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Oscillations, $SE(2)$–Snakes and Motion Control

P. S. Krishnaprasad and D. P. Tsakiris
University of Maryland at College Park
Institute for Systems Research & Department of Electrical Engineering
College Park, MD 20742
krishna@isr.umd.edu, dimitris@isr.umd.edu

Abstract

This paper is concerned with the problem of motion generation via cyclic variations in selected degrees of freedom (usually referred to as shape variables) in mechanical systems subject to nonholonomic constraints (here the classical one of a disk rolling without sliding on a flat surface). In earlier work, we identified an interesting class of such problems arising in the setting of Lie groups, and investigated these under a hypothesis on constraints, that naturally led to a purely kinematic approach. In the present work, the hypothesis on constraints does not hold, and as a consequence, it is necessary to take into account certain dynamical phenomena. Specifically we concern ourselves with the group $SE(2)$ of rigid motions in the plane and a concrete mechanical realization dubbed the $SE(2)$–snake. In a restricted version, it is also known as the Roller Racer (a patented toy).

Based on the work of Bloch, Krishnaprasad, Marsden and Murray, one recognizes in the example of this paper a balance law called the momentum equation, which is a direct consequence of the interaction of the $SE(2)$–symmetry of the problem with the constraints. The systematic use of this type of balance law results in certain structures in the example of this paper. We exploit these structures to demonstrate that the single shape freedom in this problem can be cyclically varied to produce a rich variety of motions of the $SE(2)$–snake.

In their study of the Snakeboard, a patented modification of the skateboard that also admits the group $SE(2)$ as a symmetry group, Lewis, Ostrowski, Burdick and Murray, exploited the same type of balance law as the one discussed here to generate motions. A key difference however is that, in the present paper, we have only one control variable and thus controllability considerations become somewhat more delicate.

In the present paper, we give a self–contained treatment of the geometry, mechanics and motion control of the Roller Racer.

1. Introduction

The idea of using periodic driving signals to produce rectified movement appears in a number of settings in engineering. Some of the more inventive examples are associated with the design and operation of novel actuators exploiting vibratory transduction [20], [21]. In his paper [2], Brockett develops a mathematical basis for understanding such devices. Elsewhere, in the context of robotic machines with many degrees of freedom designed to mimic snake–like movements [5], periodic variations in the shape parameters are used in an essential way to generate global movements. In [8] and [9], we have developed a general mathematical formulation to study systems of this type. The study of periodic signal generators (also called central pattern generators), as sources of timing signals to compose movements has a long history in the neurophysiology of movement dating back to the early work of Sherrington, Brown and Bernstein.

Recent studies by neurophysiologists [3] have attempted to bring together principles of motion control based on pattern generation in the spinal cord of the *lamprey*, its compliant body dynamics, and the fluid dynamics of its environment to achieve a comprehensive understanding of the swimming behavior of such anguilliform animals. These efforts have in part relied on continuum mechanical models of the body, and computational fluid dynamical (CFD) calculations. There appear to be some unifying themes that underlie this type of neural–mechanical approach to biological locomotion, and the work of the authors and others involving the study of land–based robotic machines subject to the constraint of 'no sliding'. As pointed out in [7], the connecting links between these two streams of research appear to be related to the manner in which systems of coupled oscillators are used to generate finite dimensional shape variations of the bodies of specialized robot designs, and the associated geometric–mechanical descriptions of the constraints to produce effective motion control strategies (see also the work of Collins and Stewart [4] for another dynamical systems perspective).

In the present paper, we report on a complete study of an interesting example, the (single module) $SE(2)$–snake, with a view towards deeper appreciation of the above–mentioned connections. In section 2, we present the basic geometry of the configuration space, and the applicable constraints. We also discuss a simplification that reduces the shape freedom to one variable, leading to the Roller Racer. The constraints of 'no sliding' are *insufficient* to determine the movement of the Roller Racer from shape variations alone. In section 3, a model Lagrangian and the action of $SE(2)$, the rigid motion group in the plane as a symmetry group (of the Lagrangian and the constraints) are discussed. In section 4, a balance law associated to the $SE(2)$–symmetry is derived. This is a consequence of the Lagrange D'Alembert principle (The basic results behind momentum equations are to be found in [1]). This momentum equation is the key additional data that together with the constraints allows us to generate motion control laws, the topic of section 5.

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In section 6, we consider controllability issues. Most proofs are omitted from the paper. They can be found either in [19] or in [10].

2. Configurations and Constraints

G–snakes are kinematic chains with configurations taking values in products of several copies of a Lie group G, and subject to nonholonomic constraints [9], [19]. The group G acts on the chain by diagonal action as a symmetry group. The shape space is the quotient by this action. Figure 1 illustrates an $SE(2)$–snake composed of two modules, and the configuration space Q is $SE(2)\times SE(2)\times SE(2)$.

The machine in Figure 1 is composed of three axles and linearly actuated linkages connecting each adjacent pair of axles, resulting in an assembly of two identical modules. Altering the lengths of the connecting linkages leads to changes in the shapes of component modules. The wheels mounted on each axle are independent and are *not* actuated but subject to the constraint of 'no sliding'. In this case, there are three constraints, the shape space S is $SE(2)\times SE(2)$, the constraints define a principal connection on the bundle $(Q, SE(2), S)$, away from a set of nonholonomic singularities and it is possible to generate global movement of the assembly by periodic variations in the module shapes. The entire situation can be understood at a kinematic level as long as the shapes are control variables ([8], [9], [19]).

When one of the modules is removed from the machine in Figure 1, leaving us with two axles connected by linkages and two constraints, the resulting problem is kinematically under–constrained. It is no longer possible to define a connection without using additional structure. It is this type of 1–module $SE(2)$–snake that is of interest here. Matters can be simplified by limiting the extent of shape freedom. In 1972, W.E. Hendricks was awarded U.S. patent no. $3,663,038$ for a toy illustrated in Figure 2 and dubbed the Roller Racer, that serves as one such simplification. The rider, on the seat shown, has to merely oscillate the handle–bars from side–to–side to generate forward propulsion, a behavior for which Hendricks did not claim to have an explanation.

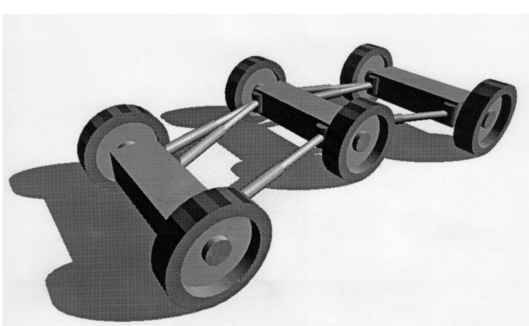

Figure 1. Two Module $SE(2)$–Snake

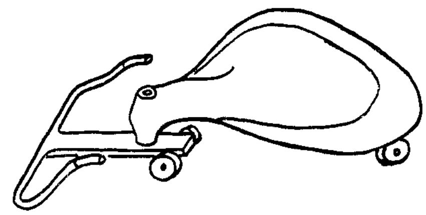

Figure 2. Roller Racer

The kinematics of the Roller Racer are given by defining θ_1 the orientation of the rear axle in the laboratory frame, $\theta_{1,2}$ the steering angle, and x_1, y_1, the coordinates of the center of mass of the rear–body (seat), assumed for simplicity to lie on the rear axle (Figure 3). There is a crucial (backward) offset d_2 of the front axle from the pivot point of the handle bars, and this offset is small compared to the length d_1 from the same pivot point to the center of the rear axle. The rider controls the steering angle which is the shape variable. Thus the configuration space is $Q = SE(2)\times S^1$, the shape space is S^1 and the group $SE(2)$ acts on the first factor of Q. The 'no sliding' constraints can be expressed in the form of the intersection $\mathcal{D}_q \subset T_qQ$ of the annihilators of the two constraint one–forms:

$$\begin{aligned} \omega_q^1 &= -\sin\theta_1 dx_1 + \cos\theta_1 dy_1 \ , \\ \omega_q^2 &= -\sin(\theta_1+\theta_{1,2})dx_1 + \cos(\theta_1+\theta_{1,2})dy_1 \\ &\quad + (d_1\cos\theta_{1,2} + d_2)d\theta_1 + d_2 d\theta_{1,2} \ . \end{aligned} \tag{1}$$

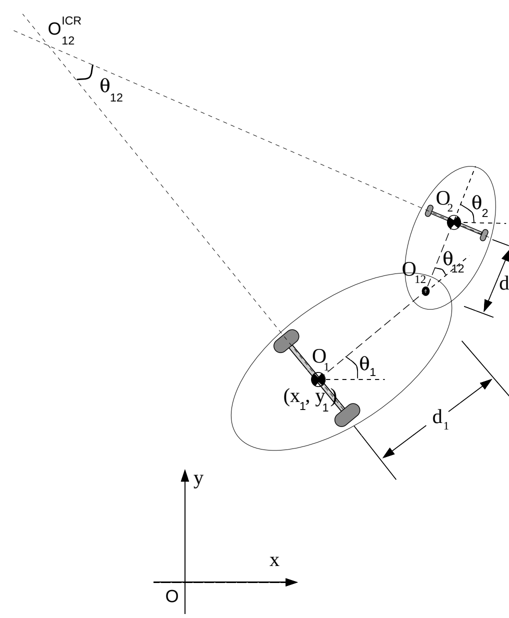

Figure 3. Roller Racer Model

It is an easy calculation to prove:

Proposition 1: The constraint distribution $\mathcal{D}_q \subset T_qQ$ is

$$\mathcal{D}_q = \mathrm{sp}\{\xi^1_Q, \xi^2_Q\} \;, \tag{2}$$

where in the case $d_2 \neq 0$:

$$\begin{aligned}\xi^1_Q &= d_2(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_{1,2}} \;, \\ \xi^2_Q &= d_2\frac{\partial}{\partial\theta_1} - (d_1\cos\theta_{1,2} + d_2)\frac{\partial}{\partial\theta_{1,2}} \;,\end{aligned} \tag{3}$$

while in the case $d_2 = 0$:

$$\begin{aligned}\xi^1_Q &= d_1\cos\theta_{1,2}(\cos\theta_1 \frac{\partial}{\partial x_1} + \sin\theta_1 \frac{\partial}{\partial y_1}) \\ &\qquad + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1} \;, \\ \xi^2_Q &= \frac{\partial}{\partial\theta_{1,2}} \;.\end{aligned} \tag{4}$$

■

3. Lagrangian and Symmetry

For the purposes of this paper, we consider a model Lagrangian (taking into account the simplifying, but reasonable hypothesis that the center of mass of the rear–body, i.e. the seat, is located on the rear axle):

$$L(\dot q) = \frac{1}{2}m_1(\dot x_1^2 + \dot y_1^2) + \frac{1}{2}I_{z_1}\dot\theta_1^2 + \frac{1}{2}I_{z_2}(\dot\theta_1 + \dot\theta_{1,2})^2 \;. \tag{5}$$

Here m_1 denotes the mass of the rear body, and we have neglected the mass of the front body containing the handle bar. Further, I_{z_1} and I_{z_2} are the moments of inertia.

The model Lagrangian is invariant under the symmetry action Φ of the group $G = SE(2)$ on the configuration space Q defined by:

$$\begin{aligned}\Phi :& G \times Q \to Q \\ & (g, (g_1, \theta_{1,2})) \mapsto (gg_1, \theta_{1,2}) \\ & ((b,c,a),(x_1, y_1, \theta_1, \theta_{1,2})) \mapsto \\ &\quad (x_1\cos a - y_1\sin a + b, x_1\sin a + y_1\cos a + c, \\ &\qquad \theta_1 + a, \theta_{1,2}) \;,\end{aligned} \tag{6}$$

where $g = g(b,c,a) \in G$. The tangent space at $q \in Q$ to the orbit of Φ is given by

$$T_q\mathrm{Orb}(q) = \mathrm{sp}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial\theta_1}\} \;. \tag{7}$$

A key point is that the above symmetry action interacts with the constraints in a nontrivial way as indicated by:

Proposition 2: Consider the intersection

$$\mathcal{S}_q = \mathcal{D}_q \cap T_q\mathrm{Orb}(q) \;.$$

In the case $d_1 \neq d_2$, the distribution $\mathcal{S}_q$ is 1–dimensional and is given by:

$$\mathcal{S}_q = \mathrm{sp}\{\xi^q_Q\} \;, \tag{8}$$

where

$$\begin{aligned}\xi^q_Q &= (d_1\cos\theta_{1,2} + d_2)(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) \\ &\qquad + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1} \;.\end{aligned} \tag{9}$$

■

4. Momentum Equation

A main result of [1] is that interactions between nonholonomic constraints and symmetries, as in Proposition 2 above, give rise to additional conditions. These are simply the consequences of the Lagrange D'Alembert principle applied to virtual displacements contained in $\mathcal{S}_q$ and are thus balance laws or momentum equations. In special cases they may reduce to conservation laws, as discussed at length in [1].

Define the nonholonomic momentum:

$$p = \sum_i \frac{\partial L}{\partial \dot q^i}(\xi^q_Q)^i \;. \tag{10}$$

The nonholonomic momentum for the Roller Racer system is

$$\begin{aligned}p &= m_1(d_1\cos\theta_{1,2} + d_2)(\dot x_1\cos\theta_1 + \dot y_1\sin\theta_1) \\ &\quad + [(I_{z_1} + I_{z_2})\dot\theta_1 + I_{z_2}\dot\theta_{1,2}]\sin\theta_{1,2} \;.\end{aligned} \tag{11}$$

Remark: The nonholonomic momentum p is nothing but the angular momentum about O_{12}^{ICR} (Figure 3).

Let

$$\Delta(\theta_{1,2}) \stackrel{\mathrm{def}}{=} (I_{z_1} + I_{z_2})\sin^2\theta_{1,2} + m_1(d_1\cos\theta_{1,2} + d_2)^2 \;. \tag{12}$$

For $d_1 \neq d_2$, we have $\Delta > 0$ for all $q \in Q$. We state:

Proposition 3: The angular velocity $\dot\theta_1$ is an affine function of the nonholonomic momentum

$$\dot\theta_1 = \frac{1}{\Delta(\theta_{1,2})}\big(\sin\theta_{1,2}\, p - \delta(\theta_{1,2})\dot\theta_{1,2}\big) \;, \tag{13}$$

where $\delta(\theta_{1,2}) \stackrel{\mathrm{def}}{=} I_{z_2}\sin^2\theta_{1,2} + m_1 d_2(d_1\cos\theta_{1,2} + d_2)$. ■

From [1], the nonholonomic momentum satisfies the following *Momentum Equation:*

$$\frac{dp}{dt} = \sum_i \frac{\partial L}{\partial \dot q^i}\left[\frac{d\xi^q}{dt}\right]^i_Q \;. \tag{14}$$

For the Roller Racer this takes the form given below:

Proposition 4: The momentum equation for the Roller Racer system is

$$\frac{dp}{dt} = A^4_1(\theta_{1,2})\dot\theta_{1,2}p + A^4_2(\theta_{1,2})\dot\theta^2_{1,2} \;, \tag{15}$$

where

$$A^4_1(\theta_{1,2}) = \frac{1}{\Delta(\theta_{1,2})}\beta(\theta_{1,2})\sin\theta_{1,2}$$

and

$$A^4_2(\theta_{1,2}) = \frac{m_1}{\Delta(\theta_{1,2})}\lambda(\theta_{1,2})\gamma(\theta_{1,2}),$$

where

$$\beta(\theta_{1,2}) \stackrel{\text{def}}{=} (I_{z_1}+I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2} + d_2),$$
$$\gamma(\theta_{1,2}) \stackrel{\text{def}}{=} -I_{z_1} d_2 + I_{z_2} d_1 \cos\theta_{1,2}\ ,$$
$$r(\theta_{1,2}) \stackrel{\text{def}}{=} d_1\cos\theta_{1,2} + d_2\ ,$$
$$\lambda(\theta_{1,2}) \stackrel{\text{def}}{=} d_1 + d_2\cos\theta_{1,2}\ ,$$
$$I \stackrel{\text{def}}{=} I_{z_1} + I_{z_2}\ .$$

Solutions of the momentum equation take the form:

$$p(t) = \Phi(t,t_0)p(t_0) + \int_{t_0}^{t} \Phi(t,\tau)A_2^4(\theta_{1,2}(\tau))\dot\theta_{1,2}^2(\tau)d\tau\ , \tag{16}$$

where the state transition matrix is

$$\Phi(t,t_0) = \exp\left[\int_{t_0}^{t} A_1^4(\theta_{1,2}(\tau))\dot\theta_{1,2}(\tau)d\tau\right] = \exp\left[\int_{\theta_{1,2}(t_0)}^{\theta_{1,2}(t)} A_1^4(\theta_{1,2})d\theta_{1,2}\right] = \sqrt{\frac{\Delta(\theta_{1,2}(t))}{\Delta(\theta_{1,2}(t_0))}}. \tag{17}$$

■

5. Reconstruction and Motion Control

For a given curve $\theta_{1,2}(.)$ in shape space, using the formula (11) for the nonholonomic momentum p, the formula (13) for θ_1 and the nonholonomic constraints given by (1), it is possible to determine the curve in the group $SE(2)$ (i.e. the motion of (x_1, y_1, θ_1)). First note that for

$$g_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & x_1 \\ \sin\theta_1 & \cos\theta_1 & y_1 \\ 0 & 0 & 1 \end{pmatrix},$$

we have $\dot g_1 = g_1 \xi_1$ with

$$\xi_1 = \xi_1^1(\theta_{1,2},\dot\theta_{1,2})\mathcal{A}_1 + \xi_2^1(\theta_{1,2},\dot\theta_{1,2})\mathcal{A}_2, \tag{18}$$

where

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

belong to $se(2)$, the Lie algebra of $SE(2)$. That the curve $g_1(.)$ is such that the curve $\xi_1(.) = g_1^{-1}(.)\dot g_1(.)$ lies in a two–dimensional subspace of $se(2)$ (spanned by $\mathcal{A}_1$ and $\mathcal{A}_2$), is equivalent to the 'no sliding' constraint

$$-\dot x_1\cos\theta_1 + \dot y_1\sin\theta_1 = 0. \tag{20}$$

Proposition 5: The reconstructed trajectory $g_1(.)$ is given by

$$\dot g_1 = g_1(\xi_1^1\mathcal{A}_1 + \xi_2^1\mathcal{A}_2),$$

with

$$\begin{pmatrix} \xi_1^1 \\ \xi_2^1 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} \sin\theta_{1,2} & -\delta(\theta_{1,2}) \\ r(\theta_{1,2}) & -\gamma(\theta_{1,2})\sin\theta_{1,2} \end{pmatrix}\begin{pmatrix} p \\ \dot\theta_{1,2} \end{pmatrix}, \tag{21}$$

where $\delta(\theta_{1,2})$ is as in Proposition 3 and $r(\theta_{1,2})$ and $\gamma(\theta_{1,2})$ are as in Proposition 4.

■

Remark: The determinant

$$\det\begin{pmatrix} \sin\theta_{1,2} & -\delta(\theta_{1,2}) \\ r(\theta_{1,2}) & -\gamma(\theta_{1,2})\sin\theta_{1,2} \end{pmatrix} = d_2\Delta(\theta_{1,2}). \tag{22}$$

Thus, for $d_2 \neq 0$, one can solve equation (21) for p and $\theta_{1,2}$, given a prescribed ξ_1^1 and ξ_2^1.

Since ξ_1^1 and ξ_2^1 are respectively the angular velocity and the heading speed of the main (rear) body of the Roller Racer, using (21) and the properties of the momentum p, one can infer some general properties of the motion of the Roller Racer. In this spirit we state

Proposition 6: Assume $d_1 > d_2$.

(a) If $\epsilon \stackrel{\text{def}}{=} \frac{I_{z_1}d_2}{I_{z_2}d_1} > 1$ and the initial value $p(0) \le 0$, then $p(t) < 0$, $\forall t$.

(b) If $\epsilon \stackrel{\text{def}}{=} \frac{I_{z_1}d_2}{I_{z_2}d_1} < 1$, $|\theta_{1,2}| < \cos^{-1}(\epsilon)$ and the initial value $p(0) \ge 0$, then $p(t) > 0$, $\forall t$.

(c) If $\epsilon \stackrel{\text{def}}{=} \frac{I_{z_1}d_2}{I_{z_2}d_1} < 1$, $|\theta_{1,2}| > \cos^{-1}(\epsilon)$ and the initial value $p(0) \le 0$, then $p(t) < 0$, $\forall t$.

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Using some easy estimates on the growth of $p(t)$ under sinusoidal shape variations $\theta_{1,2}(.)$, it is possible to strengthen the above conclusions for times larger than a transient time T.

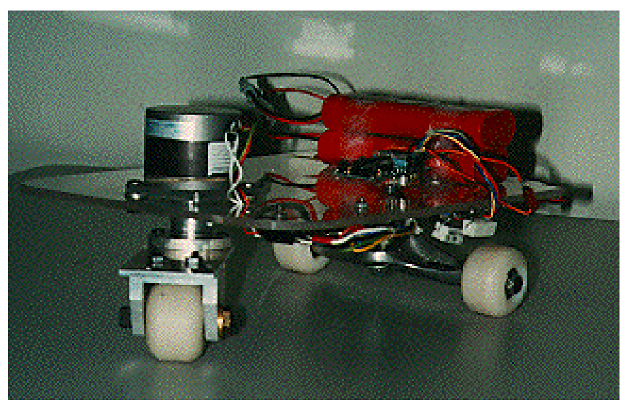

Figure 4. Roller Racer Prototype

Extensive numerical studies have been made on the model of this paper, including a parametric study of the response of the Roller Racer when subject to sinusoidal controls $\theta_{1,2}(.)$ of differing amplitudes and frequencies. Figures 5, 6, 7 and 8 summarize these results. In these figures, $\theta_{1,2}(0)$ is the average of the sinusoidal control, $\alpha_{1,2}$ is its amplitude and a choice of the Roller Racer length and inertia parameters corresponding to the case $\epsilon > 1$ in Proposition 6 is made. The shape oscillation considered in Figure 5 induces a forward translation of the system, while the one in Figure 6 produces a translation backwards. Setting the average oscillation to a value different from 0 or π, induces a rotation around the point $O_{1,2}^{\mathrm{ICR}}$ of Figure 3. As can be seen from Figure 7, there is a qualitative change in the response of the Roller Racer for higher amplitudes of shape oscillation. These results are also substantiated by results from a computer–controlled prototype (Figure 4).

6. Controllability

The Roller Racer model of this paper fits into the general framework of control systems on principal bundles. The question of accessibility can be settled by Lie bracket calculations. For this purpose, define the *base–momentum* system to be given by

$$\begin{aligned} \frac{dp}{dt} &= A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \ , \\ \frac{d\theta_{1,2}}{dt} &= \dot{\theta}_{1,2} \ , \\ \frac{d\dot{\theta}_{1,2}}{dt} &= u \end{aligned} \tag{23}$$

or, compactly

$$\dot{z} = f(z) + g(z)u \ , \tag{24}$$

where $z \stackrel{\mathrm{def}}{=} (p,\ \theta_{1,2},\ \dot{\theta}_{1,2})^\top \in M = I\!R \times S^1 \times I\!R$ and $f(z)$ and $g(z)$ are given by

$$f(z) \stackrel{\mathrm{def}}{=} \begin{pmatrix} A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \\ \dot{\theta}_{1,2} \\ 0 \end{pmatrix}$$

and $g(z) \stackrel{\mathrm{def}}{=} (0,\ 0,\ 1)^\top$.

Proposition 6: Assume $d_1 > d_2$. Then, the base–momentum system is strongly accessible at equilibria $z_e = (0,\ \theta_{1,2_e},\ 0)^\top$.

Proof Follows from checking that either

$$\mathrm{sp}\left\{g,\ [f,g],\ [[f,g],g]\right\} = I\!R^3$$

or

$$\mathrm{sp}\left\{g,\ [f,g],\ \Big[[f,g],\ [[f,g],g]\Big]\right\} = I\!R^3.$$

∎

Similarly, accessibility on the $SE(2)$–bundle over the base–momentum space can be established. Details on this and on small–time local controllability can be found in [10].

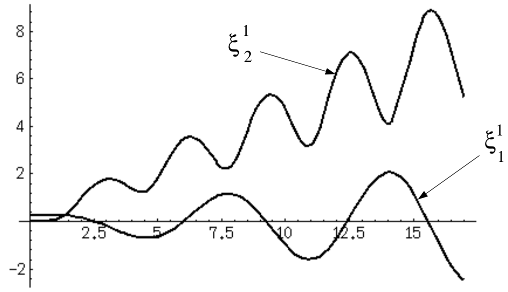

Figure 5. Case $\theta_{1,2}(0) = \pi$ and $\alpha_{1,2} = 1.0$

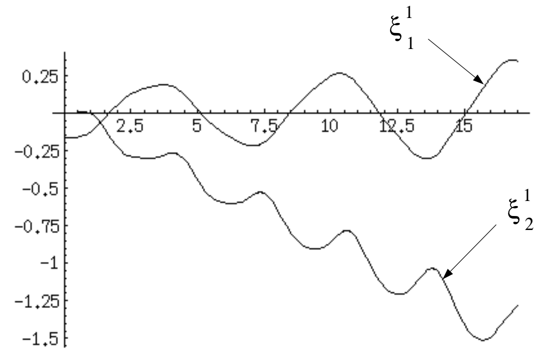

Figure 6. Case $\theta_{1,2}(0) = 0$ and $\alpha_{1,2} = 1.0$

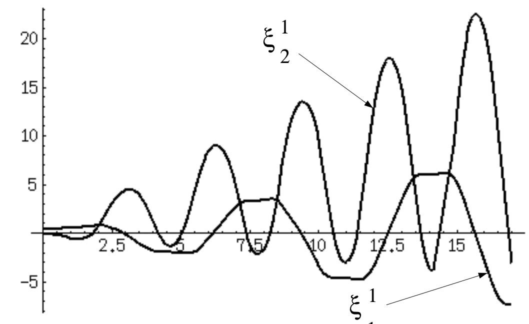

Figure 7. Case $\theta_{1,2}(0) = \pi$ and $\alpha_{1,2} = 1.5$

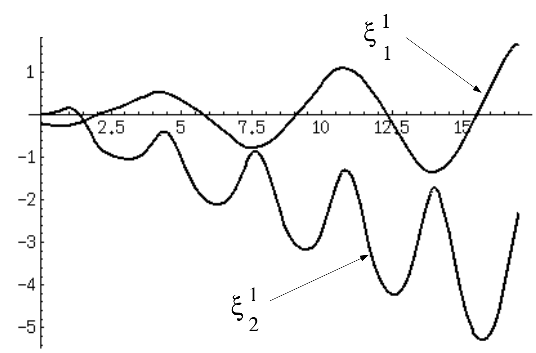

Figure 8. Case $\theta_{1,2}(0) = 0$ and $\alpha_{1,2} = 1.5$

7. Acknowledgements

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a source of inspiration. Tom Finan got us thinking about the Roller Racer. David Lindsay built a first mechanical model. Additional computer controlled models have been built by Vikram Manikonda. For an MPEG movie of one in action, see URL: *http://www.isr.umd.edu/Labs/ISL/isl.html.*

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