\omega)$ is zero; that is,

$$E[dZ(\omega)] = 0 \qquad \text{for all } \omega \tag{3.30}$$

2. The energy of the increment process $dZ(\omega)$ at different frequencies is uncorrelated; that is,

$$E[dZ(\omega)dZ^*(\nu)] = 0 \qquad \text{for } \nu \neq \omega \tag{3.31}$$

3. The expected value of $|dZ(\omega)|^2$ defines the spectrum $S(\omega)\, d\omega$; that is,

$$E[|dZ(\omega)|^2] = S(\omega)\, d\omega \tag{3.32}$$

In other words, for a wide-sense stationary discrete-time stochastic process represented by the time series $u(n)$, the increment process $dZ(\omega)$ defined by Eq. (3.29) is a *zero-mean orthogonal process*. More precisely, $dZ(\omega)$ may be viewed as a "white process" described in the frequency domain in a manner similar to the time-domain description of ordinary white noise.

Equation (3.32), in conjunction with Eq. (3.31), provides another basic definition for the power spectral density $S(\omega)$. Complex-conjugating both sides of Eq. (3.29) and using ν in place of ω, we get

$$u^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\nu n} dZ^*(\nu) \tag{3.33}$$

Hence, multiplying Eq. (3.29) by (3.33), we may express the squared magnitude of $u(n)$ as

$$|u(n)|^2 = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{jn(\omega-\nu)} dZ(\omega)\, dZ^*(\nu) \tag{3.34}$$

Next, taking the statistical expectation of Eq. (3.34), and interchanging the order of expectation and double integration, we get

$$E[|u(n)|]^2 = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{jn(\omega-\nu)} E[dZ(\omega)\, dZ^*(\nu)] \tag{3.35}$$

If we now use the two basic properties of the increment process $dZ(\omega)$ described by Eqs. (3.31) and (3.32), we may simplify Eq. (3.35) into the form

$$E[|u(n)|^2] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)\, d\omega \tag{3.36}$$

The expectation $E[|u(n)|^2]$ on the left-hand side of Eq. (3.36) is recognized as the mean-square value of the complex sample $u(n)$. The right-hand side of this equation equals the total area under the curve of the power spectral density $S(\omega)$, scaled by the factor $1/2\pi$. Accordingly, Eq. (3.36) is merely a restatement of Property 5 of the power spectral density $S(\omega)$, described by Eq. (3.19).

The Fundamental Equation

Consider the time series $u(0), u(1), \ldots, u(N-1)$, consisting of N observations (samples) of a wide-sense stationary stochastic process. The discrete-time Fourier transform of this time series is given by

$$U_N(\omega) = \sum_{n=0}^{N-1} u(n) e^{-j\omega n} \tag{3.37}$$

According to the Cramér spectral representation of the process, the observation $u(n)$ is given by Eq. (3.29). Hence, using the dummy variable ν in place of ω in Eq. (3.29), and then substituting the result in Eq. (3.37), we get

$$U_N(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=0}^{N-1} (e^{-j(\omega-\nu)n})\, dZ(\nu) \tag{3.38}$$

where we have interchanged the order of summation and integration. Define

$$K_N(\omega) = \sum_{n=0}^{N-1} e^{-j\omega n} \tag{3.39}$$

which is known as the *Dirichlet kernel*. The kernel $K_N(\omega)$ represents a geometric series with a first term of unity, a common ratio of $e^{-j\omega}$, and a total number of terms equal to N. Summing this series, we may redefine the kernel $K_N(\omega)$ as follows:

$$\begin{aligned} K_N(\omega) &= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\ &= \frac{\sin(N\omega/2)}{\sin(\omega/2)} \exp\left[-\frac{1}{2} j\omega(N-1)\right] \end{aligned} \tag{3.40}$$

Note that $K_N(0) = N$. Returning to Eq. (3.38), we may use the definition of the Dirichlet kernel $K_N(\omega)$ given in Eq. (3.39) to rewrite $U_N(\omega)$ as follows:

$$U_N(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\omega - \nu) dZ(\nu) \tag{3.41}$$

The integral equation (3.41) is a *linear* relation, referred to as the *fundamental equation* of power spectrum analysis.

An integral equation is one that involves an *unknown* function under the integral sign. In the context of power spectrum analysis as described by Eq. (3.41), the increment variable $dZ(\omega)$ is the unknown function, and $U_N(\omega)$ is known. Accordingly, Eq. (3.41) may be viewed as an example of a *Fredholm integral equation of the first kind* (Morse and Feshbach, 1953; Whittaker and Watson, 1965).

Note that $U_N(\omega)$ may be inverse Fourier transformed to recover the original data. It follows therefore that $U_N(\omega)$ is a *sufficient statistic* of the power spectral density. This property makes the use of Eq. (3.41) for spectrum analysis all the more important.

3.5 POWER SPECTRUM ESTIMATION

An issue of practical importance is how to *estimate* the power spectral density of a wide-sense stationary process. Unfortunately, this issue is complicated by the fact that there is a bewildering array of power spectrum estimation procedures, with each procedure purported to have or to show some optimum property. The situation is made worse by the fact that unless care is taken in the selection of the right method, we may end up with misleading conclusions.

Two philosophically different families of power spectrum estimation methods may be identified in the literature: *parametric methods* and *nonparametric methods*. The basic ideas behind these methods are discussed in the sequel.

Parametric Methods

In parametric methods of spectrum estimation we begin by postulating a *stochastic model* for the situation at hand. Depending on the specific form of stochastic model adopted, we may identify three different parametric approaches for spectrum estimation.

1. *Model identification procedures.* In this class of parametric methods, a rational function or a polynomial in $e^{-j\omega}$ is assumed for the transfer function of the model, and a white-noise source is used to drive the model, as depicted in Fig. 3.4. The power spectrum of the resulting model output provides the desired spectrum estimate. Depending on the application of interest, we may adopt one of the following models (Kay and Marple, 1981; Marple, 1987; Kay, 1988):
   (i) *Autoregressive (AR) model* with an all-pole transfer function.
   (ii) *Moving average (MA) model* with an all-zero transfer function.
   (iii) *Autoregressive–moving average (ARMA) model* with pole-zero transfer function.

   The resulting power spectra measured at the outputs of these models are referred to as AR, MA, and ARMA spectra, respectively. With reference to the input–output relation of Eq. (3.21), let the power spectrum $S(\omega)$ of the model input be put

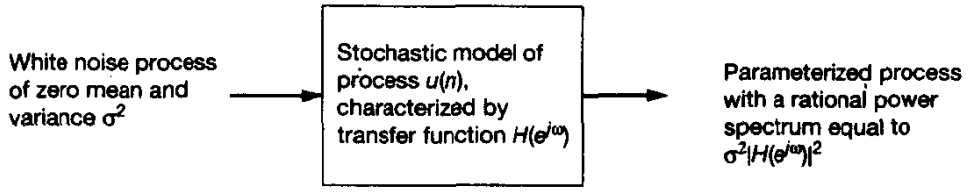

Figure 3.4 Rationale of model identification procedure for power spectrum estimation.

equal to the white noise variance σ^2. We then find that the power spectrum $S_o(\omega)$ of the model output is equal to the squared amplitude response $|H(e^{j\omega})|^2$ of the model, multiplied by σ^2. The problem thus becomes one of estimating the model parameters [i.e., parametrizing the transfer function $H(e^{j\omega})$] such that the process produced at the model output provides an acceptable representation (in some statistical sense) of the stochastic process under study. Such an approach to power spectrum estimation may indeed be viewed as a problem in *model (system) identification*.

Among the model-dependent spectra defined herein, the AR spectrum is by far the most popular. The reason for this popularity is twofold: (1) the *linear* form of the system of simultaneous equations involving the unknown AR model parameters, and (2) the availability of efficient algorithms for computing the solution.

2. *Minimum variance distortionless response method.* To describe this second parametric approach for power spectrum estimation, consider the situation depicted in Fig. 3.5. The process $u(n)$ is applied to a transversal filter (i.e., discrete-time filter with an all-zero transfer function). In the *minimum variance distortionless response (MVDR) method*, the filter coefficients are chosen so as to minimize the variance (which is the same as expected power for a zero-mean process) of the filter output, subject to the constraint that the frequency response of the filter is equal to unity at some angular frequency ω_0. Under this constraint, the process $u(n)$ is passed through the filter with *no distortion* at the angular frequency ω_0. Moreover, signals at angular frequencies other than ω_0 tend to be attenuated.
3. *Eigendecomposition-based methods.* In this final class of parametric spectrum estimation methods, the eigendecomposition of the ensemble-averaged correla-

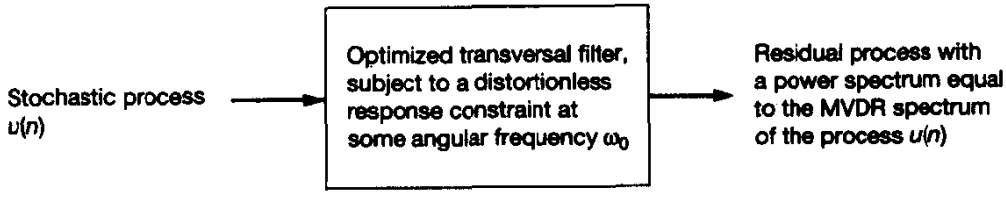

Figure 3.5 Rationale of MVDR procedure for power spectrum estimation.

tion matrix $\mathbf{R}$ of the process $u(n)$ is used to define two disjoint subspaces: *signal subspace* and *noise subspace*. This form of partitioning is then exploited to derive an appropriate algorithm for estimating the power spectrum (Schmidt, 1979, 1981). Eigenanalysis and the notion of subspace decomposition are discussed in the next chapter.

Nonparametric Methods

In nonparametric methods of power spectrum estimation, on the other hand, no assumptions are made with respect to the stochastic process under study. The starting point in the discussion is the fundamental equation (3.41). Depending on the way in which this equation is interpreted, we may distinguish two different nonparametric approaches:

1. *Periodogram-based methods.* Traditionally, the fundamental equation (3.41) is treated as a *convolution* of two frequency functions. One frequency function, $U(\omega)$, represents the discrete-time Fourier transform of an *infinitely long* time series, $u(n)$; this function arises from the definition of the increment variable $dZ(\omega)$ as the product of $U(\omega)$ and the frequency increment $d\omega$. The other frequency function is the kernel $K_N(\omega)$, defined by Eq. (3.40). This approach leads us to consider Eq. (3.12) as the basic definition of the power spectral density $S(\omega)$, and therefore the *periodogram* $|U_N(\omega)|^2/N$ as the starting point for the data analysis. However, the periodogram suffers from a *serious limitation in the sense that it is not a sufficient statistic for the power spectral density.* This implies that the phase information ignored in the use of the periodogram is essential. Consequently, the statistical insufficiency of the periodogram is inherited by any estimate that is based on or equivalent to the periodogram.
2. *Multiple-window method.* A more constructive nonparametric approach is to treat the fundamental equation (3.41) as a *Fredholm integral equation of the first kind* for the increment variable $dZ(\omega)$; the goal here is to obtain an *approximate solution* for the equation with statistical properties that are close to those of $dZ(\omega)$ in some sense (Thomson, 1982). The key to the attainment of this important goal is the use of windows defined by a set of special sequences known as *Slepian sequences*[1] or *discrete prolate spheroidal sequences*, which are fundamental to the study of time- and frequency-limited systems. The remarkable property of this family of windows is that their energy distributions add up in a very special way that collectively defines an ideal (ideal in the sense of the total in-bin versus out-of-bin energy concentration) rectangular frequency bin. This property, in turn, allows us to trade spectral resolution for improved spectral properties (i.e., reduced variance of the spectral estimate).

[1]Detailed information on Slepian sequences is given in Slepian (1978). A method for computing them, for large data length, is given in the appendix of the paper by Thomson (1982). For additional information, see the references listed in Thomson's paper. Mullis and Scharf (1991) also present an informative discussion of the role of Slepian sequences in spectrum analysis.

In general, a discrete-time stochastic process $u(n)$ has a *mixed spectrum*, in that its power spectrum contains two components: a deterministic component and a continuous component. The *deterministic component* represents the *first moment* of the increment process $dZ(\omega)$; it is explicitly given by

$$E[dZ(\omega)] = \sum_k a_k \delta(\omega - \omega_k)\, d\omega \tag{3.42}$$

where $\delta(\omega)$ is the *Dirac delta function* defined in the frequency domain. The ω_k are the angular frequencies of *periodic* or *line components* contained in the process $u(n)$, and the a_k are their amplitudes. The continuous component, on the other hand, represents the *second central moment* of the increment process $dZ(\omega)$, as shown by

$$E[|dZ(\omega) - E[dZ(\omega)]|^2] = S(\omega)\, d\omega \tag{3.43}$$

It is important that the distinction between the first and second moments is carefully noted.

Spectra computed using the parametric methods tend to have sharper peaks and higher resolution than those obtained from the nonparametric (classical) methods. The application of these parametric methods is therefore well suited for estimating the deterministic component and, in particular, for locating the frequencies of periodic components in additive white noise when the signal-to-noise ratio is high. Another well-proven technique for estimating the deterministic component is the classical method of maximum likelihood, which is discussed in Appendix D. Of course, if the physical laws governing the generation of a process match a stochastic model (e.g., AR model) in an exact manner or approximately in some statistical sense, then the parametric method corresponding to that model may be used to estimate the power spectrum of the process. If, however, the stochastic process of interest has a purely continuous power spectrum, and the underlying physical mechanism responsible for the generation of the process is unknown, then the recommended procedure is the non-parametric method of multiple windows.

In this book, we confine our attention to classes 1 and 2 of parametric methods of spectrum estimation, as their theory fits naturally under the umbrella of adaptive filters. For a comprehensive discussion of the other methods of spectrum analysis, the reader is referred to the books by Gardner (1987), Marple (1987), and Kay (1988), the paper by Thomson (1982), and a chapter contribution by Mullis and Scharf (1991).

3.6 OTHER STATISTICAL CHARACTERISTICS OF A STOCHASTIC PROCESS

In the material presented in the previous chapter and up to this point in the present chapter, we have focused our attention on a partial characterization of a discrete-time stochastic process. According to this particular characterization, we only need to specify the mean as the first moment of the process and its autocorrelation function as the second moment. Since the autocorrelation function and power spectral density form a Fourier-transform pair, we may equally well specify the power spectral density in place of the autocorrelation function. The use of second-order statistics as described herein is adequate for the study of linear adaptive filters. However, when we move on later in the book to consider

difficult applications (e.g., blind deconvolution) that are beyond the reach of linear adaptive filters, we will have to resort to the use of other statistical properties of a stochastic process.

Two particular statistical properties that bring in additional information about a stochastic process, which can prove useful in practice, are as follows:

1. *High-order statistics.* An obvious way of expanding the characterization of a stationary stochastic process is to include *higher-order statistics* (HOS) of the process. This is done by invoking the use of *cumulants* and their Fourier transforms, known as *polyspectra*. Indeed, cumulants and polyspectra of a zero-mean stochastic process may be viewed as generalizations of the autocorrelation function and power spectral density, respectively. It is important to note that higher-order statistics are only meaningful in the context of *non-Gaussian processes*. Furthermore, to exploit them, we need to use some form of nonlinear filtering.
2. *Cyclostationarity.* In an important class of stochastic processes commonly encountered in practice, the mean and autocorrelation function of the process exhibit periodicity, as in

$$\mu(t_1 + T) = \mu(t_1) \tag{3.44}$$

$$r(t_1 + T, t_2 + T) = r(t_1, t_2) \tag{3.45}$$

for all t_1 and t_2. Both t_1 and t_2 represent values of the continuous-time variable t, and T denotes period. A stochastic process satisfying Eqs. (3.44) and (3.45) is said to be *cyclostationary* in the wide sense (Franks, 1969; Gardner and Franks, 1975; Gardner, 1994). Modeling a stochastic process as cyclostationary adds a new dimension, namely, the period T, to the partial description of the process. Examples of cyclostationary processes include a modulated process obtained by varying the amplitude, phase, or frequency of a sinusoidal carrier. Note that, unlike higher-order statistics, cyclostationarity can be exploited by means of linear filtering.

In the sequel, we will discuss these two specific aspects of stochastic processes under the section headings "polyspectra" and "spectral-correlation density." As already mentioned, polyspectra provide a frequency-domain description of the higher-order statistics of a stationary stochastic process. By the same token, spectral-correlation density provides a frequency-domain description of a cyclostationary stochastic process.

3.7 POLYSPECTRA

Consider a stationary stochastic process $u(n)$ with zero mean; that is,

$$E[u(n)] = 0 \qquad \text{for all } n$$

Let $u(n)$, $u(n + \tau_1), \ldots, u(n + \tau_{k-1})$ denote the random variables obtained by observing this stochastic process at times n, $n + \tau_1, \ldots, n + \tau_{k-1}$, respectively. These random variables form the k-by-1 vector:

$$\mathbf{u} = [u(n), u(n + \tau_1), \ldots, u(n + \tau_{k-1})]^T \tag{3.46}$$

Correspondingly, define a k-by-1 vector:

$$\mathbf{z} = [z_1, z_2, \ldots, z_k]^T \tag{3.47}$$

We may then define the *k*th-*order cumulant* of the stochastic process $u(n)$, denoted by $c_k(t_1, t_2, \ldots, \tau_{k-1})$, as the coefficient of the vector $\mathbf{z}$ in the Taylor expansion of the *cumulant-generating function* (Priestley, 1981; Swami and Mendel, 1990; Gardner, 1994):

$$K(\mathbf{z}) = \ln E[\exp(\mathbf{z}^T\mathbf{u})] \tag{3.48}$$

The kth-order cumulant of the process $u(n)$ is thus defined in terms of its joint moments of orders up to k; to simplify the presentation in this section, we assume that $u(n)$ is real valued. Specifically, the second-, third-, and fourth-order cumulants are given, respectively, by

$$c_2(\tau) = E[u(n)u(n + \tau)] \tag{3.49}$$

$$c_3(\tau_1, \tau_2) = E[u(n)u(n + \tau_1)u(n + \tau_2)] \tag{3.50}$$

and

$$\begin{aligned} c_4(\tau_1, \tau_2, \tau_3) = {} & E[u(n)u(n + \tau_1)u(n + \tau_2)u(n + \tau_3)] \\ & - E[u(n)u(n + \tau_1)]E[u(n + \tau_2)u(n + \tau_3)] \\ & - E[u(n)u(n + \tau_2)]E[u(n + \tau_3)u(n + \tau_1)] \\ & - E[u(n)u(n + \tau_3)]E[u(n + \tau_1)u(n + \tau_2)] \end{aligned} \tag{3.51}$$

From the definitions given in Eqs. (3.49) to (3.51), we note the following:

1. The second-order cumulant $c_2(\tau)$ is the same as the autocorrelation function $r(t)$.
2. The third-order cumulant $c_3(\tau_1, \tau_2)$ is the same as the third-order moment $E[u(n)u(n + \tau_1)u(n + \tau_2)]$.
3. The fourth-order cumulant $c_4(\tau_1, \tau_2, \tau_3)$ is *different* from the fourth-order moment $E[u(n)u(n + \tau_1)u(n + \tau_2)u(n + \tau_3)]$. In order to generate the fourth-order cumulant, we need to know the fourth-order moment and six different values of the autocorrelation function.

Note that the kth-order cumulant $c(\tau_1, \tau_2, \ldots, \tau_{k-1})$ does not depend on time n. For this to be valid, however, the process $u(n)$ has to be stationary up to order k. A process $u(n)$ is said to be *stationary up to order k* if, for any admissible $\{n_1, n_2, \ldots, n_p\}$ all the joint moments up to order k of $\{u(n_1), u(n_2), \ldots, u(n_p)\}$ exist and equal the corresponding

joint moments up to order k of $\{u(n_1 + \tau), u(n_2 + \tau), \ldots, u(n_p + \tau)\}$ where $\{n_1 + \tau, n_2 + \tau, \ldots, n_p + \tau\}$ is an admissible set too (Priestley, 1981).

Consider next a linear time-invariant system, characterized by the impulse response h_n. Let the system be excited by a process $x(n)$ consisting of independent and identically distributed (iid) random variables. Let $u(n)$ denote the resulting system output. The kth-order cumulant of $u(n)$ is given by

$$c_k(\tau_1, \tau_2, \cdots, \tau_{k-1}) = \gamma_k \sum_{i=-\infty}^{\infty} h_i h_{i+\tau_1} \cdots h_{i+\tau_{k-1}} \tag{3.52}$$

where γ_k is the kth-order cumulant of the input process $x(n)$. Note that the summation term on the right-hand side of Eq. (3.52) has a form similar to that of a kth-order moment, except that the expectation operator has been replaced by a summation.

The *kth-order polyspectrum* (or *kth-order cumulant spectrum*) is defined by (Priestley, 1981; Nikias and Raghuveer, 1987):

$$C_k(\omega_1, \omega_2, \ldots, \omega_{k-1}) = \sum_{\tau_1=-\infty}^{\infty} \cdots \sum_{\tau_{k-1}=-\infty}^{\infty} c_k(\tau_1, \tau_2, \ldots, \tau_{k-1}) \cdot \exp[-j(\omega_1\tau_1 + \omega_2\tau_2 + \ldots + \omega_{k-1}\tau_{k-1})] \tag{3.53}$$

A sufficient condition for the existence of the polyspectrum $C_k(\omega_1, \omega_2, \ldots, \omega_{k-1})$ is that the associated kth-order cumulant $c_k(\tau_1, \tau_2, \ldots, \tau_{k-1})$ be absolutely summable, as shown by

$$\sum_{\tau_1=-\infty}^{\infty} \cdots \sum_{\tau_{k-1}=-\infty}^{\infty} |c_k(\tau_1, \tau_2, \ldots, \tau_{k-1}| < \infty \tag{3.54}$$

The *power spectrum*, *bispectrum*, and *trispectrum* are special cases of the kth-order polyspectrum defined in Eq. (3.53). Specifically, we may state the following:

1. For $k = 2$, we have the ordinary power spectrum:

$$C_2(\omega_1) = \sum_{\tau_1=-\infty}^{\infty} c_2(\tau_1) \exp(-j\omega_1\tau_1) \tag{3.55}$$

   which is a restatement of the Einstein–Wiener–Khintchine relation, namely, Eq. (3.15).

2. For $k = 3$, we have the *bispectrum*, defined by

$$C_3(\omega_1, \omega_2) = \sum_{\tau_1=-\infty}^{\infty} \sum_{\tau_2=-\infty}^{\infty} c_3(\tau_1, \tau_2) \exp[-j(\omega_1\tau_1 + \omega_2\tau_2)] \tag{3.56}$$

3. For $k = 4$, we have the *trispectrum*, defined by

$$C_4(\omega_1, \omega_2, \omega_3) = \sum_{\tau_1=-\infty}^{\infty} \sum_{\tau_2=-\infty}^{\infty} \sum_{\tau_3=-\infty}^{\infty} c_4(\tau_1, \tau_2, \tau_3) \exp[-j(\omega_1\tau_1 + \omega_2\tau_2 + \omega_3\tau_3)] \tag{3.57}$$

An outstanding property of polyspectrum is that all polyspectra of order higher than the second vanish when the process $u(n)$ is Gaussian. This property is a direct consequence of the fact that all joint cumulants of order higher than the second are zero for multivariate Gaussian distributions. Accordingly, the bispectrum, trispectrum, and all higher-order polyspectra are identically zero if the process $u(n)$ is Gaussian. Thus, higher-order spectra provide measures of the *departure of a stochastic process from Gaussianity* (Priestley, 1981; Nikias and Raghuveer, 1987).

The kth-order cumulant $c_k(\tau_1, \tau_2, \ldots, \tau_{k-1})$ and the kth-order polyspectrum $C_k(\omega_1, \omega_2, \ldots, \omega_{k-1})$ form a pair of multidimensional Fourier transforms. Specifically, the polyspectrum $C_k(\omega_1, \omega_2, \ldots, \omega_{k-1})$, is the *multidimensional discrete-time Fourier transform* of $c_k(\tau_1, \tau_2, \ldots, \tau_{k-1})$, and $c_k(\tau_1, \tau_2, \ldots, \tau_{k-1})$ is the *inverse multidimensional discrete-time Fourier transform* of $C_k(\omega_1, \omega_2, \ldots, \omega_{k-1})$.

For example, given the bispectrum $C_3(\omega_1, \omega_2)$, we may determine the third-order cumulant $c_3(\tau_1, \tau_2)$ by using the inverse two-dimensional discrete-time Fourier transform:

$$c_3(\tau_1, \tau_2) = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} C_3(\omega_1, \omega_2) \exp\left[j(\omega_1\tau_1 + \omega_2\tau_2)\right] d\omega_1 d\omega_2 \tag{3.58}$$

We may use this relation to develop an alternative definition of the bispectrum as follows. According to the *Cramér spectral representation*, we have

$$u(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega n} dZ(\omega) \qquad \text{for all } n \tag{3.59}$$

Hence, using Eq. (3.59) in (3.50), we get

$$c_3(\tau_1, \tau_2) = \left(\frac{1}{2\pi}\right)^3 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp\left[jn(\omega_1 + \omega_2 + \omega_3)\right] \cdot \exp[j(\omega_1\tau_1 + \omega_2\tau_2)] E[dZ(\omega_1)\, dZ(\omega_2)\, dZ(\omega_3)] \tag{3.60}$$

Comparing the right-hand sides of Eqs. (3.58) and (3.60), we deduce the following result:

$$E[dZ(\omega_1)\, dZ(\omega_2)\, dZ(\omega_3)] = \begin{cases} C_3(\omega_1, \omega_2)\, d\omega_1\, d\omega_2, & \omega_1 + \omega_2 + \omega_3 = 0 \\ 0, & \text{otherwise} \end{cases} \tag{3.61}$$

It is apparent from Eq. (3.61) that the bispectrum $C_3(\omega_1, \omega_2)$ represents the contribution to the mean product of three Fourier components whose *individual frequencies add up to zero*. This is an extension of the interpretation developed for the ordinary power spectrum in Section 3.3. In a similar manner we may develop an interpretation of the trispectrum.

In general, the polyspectrum $C_k(\omega_1, \omega_2, \ldots, \omega_{k-1})$ is *complex for order k higher than two*, as shown by

$$C_k(\omega_1, \omega_2, \ldots, \omega_{k-1}) = |C_k(\omega_1, \omega_2, \ldots, \omega_{k-1})| \exp[j\phi_k(\omega_1, \omega_2, \ldots, \omega_{k-1})] \tag{3.62}$$

where we note that $|C_k(\omega_1, \omega_2, \ldots, \omega_{k-1})|$ is the *magnitude* of the polyspectrum, and $\phi_k(\omega_1, \omega_2, \ldots, \omega_{k-1})$ is the *phase*. Moreover, the polyspectrum is a *periodic* function with

period 2π; that is,

$$C_k(\omega_1, \omega_2, \ldots, \omega_{k-1}) = C_k(\omega_1 + 2\pi, \omega_2 + 2\pi, \ldots, \omega_{k-1} + 2\pi) \tag{3.63}$$

Whereas the power spectral density of a stationary stochastic process is *phase blind*, the polyspectra of the process are *phase sensitive*. More specifically, the power spectral density is real-valued; referring to the input–output relation of Eq. (3.21), we clearly see that in passing a stationary stochastic process through a linear system, information about the phase response of the system is completely destroyed in the power spectrum of the output. In contrast, the polyspectrum is complex-valued, with the result that in a similar situation the polyspectrum of the output signal preserves information about the phase response of the system. It is for this reason that polyspectra provide a useful tool for the "blind" identification of an unknown system, where we only have access to the output signal and some additional information in the form of a probabilistic model of the input signal. We will have more to say on this issue in Chapter 18.

3.8 SPECTRAL-CORRELATION DENSITY

Polyspectra preserve phase information about a stochastic process by invoking higher-order statistics of the process, which is feasible only if the process is non-Gaussian. The preservation of phase information is also possible if the process is *cyclostationary* in the wide sense, as defined in Eqs. (3.44) and (3.45). This latter approach has two important advantages over the higher-order statistics approach:

- The phase information is contained in second-order cyclostationary statistics of the process; hence, the phase information can be exploited in a computationally efficient manner that avoids the use of higher-order statistics.
- Preservation of the phase information holds, irrespective of Gaussianity.

Consider then a discrete-time stochastic process $u(n)$ that is cyclostationary in the wide sense. Without loss of generality, the process is assumed to have zero mean. The ensemble-average autocorrelation function of the process $u(n)$ is defined in the usual way by Eq. (2.6), reproduced here for convenience of presentation:

$$r(n,n-k) = E[u(n)u^*(n-k)] \tag{3.64}$$

Under the condition of cyclostationarity, the autocorrelation function $r(n,n-k)$ is periodic in n for every k. Keeping in mind the discrete-time nature of the process $u(n)$, we may expand the autocorrelation function $r(n, n-k)$ into a complex Fourier series as follows (Gardner, 1994):

$$r(n, n-k) = \sum_{\{\alpha\}} r^{\alpha}(k) e^{j2\pi\alpha n - j\pi\alpha k} \tag{3.65}$$

where both n and k take on only integer values, and the set $\{\alpha\}$ includes all values of α for which the corresponding Fourier coefficient $r^{\alpha}(k)$ is not zero. The Fourier coefficient $r^{\alpha}(k)$

is itself defined by

$$r^{\alpha}(k) = \frac{1}{N}\sum_{n=0}^{N-1} r(n, n-k)e^{-j2\pi\alpha n + j\pi\alpha k} \tag{3.66}$$

where the number of samples N denotes the period. Equivalently, in light of Eq. (3.64), we may define $r^{\alpha}(k)$ as

$$r^{\alpha}(k) = \frac{1}{N}\left\{\sum_{n=0}^{N-1} E[u(n)u^*(n-k)e^{-j2\pi\alpha n}]\right\}e^{j\pi\alpha k} \tag{3.67}$$

The quantity $r^{\alpha}(k)$ is called the *cyclic autocorrelation function*, which has the following properties:

1. The cyclic autocorrelation function $r^{\alpha}(k)$ is periodic in α with period two.
2. For any α, we have from Eq. (3.67):
$$r^{\alpha+1}(k) = (-1)^k r^{\alpha}(k) \tag{3.68}$$
3. For the special case of $\alpha = 0$, Eq. (3.67) reduces to
$$r^{0}(k) = r(k) \tag{3.69}$$
where $r(k)$ is the ordinary autocorrelation function of a stationary process.

According to the Einstein–Wiener–Khintchine relations of Eqs. (3.15) and (3.16), the ordinary versions of the autocorrelation function and power spectral density of a wide-sense stationary stochastic process form a Fourier-transform pair. In a corresponding way, we may define the discrete-time Fourier transform of the cyclic autocorrelation function $r^{\alpha}(k)$ as follows (Gardner, 1994):

$$S^{\alpha}(\omega) = \sum_{k=-\infty}^{\infty} r^{\alpha}(k)e^{-j\omega k}, \qquad -\pi < \omega \leq \pi \tag{3.70}$$

The new quantity $S^{\alpha}(\omega)$ is called the *spectral-correlation density*, which is complex valued for $\alpha \neq 0$. Note that for the special case of $\alpha = 0$, Eq. (3.70) reduces to

$$S^{0}(\omega) = S(\omega) \tag{3.71}$$

where $S(\omega)$ is the ordinary power spectral density.

In light of the defining equations (3.67) and (3.70), we may set up the block diagram of Fig. 3.6 for measuring the spectral-correlation density $S^{\alpha}(\omega)$. For this measurement, it is assumed that the process $u(n)$ is *cycloergodic* (Gardner, 1994), which means that time averages may be substituted for ensemble averages "with samples taken once per period." According to the instrumentation described here, $S^{\alpha}(\omega)$ is the bandwidth-normalized version of the cross-correlation narrow-band spectral components contained in the time series $u(n)$ at the angular frequencies $\omega + \alpha\pi$ and $\omega - \alpha\pi$, in the limit as the bandwidth of these spectral components is permitted to approach zero (Gardner, 1994). Note that the two narrow-band filters in Fig. 3.6 are identical, both having a mid-band (angular) frequency ω and a bandwidth $\Delta\omega$ that is small compared to ω, but large compared to the reciprocal of

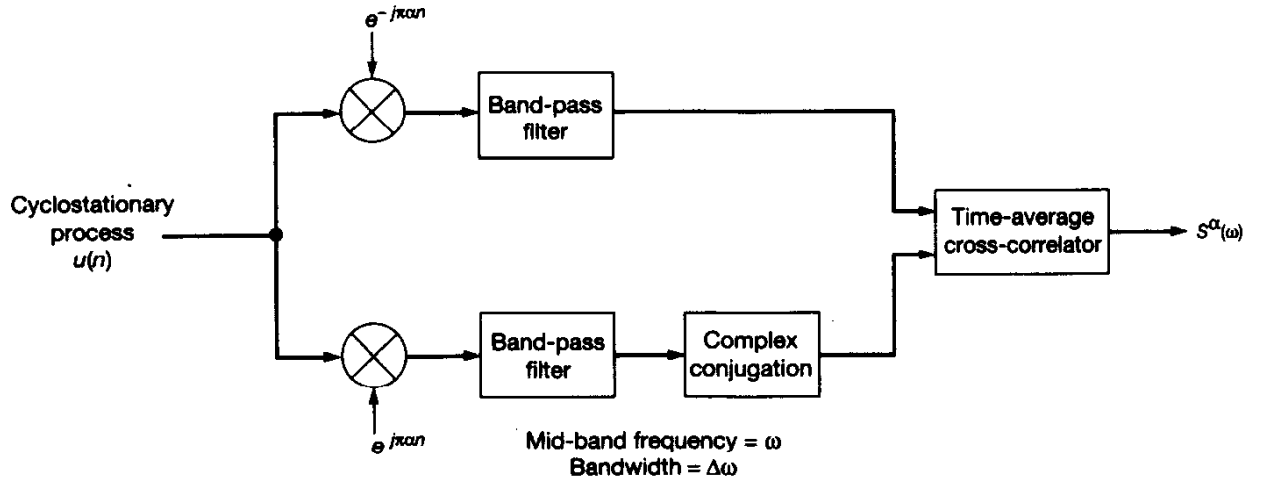

Figure 3.6 Scheme for measuring the spectral-correlation density of a cyclostationary process.

the averaging time used in the cross-correlator at the output end in Fig. 3.6. In one channel of this scheme the input $u(n)$ is multiplied by $\exp(-j\pi\alpha n)$, and in the other channel it is multiplied by $\exp(j\pi\alpha n)$; the resulting filtered signals are then applied to the cross-correlator. It is these two multiplications (prior to correlation) that provide the spectral-correlation density $S^{\alpha}(\omega)$ with a phase-preserving property for nonzero values of α.

3.9 SUMMARY AND DISCUSSION

In this chapter we discussed various aspects of spectrum analysis pertaining to a discrete-time stochastic process. In particular, we identified three distinct spectral parameters, depending on the statistical characterization of the process, as summarized here:

1. *Power spectral density,* $S(\omega)$, defined as the discrete-time Fourier transform of the ordinary autocorrelation function of a wide-sense stationary process. For such a process, the autocorrelation function is Hermitian, which makes the power spectral density $S(\omega)$ a real-valued quantity. Accordingly, $S(\omega)$ destroys phase information about the process. Despite this limitation, the power spectral density is commonly accepted as a useful parameter for displaying the correlation properties of a wide-sense stationary process.
2. *Polyspectra,* $C_k(\omega_1, \omega_2, \ldots, \omega_{k-1})$, defined as the multidimensional Fourier transform of the cumulants of a stationary process. For second-order statistics, $k = 2$, and $C_2(\omega_1)$ reduces to the ordinary power spectral density $S(\omega)$. For higher-order statistics, $k > 2$, and the polyspectra $C_k(\omega_1, \omega_1, \ldots, \omega_{k-1})$ take on complex forms. It is this property of polyspectra that makes them a useful tool for dealing with situations where knowledge of phase is a necessary requirement. However, for polyspectra to be meaningful, the process has to be non-Gaussian, and the exploitation of phase information contained in polyspectra requires the use of nonlinear filtering.
3. *Spectral-correlation density,* $S^{\alpha}(\omega)$, defined as the discrete-time Fourier transform of the cyclic autocorrelation function of a process that is cyclostationary in the wide sense. For $\alpha \neq 0$, $S^{\alpha}(\omega)$ is complex valued; for $\alpha = 0$, it reduces to $S(\omega)$. The useful feature of $S^{\alpha}(\omega)$ is that it preserves phase information, which can be exploited by means of linear filtering.

The different properties of the ordinary power spectral density, polyspectra, and spectral-correlation density give these statistical parameters their own individual areas of application.

One last comment is in order. The theories of second-order cyclostationary processes and conventional polyspectra have been brought together under the umbrella of *cyclic polyspectra*. Simply stated, cyclic polyspectra are spectral cumulants, in which the individual frequencies involved can add up to any cycle frequency α, whereas they must

add up to zero for conventional polyspectra. For a detailed treatment of cyclic polyspectra, the interested reader is referred to (Gardner and Spooner, 1994; Spooner and Gardner, 1994).

PROBLEMS

1. Consider the definition given in Eq. (3.12) for the power spectral density. Is it permissible to interchange the operation of taking the limit and that of the expectation in this equation? Justify your answer.
2. In deriving Eq. (3.25), we invoked the notion that if a wide-sense stationary process is applied to a linear, time-invariant, and stable filter, the stochastic process produced at the filter output is wide-sense stationary too. Show that, in general,

$$r_y(n, m) = \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h(i)h^*(k)r_u(n-i, m-k)$$

   which includes the result of Eq. (3.25) as a special case.
3. The mean-square value of the filter output in Eq. (3.28) assumes that the bandwidth of the filter is small compared to its midband frequency. Is this assumption necessary for the corresponding result obtained in Example 1 for a white-noise process? Justify your answer.
4. A white-noise process with a variance of 0.1 V squared is applied to a low-pass discrete-time filter whose bandwidth is 1 Hz. The process is real.
   (a) Calculate the variance of the filter output.
   (b) Assuming that the input is Gaussian, determine the probability density function of the filter output.
5. Justify the fact that the expectation of $|dZ(\omega)|^2$ has the physical significance of power.
6. Show that the third- and higher-order cumulants of a Gaussian process are all identically zero.
7. Develop a physical interpretrum $C_4(\omega_1, \omega_2, \omega_3)$ of a stationary stochastic process $u(n)$; assume that $u(n)$ is real valued.
8. Consider a linear time-invariant system whose transfer function is $H(z)$. The system is excited by a real-valued sequence $x(n)$ of independently and identically distributed (iid) random variables with zero mean and unit variance. The probability distribution of $x(n)$ is nonsymmetric.
   (a) Evaluate the third-order cumulant and bispectrum of the system output $u(n)$.
   (b) Show that the phase component of the bispectrum of $u(n)$ is related to the phase response of the system transfer function $H(z)$ as follows:

$$\arg[C_3(\omega_1, \omega_2)] = \arg[H(e^{j\omega_1})] + \arg[H(e^{j\omega_2})] - \arg[H(e^{j(\omega_1+\omega_2)})]$$

9. Equation (3.52) gives the kth-order cumulant of the output of a linear time-invariant system of impulse response h_n that is driven by a sequence $x(n)$ of independent and identically distributed random variables. Prove the validity of this equation.

10. Show that for a stochastic process $u(n)$ that is cyclostationary in the wide sense, the cyclic autocorrelation function $r^{\alpha}(k)$ satisfies the property

$$r^{\alpha}(-k) = r^{\alpha *}(k)$$

where the asterisk denotes complex conjugation.

11. Figure 3.6 describes a method for measuring the spectral-correlation density of a time series $u(n)$ that is representative of a cyclostationary process in the wide sense. For $\alpha = 0$, show that Fig. 3.6 reduces to the simpler form shown in Fig. 3.3.