Minimal Change: Relevance and Recovery Revisited

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Abstract

The operation of contraction (referring to the removal of knowledge from a knowledge base) has been extensively studied in the research field of belief change, and different postulates (e.g., the AGM postulates with recovery, or relevance) have been proposed, as well as several constructions (e.g., partial meet) that allow the definition of contraction operators satisfying said postulates. Most of the related work has focused on classical logics, i.e., logics that satisfy certain intuitive assumptions; in such logics, several nice properties and equivalences related to the above postulates and constructions have been shown to hold. Unfortunately, previous work has shown that the postulates' applicability and the related results generally fail for non-classical logics. Motivated by the fact that non-classical logics (like Description Logics or Horn logic) are increasingly being used in various applications, we study contraction for all monotonic logics, classical or not. In particular, we identify several sufficient conditions for the various postulates to be applicable, and show that, in practice, relevance is a more suitable (i.e., applicable) minimality criterion than recovery for non-classical logics. In addition, we revisit some important related results from the classical belief change literature and study conditions sufficient for them to hold for non-classical logics; the corresponding results for classical logics emerge as corollaries of our more general results. Our work is another step towards the aim of exploiting the rich belief change literature for addressing the evolution problem in a larger class of logics.

Keywords: belief revision, belief change, AGM theory, recovery, relevance, minimal change

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1. Introduction

One's beliefs are not generally static but evolve over time for various reasons: new, previously unknown, classified, or otherwise unavailable information may have become known; a new observation or experiment may reveal a new fact; or modelling errors may be identified [1]. *Belief change* [2] is the research field that deals with the problem of how idealized rational agents change their beliefs.

The most influential work on belief change is the *AGM theory*, also referred to as the *AGM paradigm* [3]. The AGM theory considers beliefs expressed using *classical logics*, i.e., logics that satisfy certain assumptions (such as supraclassicality and compactness); we refer to these assumptions as the *AGM assumptions*. The AGM assumptions are general enough to engulf many interesting logics, like classical propositional logic and first-order logic, but exclude some logics useful for various modern applications, such as Description Logics (DLs) [4], which are being used in the Semantic Web [5], Horn logic [6], the basis for many rule-based systems and applications, and intuitionistic logic [7], which is being used to formally reason about programs and programming languages.

Under the AGM paradigm, beliefs of idealized rational agents are modeled as sets of propositions closed under logical consequences, called *belief sets*. Three different change operations for belief sets were considered; here we focus on *contraction*, which causes the agent to abandon some logical proposition. Deciding what to remove during contraction is not always a purely logical decision, so contraction cannot be defined using an algebraic expression [3]. The AGM theory addressed this problem by introducing *rationality postulates* which should be satisfied by any rationally defined contraction operator, essentially defining a family of contraction operations.

Starting from these postulates, it was showed that contraction can be used to define *revision* operators, i.e., operators that add logical propositions to our knowledge in a consistent manner (and vice-versa) [3]. Further work included the proposal of constructions which allow the definition of contraction operators through an intuitive procedure, such as partial meet contraction [3] and safe contraction [8]. These results made the AGM paradigm the most influential work in the field and caused most of the belief change literature to be based on the same general intuitions and assumptions. Nevertheless, there was no complete consensus, and two main lines of criticism emerged.

The first criticism focused on the intuition regarding the AGM postulates themselves, mostly regarding the contraction postulate of *recovery* [9, 10, 11]. The recovery postulate was an attempt to formalize the Principle of Minimal Change, which states that the knowledge lost during contraction should be minimal. However, recovery was criticized as counterintuitive [11], or superfluous [10]. Hansson argued that the minimality criterion is better formalized using the *relevance* postulate [11]. Nevertheless, Hansson was able to prove that relevance and recovery are in fact equivalent in the presence of the AGM assumptions and the other contraction postulates. Hansson concludes his article claiming that recovery should be accepted as an *"emerging property, rather than as a fundamental postulate, of belief set contraction"*. In other words, Hansson claimed that “recoverability” (as expressed by the recovery postulate) is an emerging property of “minimality” (expressed by the relevance postulate).

The second line of criticism was related to the applicability of the AGM paradigm to non-classical logics. In particular, it was shown that, even though the AGM contraction postulates themselves can be recast so as to be applicable to non-classical logics, they cannot be actually used to characterize contraction, because in many interesting non-classical logics, there are no contraction operators that satisfy them [1, 12]. The recovery postulate was identified as the main reason behind this failure [1]

There have been several attempts to adapt AGM contraction to specific non-classical logics (see for example [13] for relevant logics, [14] for paraconsistent logics, [15, 16] for Horn Logics). However, they focused on particular formalisms and to our knowledge, no systematic study of the logical properties and the relation between sets of postulates and the partial meet construction for belief sets exists. For belief bases (sets not necessarily closed under logical consequence), such a study is to be found in Hansson and Wassermann [17].

The present paper focuses on the contraction operator and attempts to shed new light on the connection between relevance, recovery and the partial meet construction, by considering the applicability and relations between these two postulates and the partial meet construction for non-classical logics. In particular, we drop most of the AGM assumptions and revisit the above results by reconsidering the use of relevance as a minimality criterion. Initially, we observe that the equivalence between relevance and recovery fails when the AGM assumptions are dropped, i.e., recoverability is not the same as minimality for non-classical logics. Motivated by this fact, we consider various classes of logics and determine sufficient conditions for the applicability of the AGM postulates with recovery or relevance. In addition, we determine sufficient conditions under which partial meet can be used to construct contraction operators that satisfy the AGM postulates with recovery/relevance for various classes of logics. Finally, we apply our results to several interesting logics.

Our main conclusion is that relevance forms an interesting alternative as a minimality criterion for contraction when dealing with non-classical representation formalisms, mainly because relevance is more widely applicable for practical cases than recovery. These results are part of a larger effort towards exploiting and reusing the results of the rich belief change literature for addressing the problem of knowledge evolution in non-classical knowledge representation formalisms.

The rest of the paper is structured as follows: Section 2 contains some discussion on existing related work. Section 3 introduces the background that this paper is based upon, namely the definition of various classes of logics and the corresponding properties holding in such logics, the AGM theory [3], the relevance postulate [11] and the partial meet construction [3]. Section 4 is the main section of this paper, and describes the connection between recovery, relevance and partial meet for non-classical logics, as well as the conditions under which a contraction operator satisfying these postulates can be defined in such logics some discussion on a weaker version of relevance, namely *core-retainment*, is also included. Section 5 applies these results to show that several logics useful in applications do not admit contraction operators that satisfy the AGM postulates, but admit operators that satisfy the AGM postulates with recovery replaced by relevance which we call AGM-relevance. Section 6 provides a summary of the main results appearing in the paper, whereas Section 7 presents a number of possible extensions for future work. The proofs for all the results that appear in this paper can be found in the Appendix; the Appendix also contains several lesser results (lemmas) that do not appear in the main paper.

This paper is based on the PhD theses of two of the authors [1, 12], and some of the results have appeared in workshops and conferences ([18, 19, 20, 21]) in less depth and detail than done here.

2. Related Work

Relevance has been widely used in the context of belief base contraction in both classical [22, 23] and non-classical [24, 25, 26, 17] settings. This popularity followed the realization that recovery is not applicable for belief base contraction [23]. All these works address belief base contraction (as opposed to belief set contraction considered in this paper), and none of them considers recovery, or its relation with relevance, in their context.

A survey of works studying belief change in non-classical logics can be found in Hansson and Wasserman [27]. The limitations of the recovery postulate with regards to its applicability to non-classical logics can be found in Flouris [1], but this work focused mainly on DLs; a similar study for Horn logic appears in [28]. Again, none of these works consider the relation of recovery with relevance in their setting.

Another attempt to circumvent the non-applicability of recovery in non-classical logics is found in Lakemeyer and Lang [29], where an alternative set of postulates, suitable for four-valued logics, is proposed; the main concern in their work is to prove computational tractability instead of verifying representation theorems.

Several works also studied variations of the partial meet construction for non-classical logics. Restall and Slaney [13] proposed the use of first-degree entailment (originally introduced by Anderson [30]) instead of classical entailment; under this alternative (non-classical) semantics, recovery is inapplicable, but an adaptation of partial meet, based on non-maximal remainder sets, can be used as the basis for contraction. This approach was shown to avoid the need for taking the meet of remainder sets, because one can take one of the non-maximal remainder sets as the result of the contraction [31].

Belief change in the Horn fragment of classical propositional logic has been a hot topic in the past years. Delgrande [32] suggested two different ways in which a remainder set can be defined, one based on entailment and the other on inconsistencies. Entailment-based remainder sets (called e-remainders), are defined in the usual way, as maximal subsets of a belief set not implying a given formula (but here a belief set is closed under Horn consequence, instead of classical). The e-remainder constructions are suitable for partial meet contraction.

Inconsistency-based remainder sets (i-remainders) are defined in a slightly different way, as the maximal subsets of a belief set which are consistent with a given formula. This construction is more appropriate for defining revision in logics where negation is not always defined and was used in Ribeiro and Wassermann [24] in the context of description logics. The same paper provides a set of rationality postulates that the operations satisfy, but the postulates are not strong enough to characterize them. Basically, the operations satisfy five of the six basic AGM postulates for contraction, the exception being recovery.

Still considering Horn logics, Booth et al. [31] propose the use of infra-remainder sets, which are defined as any set between full meet and maxichoice contractions. The outcome of contraction is defined as one infra-remainder set, selected by some function. In the same paper, a representation result is given with the recovery postulate being substituted by a postulate closely resembling relevance. However, the postulate refers to infra-remainders. Another representation result was given by Booth et al. [16] showing that contraction using infra-remainders is in fact equivalent to kernel contraction.

Moreover, Zuang and Pagnucco [33, 34] worked on epistemic entrenchment and model based constructions, whereas Delgrande and Wassermann [35] developed a model-theoretic characterization of Horn contraction. Revision operations under the Horn fragment were proposed by Delgrande and Peppas [36], Adaricheva et al. [37] and Creignou et al. [38]. The latter deals also with other fragments of classical propositional logics such as *dual Horn* (allowing at most one literal per clause to be negative) and *Krom* (allowing at most two literals per clause).

3. Background

3.1. Logics

In this work, we will adopt the definition of logics given by Tarski, under which a *logic* is a pair $\langle \mathscr{L}, \mathrm{Cn} \rangle$, where $\mathscr{L}$ is the set of expressions (*propositions*) of the underlying language and $\mathrm{Cn} : 2^{\mathscr{L}} \mapsto 2^{\mathscr{L}}$ is a function mapping sets of propositions to sets of propositions (*consequence operator*). The intuitive meaning of Cn is that a set of propositions A implies exactly the propositions contained in $\mathrm{Cn}(A)$. The consequence operator is required to satisfy the following (intuitively obvious) conditions:

Inclusion: For all A, $A \subseteq \mathrm{Cn}(A)$

Iteration: For all A, $\mathrm{Cn}(A) = \mathrm{Cn}(\mathrm{Cn}(A))$

Monotonicity: For all A, B, if $A \subseteq B$ then $\mathrm{Cn}(A) \subseteq \mathrm{Cn}(B)$

A *belief set* is any set A such that $A = \mathrm{Cn}(A)$. A set of propositions A is *finitely representable* iff there is a finite set A' such that $\mathrm{Cn}(A) = \mathrm{Cn}(A')$. We say that A *implies* B iff $B \subseteq \mathrm{Cn}(A)$.

Logics that satisfy the above three properties (inclusion, iteration, monotonicity) will be called *Tarskian*. Most of the logics used in the literature are

Tarskian, the most notable exception being non-monotonic logics [39] (where monotonicity fails). In this paper, we restrict our attention to Tarskian logics, so, unless otherwise mentioned, the term logic will refer to a Tarskian logic.

One can identify several interesting subclasses of Tarskian logics that satisfy additional properties. Here, we concentrate on the following:

Compactness: A logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is *compact* iff for every $A \subseteq \mathscr{L}$, $\alpha \in \mathscr{L}$, if $\alpha \in \mathrm{Cn}(A)$ then $\alpha \in \mathrm{Cn}(A')$ for some finite $A' \subseteq A$ [40]. Compactness is important because it allows every inference to be finitely determined. Note that compactness does not imply that all belief sets are finitely representable.

Finiteness: A logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is *finite* iff there is a finite number of distinct belief sets. Obviously, all finite logics are compact [40] and their beliefs are finitely representable. Most logics are defined as infinite structures (e.g., by admitting an infinite set of atoms/constants in propositional/first-order logic respectively). However, in practice computer systems can only deal with finite structures (and logics), so the use of infinite logics usually comes with an implicit finiteness assumption (e.g., by limiting the number of atoms/constants that can be used).

Distributive: A logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is *distributive* iff for all A_1, A_2, A_3 it holds that $\mathrm{Cn}(A_1 \cup A_2) \cap \mathrm{Cn}(A_1 \cup A_3) \subseteq \mathrm{Cn}(A_1 \cup (\mathrm{Cn}(A_2) \cap \mathrm{Cn}(A_3)))$. Distributivity corresponds to a generalized version of the "Introduction of Disjunction in the Premises" property that was used in the original AGM paper [3]. It can be shown [1] that logics with this property are isomorphic to distributive lattices [41], so we use here the less verbose term "distributivity" to describe this property. Given that a lattice is distributive if and only if it is isomorphic to a lattice of sets (closed under set union and intersection) [41], the relationship of distributive logics with distributive lattices enforces a "canonical" structure to the belief sets of a distributive logic (for other interesting properties of distributive lattices, see [41]). Note that distributivity combined with monotonicity implies that $\mathrm{Cn}(A_1 \cup A_2) \cap \mathrm{Cn}(A_1 \cup A_3) = \mathrm{Cn}(A_1 \cup (\mathrm{Cn}(A_2) \cap \mathrm{Cn}(A_3)))$; as in this work we only focus on monotonic logics, we will often use the latter expression for distributivity.

Complementarity: A logic satisfies *complementarity* iff for all finitely representable $A \subseteq \mathscr{L}$ there is some finitely representable $A' \subseteq \mathscr{L}$ such that $\mathrm{Cn}(A \cup A') = \mathscr{L}$ and $\mathrm{Cn}(A) \cap \mathrm{Cn}(A') = \mathrm{Cn}(\emptyset)$ (A' is called the *complement* belief of A). This property says that every finitely representable set of propositions has a complement belief. Logics that satisfy complementarity are isomorphic to complemented lattices [1, 41]. Complementarity was studied by Flouris [1] (under a different name) and was shown (along with distributivity) to be important for our study of AGM-compliance (see also the results in Section 4 below).

Boolean A logic is *Boolean* iff it is distributive and closed under negation satisfies complementarity. The term "Boolean" stems from the fact that Boolean logics are isomorphic to Boolean lattices [1, 41].

Closure under connectives: A logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is *closed under a binary connective* $\bullet$ iff $\alpha \bullet \beta \in \mathscr{L}$ for all $\alpha, \beta \in \mathscr{L}$ [3]. The definition can be easily extended for unary and, more generally, $n-$ary connectives. Logics with this property allow compound expressions (like $\alpha \wedge (\neg \beta)$) to be formulated. Note that closure under the connective $\neg$ is similar, but not identical to complementarity. The former refers to propositions, whereas the latter refers to finitely representable beliefs. Furthermore, closure under connectives is a syntactic requirement, so, e.g., $\neg a$ is a unique proposition, and no assumptions are made as to its semantics; on the other hand, complementarity is semantic, so the complement of A as defined above has a specific semantics, and there could be several complements of a single belief (or even none at all).

Supra-classicality: A logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ that is closed under the standard logical connectives $(\wedge, \vee, \rightarrow, \neg)$ is *supra-classical* iff for all A, $\mathrm{Cn}(A)$ contains all classical consequences of A [3]. This property was used in [3] to guarantee "compatibility" with standard logical formalisms, like propositional or first-order logic. Note that supra-classicality makes sense only for logics that are closed under the standard connectives, because tautologies contain such connectives.

Deduction: A logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ that is closed under the connective $\rightarrow$ satisfies *deduction* iff for every $\alpha, \beta \in \mathscr{L}$ and every $A \subseteq \mathscr{L}$, it holds that $\alpha \in \mathrm{Cn}(A \cup \{\beta\})$ iff $\beta \rightarrow \alpha \in \mathrm{Cn}(A)$ [3]. Again, this property was used in [3] to guarantee "classical semantics" for the logics considered, and only makes sense for logics that are closed under $\rightarrow$.

As we will explain in the next subsection, all these properties (except finiteness) were included as assumptions for the logics considered in the original AGM paper [3]. Examples of logics that don't satisfy some of these assumptions will be given in Section 5.

3.2. The AGM Theory

As already mentioned, belief change addresses the problem of how idealized agents change their beliefs [2]. Belief change is a mature field of research, with a rich literature. Here, we concentrate on the most seminal work on belief change, namely the AGM theory, which is most relevant for our work, and its generalization for non-classical logics, as it appeared in Flouris' Thesis [1].

The AGM theory focuses on logics that satisfy the *AGM assumptions*, i.e., logics which are Tarskian, compact, closed under the standard logical connectives $(\wedge, \vee, \rightarrow, \neg)$, supra-classical and satisfy deduction [3]. Logics that satisfy the AGM assumptions will be called *classical*. Examples of classical logics include propositional and first-order logic. Moreover, using the fact that classical logics are closed under the standard connectives, it can be easily shown that all classical logics are also Boolean. On the other hand, several interesting logics are non-classical. In this paper, we will concentrate on the family of Description Logics [4], Horn logic [6] and intuitionistic logic [7]; these logics will be considered in more details in Section 5.

AGM theory considers three operations for belief sets: *expansion* (denoted by $+$), which causes the agent to accept a new proposition unconditionally, *revision* (denoted by $*$), which causes the agent to accept a new proposition consistently, and *contraction* (denoted by $-$), which causes the agent to abandon some proposition. All operations were originally defined between a belief set K and a proposition α. We can easily redefine these operations to apply between a belief set K and a finitely representable set A. Such a set should be interpreted conjunctively, i.e., $\{\alpha, \beta\}$ represents "α and β". As many of the logics we will consider in this paper are not closed under conjunction ($\wedge$), such as most Description Logics, this generalization is important for our purposes.

Expansion is the simplest of these operations and, given a belief set K and a finitely representable set A, it can be defined directly as: $K + A = \mathrm{Cn}(K \cup A)$.

This is not the case for revision and contraction, which cannot be explicitly defined. Let us take contraction for example: to remove $\{b\}$ from $\mathrm{Cn}(\{a, a \to b\})$, one may choose to remove, or weaken, either, or both, of the two propositions $a, a \to b$; for many of these options, one cannot devise purely logical arguments in favor of a choice. The approach chosen by the AGM trio [3] to address this problem was to define a set of rationality restrictions, called the *AGM postulates*, that determine the properties that a rational contraction (or revision) operator should satisfy. This essentially defines a class of operations which are equally plausible to serve as contraction (or revision) operators.

In this paper, we focus on contraction, so we will only present the six contraction postulates that were defined in the AGM paper [3] reformulated so as to be applicable to non-classical logics. Note that the original and reformulated versions are equivalent for classical logics. The reformulation also considered the generalized version of contraction, where the contracted belief is a finitely representable set of propositions, rather than a single proposition. The reformulated AGM postulates are:

Closure: $K - A = \mathrm{Cn}(K - A)$

Success: If $A \not\subseteq \mathrm{Cn}(\emptyset)$ then $A \not\subseteq K - A$

Inclusion: $K - A \subseteq K$

Vacuity: If $A \not\subseteq K$ then $K - A = K$

Extensionality: If $\mathrm{Cn}(A) = \mathrm{Cn}(B)$ then $K - A = K - B$

Recovery: $K \subseteq (K - A) + A$

Closure guarantees that the result of a contraction is a belief set. *Success* states that the retracted belief must be removed unless it is tautological. *Inclusion* states that no new information should be added to the original belief set during contraction. *Vacuity* deals with the limit case where the contracted belief is not a consequence of our beliefs to begin with; in this case nothing should be done. *Extensionality* guarantees that the syntax of the retracted belief does not affect the result. Finally, the most controversial postulate, *recovery*, states that if a belief is contracted and then added again, then the initial belief state should be recovered; this is meant to capture the intuition that only propositions that are somehow related to the contracted belief should be removed.

In the following, the term *AGM postulates* will refer to the above generalized version. A contraction operator satisfying the AGM postulates will be called an *AGM-compliant contraction operator*.

Note that the vacuity and success postulates are based on the fact that A is interpreted conjunctively, so, if K does not imply any of the propositions in A, then it does not imply A as a whole. Thus, it is enough to contract any one proposition from A, rather than all propositions in A; in the terminology of Fuhrmann [22], this corresponds to *choice contraction* (which is more suitable for our setting) rather than *package contraction*.

Makinson defined an operator similar to contraction, called *withdrawal* [10]. A withdrawal operator satisfies the first 5 postulates (i.e., all the AGM postulates except recovery). Thus, withdrawal operators are not required to satisfy any postulate encoding the minimality criterion. This shortcoming leads to awkward results. For example, for any given beliefs K, A such that $A \subseteq K$, the operator $K - A = \mathrm{Cn}(\emptyset)$ is a valid withdrawal operator, despite the fact that its application causes the loss of all beliefs from our knowledge, even those that are irrelevant to the contracted belief.

3.3. Relevance

Recovery is the most controversial AGM postulate and its use as a rationality postulate was criticized by many authors for having unintuitive consequences [9, 10, 11]. Given that recovery is the only AGM postulate that encodes minimality, Hansson proposed to replace recovery by a new postulate called *relevance*, which states that any proposition that is removed from our beliefs during a contraction must somehow contribute to inferring the contracted belief [11]; in other words, all removed propositions were removed for good reasons. As with the AGM postulates, the original formulation of the relevance postulate by Hansson assumed that a single proposition is contracted. A generalized version of the postulate, which is equivalent to the original one for classical logics, but is applicable for all Tarskian logics and allows the contraction of finitely representable beliefs is:

Relevance: If $\beta \in K \setminus K - A$ then there is a K' such that $K - A \subseteq K' \subset K$, $A \nsubseteq \mathrm{Cn}(K')$, but $A \subseteq \mathrm{Cn}(K' \cup \{\beta\})$.

Relevance states intuitively that *only* the sentences β that somehow "contribute" in deriving A are allowed to be removed during a contraction operation. To determine whether β "contributes" in deriving A, we check whether there is a subset K' of K which does not imply A by itself, but would imply A with the addition of β; if not, then β is obviously "irrelevant" to A, and should not be removed from K during contraction, otherwise A and β are somehow related, and β could (but need not) be removed.

Relevance is meant to capture the minimality expected from contraction operations, while avoiding some non-intuitive aspects of recovery. It has been extensively studied in the literature, especially in the context of belief base contraction [23]. However, Hansson proved that relevance and recovery are equivalent in the presence of the other AGM postulates and the AGM assumptions [11]; thus, he concluded that recovery should be accepted as an "emerging property" of belief contraction, rather than a fundamental postulate. As we will show later (Section 4), this surprising result does not hold for many non-classical logics, i.e., there are logics where relevance and recovery are not equivalent.

In the following, we will use the term *AGM-relevance postulates* to refer to the AGM postulates with recovery replaced by relevance, and the term *relevance-compliant contraction operator* to refer to an operator that satisfies the AGM-relevance postulates.

3.4. Partial Meet

The presented postulates determine the properties that a contraction operator should satisfy, but give us no clue as to how a contraction operator should be constructed. This is the role of *constructions*, like the ones presented in [3, 8]. Here, we concentrate on *partial meet contraction*, introduced in [3] (other constructions are left for future work). The idea behind partial meet contraction is that the result of a contraction operator should be the intersection of some (arbitrarily selected) sets which are maximal subsets of K that do not imply A[1].

To define the partial meet construction formally, we define the *remainder set*, which is the set of all maximal subsets of K that do not imply A:

Definition 3.1 (remainder set). *Consider a belief set K and a finitely representable set of propositions A. The* remainder set *$K \perp A$ is the set of all X such that:*

1. $X \subseteq K$
2. $A \nsubseteq Cn(X)$
3. *If* $X \subset X' \subseteq K$ *then* $A \subseteq Cn(X')$

A *selection function* is any function that chooses elements from $K \perp A$. Formally:

Definition 3.2 (selection function). *Let K be a belief set. A* selection function *for K is any function γ such that for every finitely representable A:*

1. $\emptyset \neq \gamma(K \perp A) \subseteq K \perp A$ *if* $K \perp A \neq \emptyset$

[1]Note that the original definitions from [3] have been extended to deal with sets instead of single propositions.

2. $\gamma(K \perp A) = \{K\}$ *otherwise.*

A *partial meet contraction* $-_\gamma$ is an operator defined as the intersection of the elements of $\gamma(K \perp A)$:

$$K -_\gamma A = \bigcap \gamma(K \perp A)$$

One of the central results of the AGM paper [3] was that partial meet contraction and AGM contraction postulates are equivalent, i.e., that every partial meet contraction operator is an AGM-compliant contraction operator and every AGM-compliant contraction operator can be constructed as a partial meet contraction (obviously, the same holds for relevance-compliant contraction operators [11]). This *representation theorem* for partial meet contraction holds for classical logics; in Section 4, we will show that it is not generally true for non-classical logics, and present sufficient conditions under which the theorem holds.

4. Dropping the AGM Assumptions

Many modern logical formalisms used in interesting applications do not satisfy the AGM assumptions, so the above results do not apply for them. As an example, consider the DL formalism [4], which is a unifying framework to define a family of logics (most of which are subsets of first-order logic [4]). Each DL strikes a different balance between complexity and expressivity, so they are useful for different applications. DLs are important because they constitute the major formalism for knowledge representation in the Semantic Web [5]. DLs are non-classical, because they are not closed under the standard logical connectives. The same holds for other Semantic Web languages, like RDF/S [42, 43] and OWL [44]. Similarly, propositional Horn logic [6] is a subset of propositional logic in which the satisfiability problem is polynomial and is widely used in rule-based systems and other applications that depend on low complexity. Horn logic does not satisfy complementarity, so it is not classical. Finally, intuitionistic logic [7] is similar to propositional logic, but with different semantics (e.g., the law of excluded middle fails, so it is not classical), and has been used in various applications, such as formal reasoning about the correctness of programs.

When considering logics that do not satisfy the AGM assumptions, one loses the nice structure of classical logics. As a result, for given K, A it may be the case that there is no suitable belief set to serve as a result of the operation $K - A$ and satisfy the contraction postulates. In addition, the equivalence between partial meet contraction, recovery and relevance fails. These facts can be easily shown with a simple example, such as example 4.1 below. Thus, the standard AGM postulates are not suitable for describing rational contraction operators for certain non-classical logics.

In [1], it was shown that this observation can be extended to several important logics (like many DLs, as well as OWL). Thus, we could consider replacing these postulates with another set that would be more applicable. One possible candidate is relevance [12], that could replace recovery; indeed, the logic of Example 4.1 allows defining relevance-compliant contraction operators, and the same operators can also be constructed using partial meet. As we will see later, this holds for many logics, but not all.

Example 4.1: Consider the following simple logic:

$$\begin{aligned} \mathscr{L} &= \{a, b\} \\ \mathrm{Cn}(\mathscr{L}) = \mathrm{Cn}(b) &= \mathscr{L} \\ \mathrm{Cn}(a) &= \{a\} \\ \mathrm{Cn}(\emptyset) &= \emptyset \end{aligned}$$

The logic is obviously Tarskian. It is also finite, thus compact, and distributive (easily checked by considering all possible triples of A_1, A_2, A_3). It does not satisfy complementarity ($\{a\}$ has no complement), and is not closed with respect to any connectives. Thus, it is not classical.

Moreover, one cannot define a contraction operator satisfying the AGM postulates. In particular, for the operation $\{a, b\} - \{a\}$, none of the possible results (namely $\mathscr{L}$, $\{a\}$, $\{b\}$, $\emptyset$) would satisfy all 6 AGM contraction postulates, as $\mathscr{L}$, $\{a\}$, $\{b\}$ do not satisfy success, whereas $\emptyset$ does not satisfy recovery.

If we replace recovery with relevance, then contraction operators are definable. For example, it is easy to verify that taking $\emptyset$ as the result of $\{a, b\} - \{a\}$, would satisfy the AGM-relevance postulates. Moreover, we can easily verify that for any other pair of belief sets K, A, we can satisfy the AGM-relevance postulates for the operation $K - A$ by just taking $K - A = \emptyset$ whenever $\mathrm{Cn}(K) = \mathrm{Cn}(A)$ and taking $K - A = K$ otherwise.

Finally, it can be easily verified that in all cases the result of a relevance-compliant contraction operator could be constructed as a partial meet contraction, and vice-versa. In the case of $\{a, b\} - \{a\}$ for example, it holds that $\{a, b\} \perp \{a\} = \{\emptyset\}$ (i.e., the singleton set containing just $\emptyset$), so the partial meet construction would (necessarily) give $\{a, b\} - \{a\} = \emptyset$. □

4.1. AGM-Compliance

The observation that the standard AGM postulates cannot be used for many non-classical logics led to the study of *AGM-compliant* logics, i.e., logics that admit a contraction operator satisfying the AGM postulates [1]. Formally:

Definition 4.2. *A logic is* AGM-compliant *iff at least one AGM-compliant contraction operator can be defined in this logic.*

It can be shown that a logic is AGM-compliant if and only if its beliefs can be "decomposed", with respect to any of their sub-beliefs, into two complementary (but not necessarily disjoint) sub-beliefs, as formally described in the following theorem[2]:

Theorem 4.3. *A logic $\langle \mathscr{L}, Cn \rangle$ is AGM-compliant iff for all $K, A \subseteq \mathscr{L}$, where A is finitely representable and $Cn(\emptyset) \subset Cn(A) \subset Cn(K)$, there is a $K' \subseteq \mathscr{L}$ such that $Cn(K') \subset Cn(K)$ and $K' + A = K$.*

As expected, all classical logics are AGM-compliant [1]. In fact, the following stronger result holds:

Theorem 4.4. *Every Boolean logic is AGM-compliant.*

Many popular non-Boolean logics are not AGM-compliant, such as several logics used in the Semantic Web [1, 12]. In Section 5 we will show other examples of non-AGM-compliant logics.

The next result shows that complementarity and AGM-compliance are closely related. Before that we need to define another property:

Definition 4.5 (Chain Conditions). *A* chain of belief sets *in a logic $\langle \mathscr{L}, Cn \rangle$ is a class of sets $\Gamma \subseteq 2^{\mathscr{L}}$ such that every $X \in \Gamma$ is a belief set ($X = Cn(X)$) and for every $X_1, X_2 \in \Gamma$ we have that $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$. A logic satisfies the* descending chain condition *if any chain Γ of belief sets in the logic has a minimum. A logic satisfies the* ascending chain condition *if any chain Γ of belief sets in the logic has a maximum.*

It is easy to verify that both these properties are valid for any finite logic.

Theorem 4.6. *AGM-compliant logics that satisfy the descending chain condition satisfy complementarity.*

This result is useful for AGM-compliance, because, if a logic satisfies the descending chain condition and does not satisfy complementarity, then, per Theorem 4.6 it is necessarily not AGM-compliant. An additional corollary of Theorem 4.6 is that we can show a finite logic to be non-AGM-compliant by showing that it does not satisfy complementarity.

Finally, it has been observed that a withdrawal operator (satisfying the AGM contraction postulates except recovery) can be defined in all Tarskian logics [1]. However, withdrawal operations give us no minimality guarantees, as withdrawal can be satisfied even by completely clearing the knowledge base (as shown above), an option that would not make sense for most applications. Therefore, our proposal is to replace recovery with another postulate that would provide wider applicability. A good candidate for this purpose is relevance, a proposal which is evaluated in the next subsection.

[2] Proofs for all theorems in this paper appear in Appendix A.

4.2. Relevance-Compliance

To begin our study of the feasibility of using relevance as a minimality criterion for contraction operators, we define *relevance-compliance*, which is a notion analogous to AGM-compliance:

Definition 4.7. *A logic is* relevance-compliant *iff at least one relevance-compliant contraction operator can be defined in this logic.*

We can prove that all compact logics are relevance-compliant; consequently, all finite logics are relevance-compliant also:

Theorem 4.8. *Every compact logic* $\langle \mathscr{L}, Cn \rangle$ *is relevance-compliant.*

This result is very useful, because many interesting logics are compact. In particular, all fragments of first-order logic are compact, thus relevance-compliant (this property will be heavily used in Section 5):

Corollary 4.9. *Every logic that is a subset of first-order logic is relevance-compliant.*

In addition, a Boolean logic is also guaranteed to be relevance-compliant (as well as AGM-compliant, per Theorem 4.4):

Theorem 4.10. *Every Boolean logic is relevance-compliant.*

Note that this result does not mean that relevance-compliance is a strictly stronger notion than AGM-compliance. In fact, there are logics which are AGM-compliant, but not relevance-compliant (necessarily, these logics are neither compact nor Boolean), and vice-versa. In Section 5 we will present several logics which are relevance-compliant but not AGM-compliant; it is much more difficult to find a logic that is AGM-compliant but not relevance-compliant Example 4.11 below is one such case). Theorems 4.8 and 4.10 show that most interesting logics (which are either compact or Boolean) are relevance-compliant. Thus, for practical purposes, the postulate of relevance is more widely applicable than recovery.

Example 4.11: Consider the following logic:

$$
\begin{array}{rcl}
\mathscr{L} & = & \{a, x_i, y_j \mid i, j = 1, 2, \dots\} \\
\mathrm{Cn}(a) & = & \{a\} \\
\mathrm{Cn}(y_1) & = & \{x_1, y_1\} \\
\mathrm{Cn}(y_i) & = & \{y_i\} \text{ for } i > 1 \\
\mathrm{Cn}(x_i) & = & \{x_j, y_j \mid j \leq i\} \\
\mathrm{Cn}(X) & = & \mathscr{L} \text{ if } |X| > 1 \text{ and } a \in X \\
\mathrm{Cn}(X) & = & \mathrm{Cn}(x_i) \text{ if } X \text{ is finite, } |X| > 1,\ a \notin X \\
 & & \text{and } x_i \in X \text{ or } y_i \in X \\
 & & \text{and there is no } j > i \text{ such that } x_j \in X \text{ or } y_j \in X \\
\mathrm{Cn}(X) & = & \mathscr{L} \text{ if } X \text{ is infinite} \\
\mathrm{Cn}(\emptyset) & = & \emptyset
\end{array}
$$

Intuitively, $x_1, x_2, \dots$ form a sequence of increasingly stronger propositions, such that x_i implies all x_j, y_j for $j \leq i$. Each y_i implies itself only, whereas a does not imply and is not implied by any finite set of x_i, y_j.

It is easy to show that $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is a Tarskian logic. The logic is not classical, because, e.g., it is not compact. To see this, note that for $X = \{x_i \mid i = 1, 2, \dots\}$, it holds that $a \in \mathrm{Cn}(X)$ but there is no finite subset of X that implies a. It is also not distributive, because, for example, $\mathrm{Cn}(\{x_1\} \cup \{x_2\}) \cap \mathrm{Cn}(\{x_1\} \cup \{a\}) = \mathrm{Cn}(\{x_2\})$, whereas $\mathrm{Cn}(\{x_1\} \cup (\mathrm{Cn}(\{x_2\}) \cap \mathrm{Cn}(\{a\}))) = \mathrm{Cn}(\{x_1\})$.

The logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is AGM-compliant. To see this, take any K, A such that A is finitely representable and $\mathrm{Cn}(\emptyset) \subset \mathrm{Cn}(A) \subset \mathrm{Cn}(K)$. By Theorem 4.3, it suffices to show that there is a K' such that $\mathrm{Cn}(K') \subset \mathrm{Cn}(K)$ and $K' + A = K$. The table below shows the corresponding K' for each such pair (it is obvious to see that K' satisfies the required conditions in each case). Note that there are infinitely many distinct belief sets in this logic, but they are all equivalent to one of the following sets: $\mathscr{L}$, $\{a\}$, x_i for some i, y_i for some i, or $\emptyset$, so the table contains only those types of beliefs. Some pairs K, A are omitted, because for them the relation $\mathrm{Cn}(\emptyset) \subset \mathrm{Cn}(A) \subset \mathrm{Cn}(K)$ does not hold.

The logic is not relevance-compliant. No candidate set X exists that would satisfy relevance and success for the operation $\mathscr{L} - \{a\}$. To see this, note that if X is infinite then it wouldn't satisfy success. If X is finite then there is a x_i such that $x_i \notin \mathrm{Cn}(X)$. By relevance there must exist a K' such that $a \notin \mathrm{Cn}(K')$, but $a \in \mathrm{Cn}(K' \cup \{x_i\})$. By the fact that $a \notin K'$ we have that K' must be finite, so $a \notin \mathrm{Cn}(K' \cup \{x_i\})$. □

K	A	K'
$\mathscr{L}$	$\{x_i\}$	$\{a\}$
	$\{y_i\}$	$\{a\}$
	$\{a\}$	$\{x_1\}$
$\{x_i\}$ for $i > 1$	$\{x_j\}$ for $j < i$	$\{y_i\}$
	$\{y_j\}$ for $j \leq i$	$\{x_{i-1}\}$

Table 1: AGM-compliance Proof (Example 4.11)

The next results show that Theorems 4.4, 4.10 cannot be weakened in a trivial manner, i.e., that we need both distributivity and complementarity to guarantee relevance-compliance and AGM-compliance:

Theorem 4.12. *Complementarity does not imply AGM-compliance or relevance-compliance.*

Theorem 4.13. *Distributivity does not imply AGM-compliance or relevance-compliance.*

4.3. Partial Meet, Recovery and Relevance

It has been shown that, for classical logics, partial meet contraction operators are exactly the same as AGM-compliant, or relevance-compliant operators [3, 11]. Therefore, for classical logics, relevance, recovery and partial meet are equivalent characterizations of contraction operators. However, for non-classical logics, we have (in general) no guarantees that partial meet will satisfy the AGM (or any other) postulates. For instance, applying the construction to Example 4.1 for the operation $\{a, b\} - \{a\}$ would give $\emptyset$, a result that does not satisfy recovery, but satisfies the AGM-relevance postulates. This subsection studies in depth the relationship between partial meet contraction, recovery and relevance for non-classical logics.

We start with the case of compact logics. Compact logics are not necessarily AGM-compliant, so it only makes sense to consider the relation between relevance and partial-meet contraction. In this respect, we can show that compactness is a sufficient condition for partial meet contraction and relevance-compliance to be equivalent; this explains also why relevance and partial meet give the same result in the finite (thus compact) logic of Example 4.1:

Theorem 4.14. *Let $\langle \mathscr{L}, Cn \rangle$ be compact and let K be a belief set and A be finitely representable. $K - A$ is an AGM-relevance contraction operator iff there is a selection function γ such that $K - A = \bigcap \gamma(K \perp A)$.*

For Boolean logics, the situation is more complicated. On the one hand, Boolean logics are both AGM-compliant and relevance-compliant (see Theorems 4.4, 4.10). On the other hand, partial-meet contraction does not always give rational results, because for given K, A, not all sequences of elements that are subsets of K and do not imply A have a maximal, so $K \perp A$ may "miss" some elements; at an extreme, it may even be the case that $K \perp A = \emptyset$, even if $\mathrm{Cn}(A) \neq \mathrm{Cn}(\emptyset)$ and $K \neq \mathrm{Cn}(\emptyset)$. In such cases, a partial-meet contraction would not satisfy the success postulate. This is shown in Theorem 4.15 below:

Theorem 4.15. *For Boolean logics, partial meet contraction is not necessarily equivalent with neither the AGM postulates, nor the AGM-relevance postulates.*

Despite this fact, we can show several interesting results regarding the connection between the three notions (recovery, relevance, partial meet) in Boolean logics. Let us start by investigating the relation between relevance and recovery. We say that *relevance implies recovery* (in the presence of the other AGM postulates) in a logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ iff, whenever a contraction operator $-$ satisfies relevance (and the other AGM postulates), it also satisfies recovery. The definition of *recovery implying relevance* is completely analogous. We say that relevance and recovery are *equivalent* in a logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ iff relevance implies recovery and vice-versa.

The equivalence between relevance and recovery is guaranteed in Boolean logics, as proved below:

Theorem 4.16. *If a logic $\langle \mathscr{L}, Cn \rangle$ is Boolean, $K, A \subseteq \mathscr{L}$ and A is finitely representable, relevance and recovery are equivalent (in the presence of the other AGM postulates).*

It is important to note that a logic being both AGM-compliant and relevance-compliant does not necessarily imply that the two postulates admit the same operations (even though this is true for Boolean logics, per Theorem 4.16):

Theorem 4.17. *There are logics that are relevance-compliant and AGM-compliant such that relevance does not imply recovery, and recovery does not imply relevance.*

The connection of partial-meet contraction with AGM (and relevance) contraction, is not so straightforward. In particular, the problem is that several Boolean logics do not satisfy the ascending chain condition, so we can construct infinite chains of beliefs that do not imply A, and which have no maximum; as a result, $K \perp A$ may "miss" some maximal elements and it could even be the case that $K \perp A = \emptyset$. This causes the equivalence between partial-meet contraction and AGM-compliant/relevance-compliant contraction to fail. The proof of Theorem 4.15 above exploits this fact. This problem does not appear for logics that satisfy the ascending chain condition. More specifically, the following results hold:

Theorem 4.18. *Consider a Boolean logic $\langle \mathscr{L}, Cn \rangle$ such that for all $K, A \subseteq \mathscr{L}$ where A is finitely representable, it holds that $Cn(A) = Cn(\emptyset)$ or $K \perp A \neq \emptyset$. Then, any partial-meet contraction operator is an AGM-compliant contraction operator.*

Theorem 4.19. *Consider a Boolean logic $\langle \mathscr{L}, Cn \rangle$ that satisfies the ascending chain condition. Then, any partial meet contraction operator is an AGM-compliant contraction operator, and any AGM-compliant contraction operator is also a partial meet contraction operator.*

Using Theorem 4.16, these results (Theorems 4.18 and 4.19) can be obviously reformulated for the AGM-relevance postulates.

Another interesting corollary, and one of the central positive results in this paper, states that if a logic is Boolean and compact, or it is Boolean and satisfies the ascending chain condition, then the three notions (relevance, recovery and partial meet) provide equivalent characterizations of contraction. This follows from Theorems 4.14, 4.16 (for Boolean and compact logics) and Theorems 4.16, 4.19 (for Boolean logics satisfying the ascending chain condition):

Corollary 4.20. *Consider a logic $\langle \mathscr{L}, Cn \rangle$ that is Boolean and either is compact or satisfies the ascending chain condition. Consider also two beliefs $K, A \subseteq \mathscr{L}$ such that A is finitely representable. Then the logic is both AGM-compliant and relevance-compliant, and the following are equivalent:*

- *$K - A$ is an AGM-compliant contraction operator*
- *$K - A$ is a relevance-compliant contraction operator*
- *There is a selection function γ such that $K - A = \bigcap \gamma(K \perp A)$*

4.4. Discussion

Table 2 summarizes our results, whereas Figure 1 depicts some of them graphically. In particular, Figure 1 shows that Boolean logics are both AGM-compliant and relevance-compliant, and the two postulates are equivalent characterizations of minimality in the presence of the other AGM postulates (as in classical logics). Compact logics are relevance-compliant, but not necessarily AGM-compliant; for compact logics, relevance-compliant contraction operators can be constructed using partial meet construction, and vice-versa. For logics that are both compact and Boolean, the nice properties of classical logics hold, i.e., we can equivalently use either of the three characterizations of minimality (recovery, relevance, partial meet).

Table 2 is a more complete summary. In the upper part, we show the results related to AGM-compliance and relevance-compliance. For example the lower-right cell indicates that when a logic is compact and Boolean, it is necessarily relevance-compliant, and this was shown in Theorems 4.8, 4.10. It should be emphasized that "no" in this table should be interpreted as "not necessarily", i.e., that the implication does not hold, as indicated by at least one counter-example (the corresponding theorem or counter-example is shown also on the table).

The lower part of Table 2 shows whether a certain equivalence (between AGM-compliant contraction, relevance-compliant contraction and partial meet contraction) holds for a given class of logics, as well as the theorem that shows

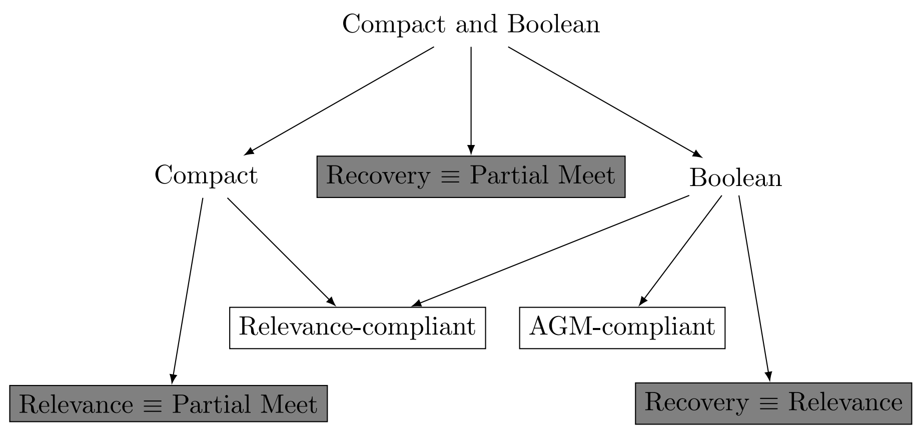

Figure 1: Diagram Summarizing Our Most Important Results

this fact. "ACC" in this table stands for "ascending chain condition". Note that compact logics are not necessarily AGM-compliant, therefore it does not make sense to determine the equivalence between recovery and relevance (or its equivalent, partial meet). We use "N/A" to denote this fact.

Compliance		
	AGM-compliance	Relevance-compliance
Compact	no (Example 4.1)	yes (Theorem 4.8)
Finite	no (Example 4.1)	yes (Theorem 4.8)
Complementarity	no (Theorem 4.12)	no (Theorem 4.12)
Distributive	no (Theorem 4.13)	no (Theorem 4.13)
Boolean	yes (Theorem 4.4)	yes (Theorem 4.10)
Compact and Boolean	yes (Theorem 4.4)	yes (Theorems 4.8, 4.10)

Equivalences			
	Recovery ≡ Partial Meet	Relevance ≡ Partial Meet	Recovery ≡ Relevance
Compact	N/A	yes (Theorem 4.14)	N/A
Boolean	when ACC holds (Theorem 4.19)	when ACC holds (Theorems 4.16, 4.19)	yes (Theorem 4.16)
Compact and Boolean	yes (Corollary 4.20)	yes (Corollary 4.20)	yes (Corollary 4.20)

Table 2: Summary of Results

The lower part of Table 2 (in particular, the second line) implies that the three characterizations of minimality in contraction operations (AGM-compliant contraction, relevance-compliant contraction, partial-meet contraction) are also equivalent for Boolean logics that satisfy the ascending chain condition (see also

Corollary 4.20).

Another corollary of our results is that recovery is not (in general) an "emerging property of relevance" (as Hansson showed for classical logics), and the two postulates capture, in general, distinct notions of minimality, i.e., minimality as expressed by recovery is different from the minimality expressed by relevance. In particular, as proved in Theorem 4.17, there exist logics which are both AGM-compliant and relevance-compliant, for which neither relevance implies recovery nor the converse, as well as logics which are AGM-compliant but not relevance-compliant (see Example 4.11) or vice-versa (see Example 4.1 and the logics in Section 5).

4.5. Some Notes on Core-retainment

In addition to relevance, Hansson also proposed *core-retainment*, a weaker version of the relevance postulate [23]. Core-retainment is similar to relevance, except that the belief set K' whose existence is required must simply be a subset of K (rather than a subset of K and a superset of $K - A$). The formal expression of core-retainment in the general setting considered in this work is as follows:

Core-retainment: If $\beta \in K \setminus K - A$ then there is a K' such that $K' \subseteq K$, $A \nsubseteq \mathrm{Cn}(K')$, but $A \subseteq \mathrm{Cn}(K' \cup \{\beta\})$.

Even though core-retainment is obviously weaker than relevance in general, it has been shown to be equivalent to relevance in the presence of the other AGM postulates and the AGM assumptions [23]. Studying core-retainment is not in the scope of this work; however, some of our results can be easily applied for core-retainment as well, based on the fact that relevance implies core-retainment in any logic.

In particular, we can deduce that every compact and every boolean logic is compliant with core-retainment (as a corollary of Theorems 4.8, 4.10 respectively). Moreover, the proof of Theorem 4.16 (see Appendix A) does not actually use the stronger requirements of relevance, but it only needs the weaker requirements of core-retainment; thus, Theorem 4.16 holds for core-retainment as well, i.e., for boolean logics core-retainment and recovery (as well as relevance of course) are equivalent (in the presence of the other AGM postulates). Similarly, Corollary 4.20 holds for core-retainment as well. Studying further the relationship of core-retainment with the other postulates is a subject of future work.

5. Applying our Results to Useful Logics

In this section we will apply our results on various well-known logics that are interesting for real-world applications and have been used for different purposes. In particular, we will consider Semantic Web languages (such as DLs [4], OWL [45, 46, 47] and RDF/S [42, 43]), Horn Logic [6] and intuitionistic logic [7]. Table 3 summarizes the results presented in the next subsections.

	OWL, RDF/S	DLs subsets of first-order logic	Horn logic	Intuitionistic logic	Classical logics
AGM-compliant	no	no	no	no	yes
Relevance-compliant	yes	yes	yes	yes	yes

Table 3: Logics, AGM-compliance and Relevance-compliance

5.1. Semantic Web Languages

Description Logics (DLs) are a family of formalisms that deal with conceptual knowledge [4]. DLs are being used in the Semantic Web as the main formalism for knowledge representation. The family of DLs is very flexible; it contains several different logics (all defined in a similar manner) each allowing the use of different sets of operators and connectives to form *DL axioms* (corresponding to logical propositions). DL axioms are used to express relationships between concepts (corresponding to unary predicates), roles (corresponding to binary predicates) and individuals (constants). The complexity of the expressible relationships is determined by the available operators and connectives (see [4] for details). As a result, each DL strikes a different balance between complexity and expressivity, which makes them suitable for a variety of applications.

DLs were chosen as the formal background to describe OWL [45, 46, 47], which is the standard language to represent ontologies on the web. OWL comes in several different variants (called "flavors" or "profiles"), such as OWL DL, OWL Lite, OWL 2 QL, OWL 2 EL and others. OWL builds on RDF/S [42, 43], a simple formalism for expressing basic relationships between classes (concepts in DL terminology), properties (roles in DL terminology) and individuals.

The family of DLs is vast, so it is impossible to make any general statements regarding the properties (from the list in Subsection 3.1) that they satisfy. As a rule of thumb, one could say that most DLs are fragments of first-order logic, thus compact. Note however that some of the operators admitted in some DLs (e.g., transitive closure of roles) cannot be expressed in first-order logic. DLs usually admit an infinite number of constants. Highly expressive DLs are often Boolean (provided that they admit operators on roles, as well as on concepts), but most do not satisfy distributivity or complementarity because they don't admit complex roles. DLs are not closed under any of the standard connectives, so supra-classicality and deduction are not applicable here.

Regarding AGM-compliance, one can find a detailed study in Flouris' thesis [1], where most DLs were shown to be non-AGM-compliant. Examples of non-AGM-compliant DLs include logics behind various versions of OWL (such as $\mathcal{SHIF}(\mathbf{D})$, $\mathcal{SHOIN}(\mathbf{D})$ and $\mathcal{SROIQ}$). In addition, Ribeiro's thesis [12] showed that RDF/S and various OWL profiles are not AGM-compliant either. An example of an AGM-compliant DL is $\mathcal{ALCO}(\cap, \cup, \neg(full))$ [1]. The reader is referred to Baader et al. [4] for details on the above DLs.

On the other hand, most DLs are fragments of first-order logic, thus, they are compact [4]. The same holds for all OWL flavors and RDF/S. Thus, all these logics are relevance-compliant:

Corollary 5.1. *Every DL that is a subset of first-order logic, as well as RDF/S and all OWL flavors are relevance-compliant.*

5.2. Horn Logic

Horn Logic was proposed as an efficient restriction of propositional logic, because its satisfiability problem is polynomial. Although much less expressive, Horn logic is widely used in various artificial intelligence applications which require low computational complexity and are willing to trade expressive power for efficiency (e.g., PROLOG[4] or rule-based systems).

Horn logic is compact, as a subset of first-order logic, and infinite. It does not satisfy distributivity. To see this, take a logic having only 3 atoms (a, b, c) and the beliefs $A_1 = \{a \wedge b \to c\}$, $A_2 = \{a \to c\}$, $A_3 = \{b\}$. We note that $\mathrm{Cn}(A_2) \cap \mathrm{Cn}(A_3)$ corresponds to the proposition $a \to b \vee c$, which is not a Horn proposition. The strongest belief that can be composed by Horn propositions and is weaker than both A_2, A_3 is the tautology $\top$. Using this fact, the reader can easily verify that Horn logic is not distributive. The underlying reason behind this failure is that the disjunction of Horn propositions is not necessarily a Horn proposition.

Similarly, even though the complement of a Horn proposition can be expressed as a finite set of Horn propositions (e.g., the complement of $\{a \wedge b \to c\}$ is $\{a, b, \neg c\}$), this is not true for sets of Horn propositions (e.g., the complement of $\{\neg a, \neg b\}$ is $\{a \vee b\}$, i.e., non-Horn, and there is no belief that can be composed of Horn propositions that has the properties of a complement of $\{\neg a, \neg b\}$); thus Horn logic does not satisfy complementarity either. The application of $\neg$, $\vee$ or $\to$ upon Horn propositions does not necessarily result to a proposition that is expressible using (sets of) Horn propositions; thus, Horn logic is not closed under $\neg$, $\vee$ or $\to$.

Recent studies showed that, even though AGM-compliant contraction is not generally possible in Horn logic (i.e., Horn logic is not AGM-compliant), there are belief sets for which AGM-compliant contraction can be defined and a precise characterization of such sets was given in Langlois et al. [28]. Since Horn logic is compact, it is relevance-compliant:

Corollary 5.2. *Horn logic is relevance-compliant.*

The compactness of Horn logic also implies (by Theorem 4.14) that all relevance-compliant operations can be constructed using partial-meet (and vice-versa). In this respect, it is worth mentioning that it has been argued that partial-meet construction (and consequently relevance-contraction) in Horn logic have undesirable properties [32, 15, 16]. Exploring the consequences of this claim and possible alternatives is not in the scope of this work.

[4] `http://www.swi-prolog.org/`

5.3. Intuitionistic Logic

Propositional intuitionistic logic [7] is the logic used by the constructivist mathematicians. It can be described as classical logic missing the law of excluded middle ($\alpha \vee \neg\alpha$ is not a theorem in intuitionistic logic)[5]. In the context of computer science, it has been used as a meta-logic for reasoning about a variety of computational phenomena, including reasoning about programs and programming languages. Intuitionistic logic has the same language ($\mathscr{L}$) as propositional logic, but the connectives have different semantics and are interpreted in a different way (so Cn is different).

Like Horn logic, intuitionistic logic is compact [48] and infinite. Intuitionistic logic is distributive, because it is isomorphic to a distributive lattice called Heyting Algebra[6]. Due to the different semantics of the negation connective ($\neg$), intuitionistic logic does not satisfy complementarity. Even though intuitionistic logic is closed under the standard connectives, the connectives (especially negation) do not have the standard semantics, so supra-classicality is not satisfied; however, deduction holds, as shown by Epstein [49].

We can show that intuitionistic logic is relevance-compliant, but not AGM-compliant:

Theorem 5.3. *Intuitionistic logic is relevance-compliant, but not AGM-compliant.*

6. Summary of Results

The aim of this paper was to shed more light into the applicability of well-known results from the belief change literature into logics that were not in their original scope (i.e., non-classical logics). Our work is motivated by the fact that the problem of evolution is ubiquitous in all forms of knowledge, and several modern knowledge representation formalisms, such as logics used in the Semantic Web (DLs, OWL, RDF/S), logics used in rule-based systems and applications (Horn logic), or logics used to reason about various computational phenomena (intuitionistic logic) are not classical. Being able to exploit the rich belief change literature would greatly aid research efforts addressing the problem of evolution in such formalisms.

Our work focused on the contraction operator, and, in particular, on three different characterizations of contraction operators. The first was proposed by the AGM trio [3], and characterized contraction through a set of postulates, the *AGM postulates*, that included the debated postulate of recovery [9, 10, 11]. The second characterization proposed the replacement of recovery in the AGM postulates with the postulate of *relevance* [11]. The third characterization was actually a constructive way to define contraction operators (called *partial meet*

[5]See `http://plato.stanford.edu/entries/logic-intuitionistic` for a brief introduction for intuitionistic logic.

[6]`http://en.wikipedia.org/wiki/Heyting_algebra`

contraction [3]). In classical logics, it can be shown that all three characterizations are equivalent, in the sense that they allow the definition of the same contraction operators; however, this is not true for non-classical logics. Moreover, there are non-classical logics where the proposed postulates fail altogether, i.e., there are logics where one cannot define neither an AGM-compliant contraction operator nor a relevance-compliant contraction operator.

We addressed these problems by studying different useful classes of (non-classical) logics and formulating sufficient conditions for the applicability of the two different sets of postulates in such logics; in addition, we gave sufficient conditions for the different characterizations to be equivalent, and proved some negative results using counter-examples. The most important of our results are summarized in Figure 1 and Table 2. Note that the results for classical logics that appear in the belief change literature are corollaries of our more general results.

For practical purposes, our main conclusion is that relevance has wider applicability in practically useful representation formalisms. To support this conclusion, we listed several practically useful representation formalisms where we can apply the AGM-relevance postulates but not the AGM postulates (see Table 3).

7. Extensions and Future Research Goals

As already mentioned, this paper can be seen as the continuation of a larger effort to adapt results from the rich belief change literature in modern logics that were not in its original scope. In this respect, there are several results that are related to the AGM theory, or its alternatives, that were left out of this study and will be considered in future work (such as core-retainment, briefly discussed in Subsection 4.5).

One important branch of future work should include the so-called *AGM supplementary postulates*, which were proposed in the AGM paper [3] to control how a contraction operator should behave when a contracted proposition is "broken up" in its constituents; in other words, supplementary postulates describe the relation between $K - \alpha$, $K - \beta$ and $K - (\alpha \wedge \beta)$. Some preliminary results in this direction have appeared in [1], where some special cases of logics admitting operators satisfying all AGM postulates (basic and supplementary) were identified. However, no complete characterization exists to date; also, the combination of the supplementary postulates with the relevance postulate has not been considered.

The main difficulty with the supplementary postulates is that, unlike the basic ones, they have a "global" character, in the sense that they force the result of each contraction (say $K - \alpha$) to affect (and to be affected by) the result of all others (e.g., by $K - (\alpha \wedge \beta)$ for all β). Thus, a contraction operator satisfying both the basic and the supplementary postulates should be defined via a complex unifying scheme that determines the result of $K - \alpha$ for all α, rather than for each α in isolation. Defining such a scheme in the general setting considered in this paper has been proven difficult. An additional difficulty in our more general setting is that the supplementary postulates depend on the internal structure of sentences, a feature which complicates things when operators like $\wedge$ are unavailable.

Another postulate that could be considered as an alternative to recovery is *optimal recovery*, which is a weaker version of recovery and was proposed in [50]. Under this postulate, the contraction of a proposition, followed by its re-addition (i.e., $(K - A) + A$), should result in the strongest possible belief set; in the case of AGM-compliant logics, this is K itself, so the postulate is equivalent to the original one in this case, but weaker in general.

The main difficulty with this postulate is that it is hard to guarantee (in general) the existence of the required maximal belief set. Thus, it seems like compliance with this postulate is related in some way to the ascending chain condition; uncovering the exact conditions for compliance with optimal recovery is in our future plans.

Of course, future study should also include the operation of *revision* [3]. Revision is based on a similar set of AGM postulates (basic and supplementary), and can be defined via contraction through the so-called *Levi identity* [3]. Even though revision is generally considered a more useful operator for practical purposes [23], it is often viewed as a "derived" operator (via contraction and the Levi identity), so most works focus on the study of contraction.

Unfortunately, this viewpoint is problematic in our generalized setting, because the Levi identity uses in a crucial way the negation of propositions. Earlier efforts showed that even the original revision postulates themselves [3] are hard to adapt for the generalized setting [1]. Various proposals to overcome this problem can be found in works by Qi and Du [51] and Ribeiro and Wassermann [52]. We plan to work further on this problem in the future, along the lines of Ribeiro and Wassermann [52].

A. Proofs

Proof of Theorem 4.3: A logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is AGM-compliant iff for all $K, A \subseteq \mathscr{L}$, where A is finitely representable and $\mathrm{Cn}(\emptyset) \subset \mathrm{Cn}(A) \subset \mathrm{Cn}(K)$, there is a $K' \subseteq \mathscr{L}$ such that $\mathrm{Cn}(K') \subset \mathrm{Cn}(K)$ and $K' + A = K$.

($\Rightarrow$) Suppose that $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is AGM-compliant and set $K' = K - A$. Then, by closure and inclusion, $\mathrm{Cn}(K') = K' \subseteq K \subseteq \mathrm{Cn}(K)$. If $\mathrm{Cn}(K') = \mathrm{Cn}(K)$ then $A \subset \mathrm{Cn}(K')$, a contradiction by success. So $\mathrm{Cn}(K') \subset \mathrm{Cn}(K)$. By recovery, $K \subseteq K' + A$ and since $K' \subseteq K, A \subseteq K$ it follows that $K = K' + A$.
($\Leftarrow$) If $\mathrm{Cn}(A) = \mathrm{Cn}(\emptyset)$ or $\mathrm{Cn}(A) \nsubseteq \mathrm{Cn}(K)$ then take $K - A = \mathrm{Cn}(K)$; we can easily show that this result satisfies all AGM postulates. If $\mathrm{Cn}(A) = \mathrm{Cn}(K)$ then take $K - A = \mathrm{Cn}(\emptyset)$; again, we can easily show that this result satisfies all AGM postulates. For the more interesting case where $\mathrm{Cn}(\emptyset) \subset \mathrm{Cn}(A) \subset \mathrm{Cn}(K)$, take a $K - A = \mathrm{Cn}(K')$ where K' is such that $\mathrm{Cn}(K') \subset \mathrm{Cn}(K)$ and $K' + A = K$ (which is guaranteed to exist by the hypothesis). Then, closure is obviously satisfied. If $A \subseteq K'$ then $K' + A = K' \subset K$ a contradiction, so success is satisfied. Inclusion, vacuity and recovery are trivially satisfied by the hypotheses. Extensionality can be guaranteed as long as we select the same K' for all A' for which $\mathrm{Cn}(A) = \mathrm{Cn}(A')$ (note that this is always possible to do, because if K' satisfies the hypotheses for A it also satisfies them for A' and vice-versa).

□

Let us denote by $K^-(A)$ the following set of beliefs: $K^-(A) = \{X \mid X = \mathrm{Cn}(X), A \nsubseteq X, \mathrm{Cn}(A \cup X) = \mathrm{Cn}(K)\}$. The following can be shown for $K^-(A)$:

Lemma A.1. *A logic $\langle \mathscr{L}, Cn \rangle$ is AGM-compliant iff for every K, A such that A is finitely representable and $Cn(\emptyset) \subset Cn(A) \subset Cn(K)$ we have that $K^-(A) \neq \emptyset$.*

Proof: ($\Rightarrow$) Suppose that $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is AGM-compliant and take K, A as above. By Theorem 4.3, there is some K' such that $\mathrm{Cn}(K') \subset \mathrm{Cn}(K)$ and $K' + A = K$. Suppose that $A \subseteq K'$ then $K' + A = K' \subset \mathrm{Cn}(K)$ a contradiction, so $A \nsubseteq K'$, thus $K' \in K^-(A)$.
($\Leftarrow$) Take some K, A as in the lemma's hypotheses and set $K' \in K^-(A)$. Since $\mathrm{Cn}(A \cup K') = \mathrm{Cn}(K)$, it follows that $\mathrm{Cn}(K') \subseteq \mathrm{Cn}(K)$. If $K' = \mathrm{Cn}(K') = \mathrm{Cn}(K)$ then $A \subseteq K'$, a contradiction by definition, so $\mathrm{Cn}(K') \subset \mathrm{Cn}(K)$. Thus, by Theorem 4.3, $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is AGM-compliant. □

Lemma A.2. *In a distributive logic $\langle \mathscr{L}, Cn \rangle$, if some belief A has a complement, then it is unique, modulo Cn.*

Proof: Consider some belief A, and two complements B, C. Then:
$\mathrm{Cn}(B) = \mathrm{Cn}(B \cup (\mathrm{Cn}(A) \cap \mathrm{Cn}(C))) =$
$= \mathrm{Cn}(B \cup A) \cap \mathrm{Cn}(B \cup C) = \mathrm{Cn}(B \cup C)$.
Similarly, we can show that $\mathrm{Cn}(C) = \mathrm{Cn}(B \cup C)$, so $\mathrm{Cn}(B) = \mathrm{Cn}(C)$. □

Using this lemma, we can use the notation $\neg A$ for the complement of a belief A in distributive logics.

Lemma A.3. *Consider a Boolean logic $\langle \mathscr{L}, Cn\rangle$. If A is a finitely representable belief, and K a belief such that $Cn(\emptyset) \subset Cn(A) \subset Cn(K)$, then $Cn(K) \cap Cn(\neg A) \in K^{-}(A)$.*

Proof: Set $B = \mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A)$. Obviously, $B = \mathrm{Cn}(B)$.
Moreover, $\mathrm{Cn}(A \cup B) = \mathrm{Cn}(A \cup (\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A)))$. By distributivity this is equal to $\mathrm{Cn}(A \cup K) \cap \mathrm{Cn}(A \cup \neg A) = \mathrm{Cn}(K)$.
If $A \subseteq B$ then $\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A) = B = \mathrm{Cn}(B) = \mathrm{Cn}(B \cup A) = \mathrm{Cn}(K)$, so $\mathrm{Cn}(\neg A) \supseteq \mathrm{Cn}(K) \supset \mathrm{Cn}(A)$, which can only be true if $\mathrm{Cn}(A) = \mathrm{Cn}(\emptyset)$, a contradiction. Thus, $A \nsubseteq B$.
We conclude that $B \in K^{-}(A)$. □

Proof of Theorem 4.4: Every Boolean logic is AGM-compliant.

Obvious by Lemma A.3 together with Theorem 4.3. □

Proof of Theorem 4.6: AGM-compliant logics that satisfy the descending chain condition satisfy complementarity.

Suppose that the logic is $\langle \mathscr{L}, \mathrm{Cn}\rangle$, and take any $A \subseteq \mathscr{L}$ such that $\mathrm{Cn}(\emptyset) \subset \mathrm{Cn}(A) \subset \mathrm{Cn}(\mathscr{L})$. By the AGM-compliance of $\langle \mathscr{L}, \mathrm{Cn}\rangle$, we have that $\mathscr{L}^{-}(A) \neq \emptyset$. By the descending chain condition, there is some minimal $X \in \mathscr{L}^{-}(A)$. We claim that X is a complement of A. Since $X \in \mathscr{L}^{-}(A)$ then $X + A = \mathscr{L}$. Now suppose that $\mathrm{Cn}(X) \cap \mathrm{Cn}(A) \neq \mathrm{Cn}(\emptyset)$. Then, $\mathrm{Cn}(\emptyset) \subset \mathrm{Cn}(X) \cap \mathrm{Cn}(A) \subset \mathrm{Cn}(X)$. Since the logic is AGM-compliant, there is $Y \in X^{-}(\mathrm{Cn}(X) \cap \mathrm{Cn}(A))$. Thus, $X = Y + (\mathrm{Cn}(A) \cap \mathrm{Cn}(X)) \subseteq Y + A$. However, since $A \subseteq Y + A$, we have that $X + A \subseteq Y + A$. Hence $Y + A = \mathscr{L}$.
In this case $\mathrm{Cn}(Y) \subset \mathrm{Cn}(X)$ and $Y + A = \mathscr{L}$. It follows that X is not minimal in $\mathscr{L}^{-}(A)$, a contradiction. Hence $\mathrm{Cn}(X) \cap \mathrm{Cn}(A) = \mathrm{Cn}(\emptyset)$.
For the limit cases where $\mathrm{Cn}(A) = \mathrm{Cn}(\emptyset)$ or $\mathrm{Cn}(A) = \mathrm{Cn}(\mathscr{L})$ take $X = \mathscr{L}$, $X = \emptyset$ respectively; it is trivial to show that X is the complement of A. □

Definition A.4. *A logic $\langle \mathscr{L}, Cn\rangle$ satisfies the* upper-bound property *iff for every $K \subseteq \mathscr{L}$, every $X \subseteq K$ and every $\alpha \in \mathscr{L}$ for which $\alpha \notin Cn(X)$, there is a X' such that $X \subseteq X'$ and $X' \in K \perp \alpha$.*

Definition A.5. *A logic $\langle \mathscr{L}, Cn\rangle$ satisfies the* generalized upper-bound property *iff for every $K \subseteq \mathscr{L}$, every $X \subseteq K$ and every finitely representable $A \subseteq \mathscr{L}$ for which $A \nsubseteq Cn(X)$, there is a X' such that $X \subseteq X'$ and $X' \in K \perp A$.*

Lemma A.6. *Every compact logic satisfies the generalized upper bound property.*

Proof: Take some $K \subseteq \mathscr{L}$, a $X \subseteq K$ and some finitely representable $A \subseteq \mathscr{L}$ for which $A \nsubseteq \text{Cn}(X)$. Arrange the elements of K in a sequence $\beta_1, \beta_2, \ldots$[7]. Let $X_0 = X$ and for all $i \geq 1$ define X_i as follows:

$$X_i = \begin{cases} X_{i-1} & \text{if } A \subseteq \text{Cn}(X_{i-1} \cup \{\beta_i\}) \\ X_{i-1} \cup \{\beta_i\} & \text{otherwise.} \end{cases}$$

Let $K' = \bigcup_i X_i$. Since A is finitely representable, there is a finite A', say $A' = \{\alpha_1, \ldots \alpha_n$, which is equivalent to A. Now suppose that $A \subseteq \text{Cn}(K')$; then $A' \subseteq \text{Cn}(K')$. By compactness it follows that for every α_j with $1 \leq j \leq n$ there is a finite K_j such that $K_j \subseteq K'$ and $\alpha_j \in \text{Cn}(K_j)$. It follows that $A' \subseteq \text{Cn}(\bigcup_j K_j)$ and, since $\bigcup_j K_j$ is finite, $A' \subseteq \text{Cn}(X_i)$ for some i, so $A \subseteq \text{Cn}(X_i)$ which is a contradiction by the construction of X_i. It follows that $A \nsubseteq \text{Cn}(K')$. Furthermore, if we suppose that there exists some K'' such that $K' \subset K'' \subseteq K$ then there is $\beta \in K$ such that $\beta \in K''$ and $\beta \notin K'$. By construction we have that $A \subseteq \text{Cn}(K' \cup \{\beta\})$, hence, $A \subseteq \text{Cn}(K'')$.
Of course $X \subseteq K' \subseteq K$ which concludes the proof because $X \subseteq K' \in K \perp A$. □

Lemma A.7. *Let $\langle \mathscr{L}, Cn \rangle$ be a compact logic. Let $K \subseteq \mathscr{L}$ be a belief set and $A \subseteq \mathscr{L}$ be finitely representable. Then $K \perp A = \emptyset$ iff $A \subseteq Cn(\emptyset)$.*

Proof: Follows directly from Lemma A.6 and the definitions. □

Lemma A.8. *Suppose that $\langle \mathscr{L}, Cn \rangle$ is compact and let K be a belief set. Then $K \perp A = K \perp B$ iff for every $K' \subseteq K$ it holds that $A \subseteq Cn(K')$ iff $B \subseteq Cn(K')$.*

Proof: Suppose that there is $K' \subseteq K$ such that $A \subseteq \text{Cn}(K')$ and $B \nsubseteq \text{Cn}(K')$. By the upper-bound property there is a K'' such that $K' \subseteq K''$ and $K' \in K \perp B = K \perp A$, a contradiction. Thus, $B \subseteq \text{Cn}(K')$. Similarly, we can prove that if $B \subseteq \text{Cn}(K')$ then $A \subseteq \text{Cn}(K')$.
The converse is trivial. □

Lemma A.9. *Consider a compact logic $\langle \mathscr{L}, Cn \rangle$, a belief set K and a finitely representable set A, such that $Cn(A) \neq Cn(\emptyset)$. Take any $Y \subseteq K \perp A$, $Y \neq \emptyset$. Then the operation $K - A = \bigcap_{X \in Y} X$ satisfies the AGM-relevance postulates.*

Proof: *Inclusion* and *closure* follow from the definition of Y. For *vacuity* notice that if $A \nsubseteq K$ then $K \perp A = \{K\}$. *Extensionality* follows from the fact that if $\text{Cn}(A) = \text{Cn}(B)$ then $K \perp A = K \perp B$. *Success* follows from Lemma A.7 and the definition of remainder sets. Finally, for *relevance*, take any $K' \in Y$ and any $\beta \in K$ such that $\beta \notin K - A$. By the definition of $K - A$ and remainder sets,

[7] This step depends on the language being enumerable which is a reasonable assumption. If the language is not enumerable then the proof follows using transfinite induction and will depend on some version of the axiom of choice, but this case will be omitted here.

we have that $K - A \subseteq K' \subseteq K$ and that $A \nsubseteq \mathrm{Cn}(K')$, but $A \subseteq \mathrm{Cn}(K' \cup \{\beta\})$.
$\square$

Proof of Theorem 4.8: Every compact logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is relevance-compliant.

Follows directly from Lemma A.9; we only need to additionally consider the trivial case where $\mathrm{Cn}(A) = \mathrm{Cn}(\emptyset)$. $\square$

Lemma A.10. *Assume that $\langle \mathscr{L}, Cn \rangle$ is Boolean. If $\beta \in Cn(K)$ and $-$ satisfies recovery then $Cn(\neg A) \cap Cn(\beta) \subseteq K - A$.*

Proof: By recovery $\beta \in K - A + A = K$. It follows that $\mathrm{Cn}(\beta) \subseteq K - A + A$ and $\mathrm{Cn}(\beta) \cap \mathrm{Cn}(\neg A) \subseteq K - A + A \cap K - A + (\neg A)$. By distributivity $\mathrm{Cn}(\neg A) \cap \mathrm{Cn}(\beta) \subseteq K - A + (\mathrm{Cn}(A) \cap \mathrm{Cn}(\neg A)) = K - A$. $\square$

Lemma A.11. *If a logic $\langle \mathscr{L}, Cn \rangle$ is Boolean, $K, A \subseteq \mathscr{L}$ and A is finitely representable, recovery implies relevance (in the presence of the other AGM postulates).*

Proof: Take $K - A$ the result of a contraction operation that satisfies the AGM postulates (by Theorem 4.4 there is such an operation). We will show that it also satisfies relevance.
To prove that the logic satisfies relevance we must have that if $\beta \in K$ and $\beta \notin K - A$ then there is K' such that: 1) $K - A \subseteq K' \subseteq K$, 2) $A \subseteq K' + \{\beta\}$ and 3) $A \nsubseteq \mathrm{Cn}(K')$. We are going to prove this for $K' = K - A + (\mathrm{Cn}(A) \cap \mathrm{Cn}(\neg\{\beta\}))$.
For the case where $A \nsubseteq K$ we have by *vacuity* that $K - A = K$. It follows that there is no β satisfying both conditions. Hence, relevance is trivially satisfied.
Let's consider that $A \subseteq K$:

1) $K - A \subseteq K'$ follows from the construction. To see that $K' \subseteq K$, notice that by *inclusion* $K - A \subseteq K$. It follows that $K - A \cup (\mathrm{Cn}(A) \cap \mathrm{Cn}(\neg\{\beta\})) \subseteq K + A = K$, because $A \subseteq K$.

2) $(K - A \cup (\mathrm{Cn}(A) \cap \mathrm{Cn}(\neg\{\beta\}))) + \{\beta\} = K - A \cup ((A + \{\beta\}) \cap (\neg\{\beta\} + \{\beta\})) = K - A \cup (A + \{\beta\})$. Hence $A \subseteq K' + \{\beta\}$.

3) Suppose that $A \subseteq K - A + (\mathrm{Cn}(A) \cap \mathrm{Cn}(\neg\{\beta\})) \subseteq K - A + \neg\{\beta\}$. It follows that $\mathrm{Cn}(A) \cap \mathrm{Cn}(\beta) \subseteq K - A + \neg\{\beta\} \cap (K - A + \{\beta\})$. By distributivity we have that $\mathrm{Cn}(A) \cap \mathrm{Cn}(\beta) \subseteq K - A + (\mathrm{Cn}(\beta) \cap \mathrm{Cn}(\neg\{\beta\}))) = K - A$.

Since $\beta \in \mathrm{Cn}(K)$ by Lemma A.10 $\mathrm{Cn}(\neg A) \cap \mathrm{Cn}(\beta) \subseteq K - A$. Since $\mathrm{Cn}(\neg A) \cap \mathrm{Cn}(\beta) \subseteq K - A$ and $\mathrm{Cn}(A) \cap \mathrm{Cn}(\beta) \subseteq K - A$, we have that $K - A \supseteq (\mathrm{Cn}(\neg A) \cap \mathrm{Cn}(\beta)) \cup (\mathrm{Cn}(A) \cap \mathrm{Cn}(\beta)) = (\mathrm{Cn}(A) \cap \mathrm{Cn}(\neg A)) + \{\beta\} = \mathrm{Cn}(\beta)$. However, this contradicts the definition of β. Hence, $A \nsubseteq K'$. $\square$

Proof of Theorem 4.10: Every Boolean logic is relevance-compliant.

Follows from Theorem 4.4 and Lemma A.11. $\square$

Proof of Theorem 4.12: Complementarity does not imply AGM-compliance or relevance-compliance.

Consider the following logic:

$$\begin{aligned}
\mathscr{L} &= \{a, b, c\} \\
\mathrm{Cn}(\mathscr{L}) = \mathrm{Cn}(\{a, c\}) = \mathrm{Cn}(\{a, b\}) &= \mathscr{L} \\
\mathrm{Cn}(\{b, c\}) &= \{b, c\} \\
\mathrm{Cn}(c) &= \{b, c\} \\
\mathrm{Cn}(a) &= \{a\} \\
\mathrm{Cn}(b) &= \{b\} \\
\mathrm{Cn}(\emptyset) &= \emptyset
\end{aligned}$$

The logic is obviously Tarskian and satisfies complementarity (a is a complement of b and c, whereas b is a complement of a). However, it is not AGM-compliant, because it can be easily verified that there is no result that would satisfy the AGM postulates for the operation $\{c\} - \{b\}$. Thus, this logic shows that complementarity does not imply AGM-compliance.

Regarding relevance-compliance, consider the logic of Example 4.11. This logic satisfies complementarity, which can be verified by noticing that $\mathrm{Cn}(a)$ is the complement of every belief B which is neither tautological ($\mathrm{Cn}(B) = \mathrm{Cn}(\emptyset)$), trivial ($\mathrm{Cn}(B) = \mathscr{L}$), or equivalent to $\mathrm{Cn}(a)$, whereas x_1 is the complement of a. However, as was shown in Example 4.11, the logic is not relevant-compliant. □

Proof of Theorem 4.13: Distributivity does not imply AGM-compliance or relevance-compliance.

Consider the following logic:

$$\begin{aligned}
\mathscr{L} &= \{z, y, x_i \mid i = 1, 2, \dots\} \\
\mathrm{Cn}(\mathscr{L}) &= \mathscr{L} \\
\mathrm{Cn}(y) &= \mathscr{L} \\
\mathrm{Cn}(z) &= \{z, x_i \mid i = 1, 2, \dots\} \\
\mathrm{Cn}(x_i) &= \{x_j \mid j \leq i\} \\
\mathrm{Cn}(X) &= \mathrm{Cn}(y) \text{ if } y \in X \\
\mathrm{Cn}(X) &= \mathrm{Cn}(z) \text{ if } z \in X, y \notin X \\
\mathrm{Cn}(X) &= \mathrm{Cn}(x_i) \text{ if } X \text{ is finite, } z, y \notin X \\
& \text{ and } x_i \in X \\
& \text{ and there is no } j > i \text{ such that } x_j \in X \\
\mathrm{Cn}(X) &= \mathrm{Cn}(z) \text{ if } X \text{ is infinite, } z, y \notin X \\
\mathrm{Cn}(\emptyset) &= \emptyset
\end{aligned}$$

Intuitively, $x_1, x_2, \dots$ form a sequence of increasingly stronger propositions, such that x_i implies all x_j for $j \leq i$. The proposition z is the upper bound of the sequence $\mathrm{Cn}(x_i)$, in the sense that it implies all x_i, and, even though it is not implied by any finite set $\{x_{i_1}, \ldots, x_{i_n}\}$, it is implied by infinite sets of the form $\{x_i \mid i \in I\}$. Finally, y is a trivial statement, implying the entire $\mathscr{L}$.

It is easy to see that the logic is Tarskian. Due to the fact that its beliefs form a chain, the logic can be shown to be distributive, by simply considering any three beliefs A_1, A_2, A_3 in the chain: for example, if A_1 implies both A_2, A_3 then $\mathrm{Cn}(A_1 \cup A_2) \cap \mathrm{Cn}(A_1 \cup A_3) = \mathrm{Cn}(A_1 \cup (\mathrm{Cn}(A_2) \cap \mathrm{Cn}(A_3))) = Cn(A_1)$; we can easily show the same for the other cases.

Now consider the operation $\mathscr{L} - \{x_1\}$. The only result that would satisfy success would be $\emptyset$, which obviously does not satisfy recovery, so the logic is not AGM-compliant.

Regarding relevance, consider the operation $\mathscr{L} - \{z\}$. To satisfy success, one should take some finite set of the form $X = \{x_{i_1}, \ldots, x_{i_n}\}$ as the result of the operation. We will show that this result would not satisfy relevance. Indeed, since X is finite, there is some x_j such that $x_j \notin \mathrm{Cn}(X)$. Let's take $\beta = x_j$ and any K' such that $X \subseteq K' \subset \mathscr{L}$ such that $\{z\} \nsubseteq K'$. Since $\{z\} \nsubseteq K'$, it follows that K' consists of a finite number of x_i, so it cannot be the case that $\{z\} \subseteq \mathrm{Cn}(K' \cup \{x_j\})$, which completes the proof. □

Proof of Theorem 4.14: Let $\langle \mathscr{L}, \mathrm{Cn} \rangle$ be compact and let K be a belief set and A be finitely representable. $K - A$ is an AGM-relevance contraction operator iff there is a selection function γ such that $K - A = \bigcap \gamma(K \perp A)$.

($\Rightarrow$) To see that every operator $-$ satisfying the six postulates can be constructed as a partial meet contraction, just define the selection function γ as $\gamma(K \perp A) = \{K\}$ when $K \perp A = \emptyset$ and $\gamma(K \perp A) = \{X \in K \perp A : K - A \subseteq X\}$ otherwise.

First we need to prove that γ is well defined i.e. if $K \perp A = K \perp B$ then $\gamma(K \perp A) = \gamma(K \perp B)$. If $A \nsubseteq K$ then by Lemma A.8, $B \nsubseteq K$ and by *vacuity* we have that $K - A = K - B = \{K\}$. It follows that $\gamma(K \perp A) = \gamma(K \perp B)$.

If $A \subseteq K$ then $\mathrm{Cn}(A) \subseteq \mathrm{Cn}(K)$ and by Lemma A.8 we have that $B \subseteq \mathrm{Cn}(A)$, since $A \subseteq \mathrm{Cn}(A)$. Analogously, $A \subseteq \mathrm{Cn}(B)$. It follows that $\mathrm{Cn}(A) = \mathrm{Cn}(B)$ and then, by *extensionality* $K - A = K - B$. Hence $\gamma(K \perp A) = \gamma(K \perp B)$.

Moreover, from *success*, *inclusion* and Lemma A.6, it follows that if $K \perp A \neq \emptyset$, then $\gamma(K \perp A) \neq \emptyset$, i.e., γ is a selection function.
Now we prove that $K - A = \bigcap \gamma(K \perp A)$. First, suppose that $A \subseteq \mathrm{Cn}(\emptyset)$; it follows by definition that $K - A = \bigcap \gamma(K \perp A) = \{K\}$. Now suppose that $A \nsubseteq \mathrm{Cn}(\emptyset)$, we have by the definition of γ that $K - A \subseteq \bigcap \gamma(K \perp A)$. To prove that $\bigcap \gamma(K \perp A) \subseteq K - A$ we will prove that if $\beta \notin K - A$ then $\beta \notin \bigcap \gamma(K \perp A)$. If $\beta \notin K$ then (by definition) $\beta \notin \bigcap \gamma(K \perp A)$, so let us suppose that $\beta \in K$. By *relevance*, we have that there is a K' such that $K - A \subseteq K' \subseteq K$, $A \nsubseteq \mathrm{Cn}(K')$, but $A \subseteq \mathrm{Cn}(K' \cup \{\beta\})$. By Lemma A.6 we have that there is K'' such that $K' \subseteq K'' \in K \perp A$, but it is easy to see that $\beta \notin K''$ (otherwise we would have $A \subseteq \mathrm{Cn}(K'')$). It follows from $K - A \subseteq K' \subseteq K''$ that $K'' \in \gamma(K \perp A)$ and we conclude that $\beta \notin \bigcap \gamma(K \perp A)$.
($\Leftarrow$) Obvious by Lemmas A.9, A.7. □

Proof of Theorem 4.15: For Boolean logics, partial meet contraction is not necessarily equivalent with neither the AGM postulates, nor the AGM-relevance postulates.

Let $\mathcal{R}$ be the set of real numbers, and $\mathcal{Q}$ the set of rational numbers. Consider the following logic:

$$\begin{aligned}\mathscr{L} &= \{x_{q_1,q_2} \mid q_1, q_2 \in \mathcal{Q} \cup \{-\infty, +\infty\}\} \\ \mathrm{Cn}(A) &= \{x_{q_1,q_2} \mid (q_1, q_2) \cap (\mathcal{R} \setminus \mathcal{Q}) \subseteq \bigcup_{x_{q'_1,q'_2} \in A} ((q'_1, q'_2) \cap (\mathcal{R} \setminus \mathcal{Q}))\}\end{aligned}$$

In the above formulas (q_1, q_2) stands for the open interval between the numbers $q_1, q_2 \in \mathcal{Q} \cup \{-\infty, +\infty\}$. Note that $\mathscr{L}$ is comprised of elements x_{q_1,q_2} representing sets of a special form, namely subsets of irrational numbers that are contained within some open interval bounded by any two rational numbers, or infinity. Intuitively, the semantics of implication (as encoded by Cn) is the same as the subset relation of sets. More specifically, if we view each $x_{q_1,q_2} \in \mathscr{L}$ as the set of the irrational numbers in (q_1, q_2) and a belief as the union of all the sets represented by the x_{q_1,q_2} that it contains, then A implies B iff the set that A corresponds to is larger (per the subset relation) than the set that B corresponds to. For example, $x_{1,3} \in \mathrm{Cn}(\{x_{0,2}, x_{2,4}\})$ and $x_{0,1} \in \mathrm{Cn}(\{x_{0,2}\})$.

It is easy to show that $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is a Tarskian logic.

In addition, $\langle \mathscr{L}, \mathrm{Cn} \rangle$ satisfies complementarity. Indeed, take any singleton $A = \{x_{q_1,q_2}\}$; for the set $B = \{x_{-\infty,q_1}, x_{q_2,+\infty}\}$ it holds that $\mathrm{Cn}(A \cup B) = \mathscr{L}$ and $\mathrm{Cn}(A) \cap \mathrm{Cn}(B) = \emptyset = \mathrm{Cn}(\emptyset)$. Thus B is the complement of A. It is easy to extend this proof when A is finitely representable and for the special cases where q_1 and/or q_2 are $-\infty, +\infty$.

Finally, $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is distributive. Indeed, take A_1, A_2, A_3 and some $x_{q_1,q_2} \in \mathrm{Cn}(A_1 \cup A_2) \cap \mathrm{Cn}(A_1 \cup A_3)$. Using the definition of $\mathrm{Cn}(A)$, and applying the standard set theoretic properties of $\cup, \cap$ in the subscript of $\bigcup$, it follows that $x_{q_1,q_2} \in \mathrm{Cn}(A_1 \cup (\mathrm{Cn}(A_2) \cap \mathrm{Cn}(A_3)))$.

Therefore, $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is Boolean (thus AGM-compliant and relevance-compliant).

Now take any finitely representable set A. We will show that $\mathscr{L} \perp A = \emptyset$. Indeed, suppose that there is some $B \in \mathscr{L} \perp A$. Then $A \nsubseteq \mathrm{Cn}(B)$, so there is some $x_{q_1,q_2} \in A$ such that $x_{q_1,q_2} \notin \mathrm{Cn}(B)$. Set now $q_3 = (q_1 + q_2)/2$; obviously $q_3 \in \mathcal{Q}$. Take the set $C = \mathrm{Cn}(B \cup \{x_{q_1,q_3}\})$. It holds that $\mathrm{Cn}(B) \subset \mathrm{Cn}(C) \subset \mathscr{L}$, and $A \nsubseteq \mathrm{Cn}(C)$, a contradiction by the definition of B. We conclude that $\mathscr{L} \perp A = \emptyset$ for all finitely representable sets A.

Therefore, a partial meet contraction operator would give that $\mathscr{L} - A = \mathscr{L}$, a result which does not satisfy the AGM postulates or the AGM-relevance postulates whenever $\mathrm{Cn}(A) \neq \mathrm{Cn}(\emptyset)$ (e.g., for $A = \{x_{-\infty,0}\}$). □

Proof of Theorem 4.16: If a logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ is Boolean, $K, A \subseteq \mathscr{L}$ and A is finitely representable, relevance and recovery are equivalent (in the presence of the other AGM postulates).

(⇒) We will show that if $\beta \notin K - A + A$ then $\beta \notin K$. So, suppose that $\beta \notin K - A + A$, but $\beta \in K$. Then, $X = \mathrm{Cn}(\neg A) \cap \mathrm{Cn}(\beta) \subseteq K$. By distributivity, $X + A = (\neg A + A) \cap (A + \beta) = A + \beta$. It follows that $\beta \in X + A$, but since $\beta \notin K - A + A$, so we have that $X \nsubseteq K - A$. It follows that there is some $x \in X$ such that $x \in K$, but $x \notin K - A$. By *relevance* there is $K' \subseteq K$ such that $A \nsubseteq \mathrm{Cn}(K')$, but $A \subseteq K' + x$. It follows that $A \subseteq K' + X$. Now notice that if $A \subseteq K' + X = (K' + \neg A) \cap (K' + \beta)$, then $A \subseteq K' + \neg A$. In this case, $A \subseteq (K' + \neg A) \cap (K' + A) = \mathrm{Cn}(K' \cap (A + \neg A)) = \mathrm{Cn}(K' \cap \mathscr{L}) = \mathrm{Cn}(K')$ which is a contradiction. Hence, $X \nsubseteq \mathrm{Cn}(K)$ and $\mathrm{Cn}(\beta) \nsubseteq \mathrm{Cn}(K)$ or equivalently $\beta \notin \mathrm{Cn}(K)$.
(⇐) Follows from Lemma A.11. □

Proof of Theorem 4.17: There are logics that are relevance-compliant and AGM-compliant such that relevance does not imply recovery, and recovery does not imply relevance.

Consider the following logic:

$$
\begin{aligned}
\mathscr{L} &= \{a, b, c\} \\
\mathrm{Cn}(\mathscr{L}) = \mathrm{Cn}(\{b, c\}) = \mathrm{Cn}(\{a, c\}) &= \mathscr{L} \\
\mathrm{Cn}(\{a, b\}) &= \{a, b\} \\
\mathrm{Cn}(a) &= \{a\} \\
\mathrm{Cn}(b) &= \{b\} \\
\mathrm{Cn}(c) &= \{c\} \\
\mathrm{Cn}(\emptyset) &= \emptyset
\end{aligned}
$$

The logic is obviously Tarskian and finite, thus compact, so it is also relevance-compliant by Theorem 4.8.

It can also be shown to be AGM-compliant. To see this, take any K, A such that $\mathrm{Cn}(\emptyset) \subset \mathrm{Cn}(A) \subset \mathrm{Cn}(K)$. By Theorem 4.3, it suffices to show that there is a K' such that $\mathrm{Cn}(K') \subset \mathrm{Cn}(K)$ and $K' + A = K$. The table below shows the corresponding K' for each such pair (it is obvious to see that K' satisfies the required conditions in each case). Note that in this logic there are six distinct belief sets, namely: $\mathscr{L}$, $\{a, b\}$, $\{a\}$, $\{b\}$, $\{c\}$ and $\emptyset$, and some pairs are omitted from the table, because for them the relation $\mathrm{Cn}(\emptyset) \subset \mathrm{Cn}(A) \subset \mathrm{Cn}(K)$ does not hold.

So the logic is both AGM-compliant and relevance-compliant. Now take the contraction operation $\mathscr{L} - \{a, b\}$. If we accept $\{a\}$ as the result of this operation, then relevance is satisfied, but not recovery; if we accept $\{c\}$ as the result, then recovery is satisfied, but not relevance. So relevance does not imply recovery, or vice-versa.

□

Lemma A.12. *Consider a Boolean logic $\langle \mathscr{L}, Cn \rangle$. If A is a finitely representable belief, and K a belief such that $Cn(\emptyset) \subset Cn(A) \subset Cn(K)$, then for any $C \in K^{-}(A)$ it holds that $Cn(K) \cap Cn(\neg A) \subseteq C$.*

K	A	K'
$\mathscr{L}$	$\{a,b\}$	$\{c\}$
	$\{c\}$	$\{a,b\}$
	$\{a\}$	$\{c\}$
	$\{b\}$	$\{c\}$
$\{a,b\}$	$\{a\}$	$\{b\}$
	$\{b\}$	$\{a\}$

Table A.4: AGM-compliance Proof (Theorem 4.17)

Proof:
$\mathrm{Cn}(C \cup A) \cap \mathrm{Cn}(C \cup (\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A))) = \mathrm{Cn}(C \cup (A \cap \mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A))) = C$.
Moreover, $C \subseteq \mathrm{Cn}(K)$, $\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A) \subseteq \mathrm{Cn}(K)$ so
$\mathrm{Cn}(C \cup (\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A))) \subseteq \mathrm{Cn}(K)$ and $\mathrm{Cn}(C \cup A) = \mathrm{Cn}(K)$ by definition.
Thus, $\mathrm{Cn}(C \cup A) \cap \mathrm{Cn}(C \cup (\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A))) =$
$\mathrm{Cn}(K) \cap \mathrm{Cn}(C \cup (\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A))) = \mathrm{Cn}(C \cup (\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A)))$.
We conclude that $C = \mathrm{Cn}(C \cup (\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A)))$, thus $\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A) \subseteq C$. □

Lemma A.13. *Consider a Boolean logic $\langle \mathscr{L}, Cn \rangle$ and let A be a finitely representable belief, and K a belief such that $Cn(\emptyset) \subset Cn(A) \subset Cn(K)$. Take some C such that $C = Cn(C)$. Then it holds that $C \in K^-(A)$ iff $Cn(K) \cap Cn(\neg A) \subseteq C \subset Cn(K)$.*

Proof: ($\Rightarrow$) Suppose that $C \in K^-(A)$. Then $\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A) \subseteq C$ by Lemma A.12, and, by definition, $C \subset \mathrm{Cn}(K)$.
($\Leftarrow$) Suppose that C is such that $\mathrm{Cn}(K) \cap \neg A \subseteq C \subset \mathrm{Cn}(K)$.
Then, $\mathrm{Cn}(C \cup A) \subseteq \mathrm{Cn}(K \cup A) = \mathrm{Cn}(K)$ and
$\mathrm{Cn}(C \cup A) \supseteq \mathrm{Cn}(A \cup (\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A))) = \mathrm{Cn}(K)$ because $\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A) \in K^-(A)$ by Lemma A.3.
Finally, suppose that $A \subseteq C$. Then $C = A \cup C = \mathrm{Cn}(K)$, a contradiction. We conclude that $C \in K^-(A)$. □

Lemma A.14. *Consider a Boolean logic $\langle \mathscr{L}, Cn \rangle$ and let A be a finitely representable belief, and K a belief such that $Cn(\emptyset) \subset Cn(A) \subset Cn(K)$. Then $C \in K \perp A$ implies that $C \in K^-(A)$.*

Proof: Suppose that $C \subset \mathrm{Cn}(C)$. Then for $C' = \mathrm{Cn}(C)$, the third condition of the definition of the remainder set does not hold, so $C = \mathrm{Cn}(C)$.
By Lemma A.13, it suffices to show that $\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A) \subseteq C \subset \mathrm{Cn}(K)$. Initially, we observe that $C \subset \mathrm{Cn}(K)$ by definition. Let us suppose, for the sake of contradiction, that $\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A) \nsubseteq C$, and set $D = \mathrm{Cn}(C \cup (\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A)))$.
Then, obviously, $C \subset D$ and $D \subseteq \mathrm{Cn}(K)$, so, given that $C \in K \perp A$, it follows that $A \subseteq D$. Obviously, it also holds that $A \subseteq \mathrm{Cn}(C \cup A)$. Thus:
$A \subseteq D \cap \mathrm{Cn}(C \cup A) =$ (by definition)

$\mathrm{Cn}(C \cup (\mathrm{Cn}(K) \cap \neg A)) \cap \mathrm{Cn}(C \cup A) =$ (by distributivity)
$\mathrm{Cn}(C \cup (\mathrm{Cn}(K) \cap \neg A) \cap A) =$ (by definition)
$\mathrm{Cn}(C) = C$, which is a contradiction, by the definition of C.
We conclude that $\mathrm{Cn}(K) \cap \mathrm{Cn}(\neg A) \subseteq C \subset \mathrm{Cn}(K)$, so by Lemma A.13 $C \in K^{-}(A)$. □

Lemma A.15. *Consider a Boolean logic $\langle \mathscr{L}, Cn \rangle$ and let A be a finitely representable belief, and K a belief such that $Cn(\emptyset) \subset Cn(A) \subset Cn(K)$. Then, for any $\Gamma \subseteq K \perp A$, $\Gamma \neq \emptyset$ it holds that $\bigcap \Gamma \in K^{-}(A)$.*

Proof: By Lemma A.14, for any $X \in \Gamma$ it holds that $X \in K^{-}(A)$, so $X \supseteq \mathrm{Cn}(K) \cap \neg A$. Thus, $\bigcap \Gamma \supseteq \mathrm{Cn}(K) \cap \neg A$, i.e., $\bigcap \Gamma \in K^{-}(A)$. □

Proof of Theorem 4.18: Consider a Boolean logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ such that for all $K, A \subseteq \mathscr{L}$ where A is finitely representable, it holds that $\mathrm{Cn}(A) = \mathrm{Cn}(\emptyset)$ or $K \perp A \neq \emptyset$. Then, any partial-meet contraction operator is an AGM-compliant contraction operator.

Obvious, by Lemma A.15 and the definitions. □

Proof of Theorem 4.19: Consider a Boolean logic $\langle \mathscr{L}, \mathrm{Cn} \rangle$ that satisfies the ascending chain condition. Then, any partial meet contraction operator is an AGM-compliant contraction operator, and any AGM-compliant contraction operator is also a partial meet contraction operator.

A logic that is Boolean and satisfies the ascending chain condition is isomorphic to a complete and atomic Boolean lattice, which is isomorphic to a complete and atomic Boolean algebra, which, in turn, is isomorphic to the power set of some set, ordered under standard set inclusion ($\subseteq$). Proving the theorem for the power set is easy; the result follows by the isomorphism. □

Proof of Theorem 5.3: Intuitionistic logic is relevance-compliant, but not AGM-compliant.

Consider an enumerable set $\mathbb{P} = \{p_0, p_1 \ldots\}$ of propositional variables. A *Kripke model* for intuitionistic logic is a triple $M = \langle W, R, V \rangle$ where W is any enumerable set whose elements are called *worlds*, R is a partial order over W called the *accessibility relation* and $V : \mathbb{P} \to 2^{W}$ a function that associates propositional variables to sets of worlds.

Given a Kripke model M, we define the relation $\models_M \subseteq W \times \mathscr{L}$ as follows:

- $w \models_M p$ iff $w \in V(p)$.
- $w \models_M \alpha \vee \beta$ iff $w \models_M \alpha$ or $w \models_M \beta$.
- $w \models_M \alpha \wedge \beta$ iff $w \models_M \alpha$ and $\models_M \beta$.
- $w \models_M \neg\alpha$ iff $w' \not\models_M \alpha$ for every $w \leq w'$.
- $w \models_M \alpha \to \beta$ iff $w' \not\models_M \beta$ for all $w \leq w'$ such that $w' \models_M \alpha$.

We write $w \vDash_M A$ iff $w \vDash_M \beta$ for every $\beta \in A$. The notation $A \vDash_M \alpha$ will be used when for every world $w \in W$ we have that $w \vDash_M A$ implies $w \vDash_M \alpha$. Furthermore, $A \vDash \alpha$ is written when $A \vDash_M \alpha$ for every model M and $\vDash_M \alpha$ abbreviates $\emptyset \vDash_M \alpha$. Finally $\mathrm{Cn}(A) := \{\alpha \in \mathscr{L} : A \vDash \alpha\}$.

Let $K = \{\neg p\}$ and $A = \{\neg p \vee p\}$. We will show that: 1) $\mathrm{Cn}(\emptyset) \subset \mathrm{Cn}(A) \subset \mathrm{Cn}(K)$ and 2) There is no X such that $\mathrm{Cn}(X) \subset \mathrm{Cn}(K)$ and $X + A = K$.

1) Using natural deduction we have that $p \vee \neg p \in \mathrm{Cn}(\neg p)$, by introduction of disjunction $\frac{\neg p}{p \vee \neg p}$.

To show that $\mathrm{Cn}(p \vee \neg p) \neq \mathrm{Cn}(\neg p)$ we will use the Kripke semantics of intuitionistic logic. We have to show a model $M = \langle V, W, R\rangle$ such that $\vDash_M p \vee \neg p$, but $\nvDash_M \neg p$. The following model does this: $W = \{w\}$, $R = \{(w, w)\}$ and $V(p) = \{w\}$.

In order to prove that $p \vee \neg p \notin \mathrm{Cn}(\emptyset)$, we have to show a model $M = \langle V, W, R\rangle$ such that $\nvDash_M p \vee \neg p$. Consider the following model that accomplish this requirement: $W = \{w_0, w_1\}$, $R = \{(w_0, w_0), (w_0, w_1), (w_1, w_1)\}$ and $V(p) = \{w_1\}$.

2) We will show that for any X such that $\mathrm{Cn}(X) \subset \mathrm{Cn}(\neg p)$ there is a model M such that $\vDash_M X \cup \{p \vee \neg p\}$, but $\nvDash_M \neg p$. From this it follows trivially that $\mathrm{Cn}(\neg p) \neq \mathrm{Cn}(X \cup \{p \vee \neg p\})$.

Take X such that $\mathrm{Cn}(X) \subset \mathrm{Cn}(\neg p)$. Since $\neg p \notin \mathrm{Cn}(X)$ there is a model $M' = \langle V', W', R'\rangle$ of X ($\vDash_{M'} X$) such that there is a $w \in W'$ with $(w_0, w) \in R'$ and $w \in V'(p)$. Now, consider a model $M = \langle V, W, R\rangle$ such that $W = \{w \in W' : w \in V'(p)\}$, $V(q) = V'(q) \cap W$ for all q and $R = \{(w_1, w_2) \in W \times W : (w_1, w_2) \in R'\}$. Then, we have that, $\vDash_M X$ and $\vDash_M p$. It follows that $\vDash_M p \vee \neg p \cup \{X\}$, but $\nvDash_M \neg p$.

On the other hand, intuitionistic logic is compact [48], so it is relevance-compliant. □

References

[1] G. Flouris, On belief change and ontology evolution, Ph.D. thesis, University of Crete (2006).

[2] P. Gardenfors, Knowledge in Flux: Modeling the Dynamics of Epistemic States, MIT Press, Cambridge, 1988.

[3] C. Alchourrón, P. Gärdenfors, D. Makinson, On the logic of theory change, Journal of Symbolic Logic 50 (2) (1985) 510–530.

[4] F. Baader, D. Calvanese, D. McGuinness, D. Nardi, P. Patel-Schneider (Eds.), The Description Logic Handbook, Cambridge University Press, 2003.

[5] T. Berners-Lee, J. Hendler, O. Lassila, The semantic web, Scientific American May 2001 (2001) 29–37.

[6] A. Horn, On sentences which are true of direct unions of algebras, Journal of Symbolic Logic 16 (1951) 14–21.

[7] A. Heyting, Intuitionism: An Introduction, North-Holland Pub. Co., Amsterdam, 1956.

[8] C. Alchourron, D. Makinson, On the logic of theory change: Safe contraction, Studia Logica 44 (1985) 405–422.

[9] R. Niederée, Multiple contraction. a further case against Gärdenfors' principle of recovery, in: Proceedings of the Workshop on The Logic of Theory Change, 1991, pp. 322–334.

[10] D. Makinson, On the status of the postulate of recovery in the logic of theory change, Journal of Philosophical Logic 16 (1987) 383–394.

[11] S. O. Hansson, Belief contraction without recovery, Studia Logica 50 (2) (1991) 251–260.

[12] M. M. Ribeiro, Belief revision in description logics and other non-classical logics, Ph.D. thesis, University of São Paulo, text in portuguese (September 2010).

[13] G. Restall, J. Slaney, Realistic belief revision, in: Proceedings of the 2^{nd} World Conference on Foundations of Artificial Intelligence (WOCFAI-95), 1995, pp. 367–378.

[14] N. D. Costa, O. Bueno, Belief change and inconsistency, Logique & Analyse (161-162-163) (1998) 31–56.

[15] J. P. Delgrande, R. Wassermann, Horn clause contraction functions: Belief set and belief base approaches, in: Principles of Knowledge Representation and Reasoning (KR-10), AAAI Press, 2010.

[16] R. Booth, T. Meyer, I. Varzinczak, R. Wassermann, On the link between partial meet, kernel, and infra contraction and its application to Horn logic, Journal of Artificial Intelligence Research (JAIR) 42 (2011) 31–53.

[17] S. O. Hansson, R. Wassermann, Local change, Studia Logica 70 (1) (2002) 49–76.

[18] G. Flouris, D. Plexousakis, G. Antoniou, Generalizing the AGM postulates: Preliminary results and applications., in: J. P. Delgrande, T. Schaub (Eds.), Proceedings of the 10^{th} International Workshop on Non-Monotonic Reasoning 2004 (NMR-04), 2004, pp. 171–179.

[19] G. Flouris, D. Plexousakis, G. Antoniou, On applying the AGM theory to DLs and OWL, in: Proceedings of the 4^{th} International Semantic Web Conference (ISWC-05), 2005, pp. 353–367.

[20] M. Ribeiro, R. Wassermann, G. Antoniou, G. Flouris, J. Pan, Belief contraction in web-ontology languages, in: Proceedings of the 3^{rd} International Workshop on Ontology Dynamics (IWOD-09), 2009, short paper.

[21] M. M. Ribeiro, R. Wassermann, First steps towards revising ontologies, in: Proceedings of the 2^{nd} Workshop on Ontologies and their Applications (WONTO 2006), 2006.

[22] A. Fuhrmann, S. O. Hansson, A survey of multiple contractions, Journal of Logic, Language and Information 3 (1) (1994) 39–75.

[23] S. O. Hansson, A Textbook of Belief Dynamics, Theory Change and Database Updating, Kluwer Academic, 1999.

[24] M. M. Ribeiro, R. Wassermann, Base revision for ontology debugging, Journal of Logic and Computation 19 (5) (2009) 721–743.

[25] G. Qi, J. Z. Pan, An analysis of approaches to resolving inconsistencies in DL-based ontologies, in: Proceedings of the International Workshop on Ontology Dynamics (IWOD-07), 2007.

[26] C. Halaschek-Wiener, Y. Katz, B. Parsia, Belief base revision for expressive Description Logics, in: Proceedings of the 2^{nd} International Workshop on OWL: Experiences and Directions (OWLED-06), 2006.

[27] R. Wassermann, On AGM for non-classical logics, Journal of Philosophical Logic 40 (2011) 271–294.

[28] M. Langlois, R. H. Sloan, B. Szörényi, G. Turán, Horn complements: Towards Horn-to-Horn belief revision, in: D. Fox, C. P. Gomes (Eds.), Proceedings of the 23^{rd} AAAI Conference on Artificial Intelligence (AAAI 2008), AAAI Press, Chicago, Illinois, USA, 2008, pp. 466–471.

[29] G. Lakemeyer, W. Lang, Belief revision in a non-classical logic, in: Proceedings of the KI-96: Advances in Artificial Intelligence, Vol. 1137 of Lecture Notes in Computer Science, Springer Berlin / Heidelberg, 1996, pp. 199–211.

[30] A. Anderson, N. Belnap, First degree entailments, Mathematische Annalen 149 (4) (1963) 302–319.

[31] R. Booth, T. Meyer, I. J. Varzinczak, New steps in propositional Horn contraction, in: Proceedings of the 21^{st} International Joint Conference of Artificial Intelligence (IJCAI-09), 2009.

[32] J. P. Delgrande, Horn clause belief change: Contraction functions, in: G. Brewka, J. Lang (Eds.), Proceedings of the 11^{th} International Conference on Principles of Knowledge Representation and Reasoning (KR-08), AAAI Press, 2008, pp. 156–165.

[33] Z. Q. Zhuang, M. Pagnucco, Horn contraction via epistemic entrenchment, in: Proceedings of the 12th European conference on Logics in artificial intelligence, JELIA'10, Springer-Verlag, Berlin, Heidelberg, 2010, pp. 339–351.
URL http://dl.acm.org/citation.cfm?id=1888743.1888774

[34] Z. Q. Zhuang, M. Pagnucco, Model based horn contraction, in: In Proceedings of the XIII Principles of Knowledge Representation and Reasoning, 2012.

[35] J. P. Delgrande, R. Wassermann, Horn clause contraction functions: Belief set and belief base approaches, in: In Proceedings of the XII Principles of Knowledge Representation and Reasoning, AAAI Press, 2010.

[36] J. P. Delgrande, P. Peppas, Revising horn theories, in: Proceedings of the 22nd International Joint Conference on Artificial Intelligence, IJCAI/AAAI, Barcelona, Catalonia, Spain, 2011, pp. 839–844.

[37] K. V. Adaricheva, R. H. Sloan, B. Szörényi, G. Turán, Horn belief contraction: Remainders, envelopes and complexity, in: In Proceedings of the XIII Principles of Knowledge Representation and Reasoning, AAAI Press, 2012.

[38] N. Creignou, O. Papini, R. Pichler, S. Woltran, Belief revision within fragments of propositional logic, in: In Proceedings of the XIII Principles of Knowledge Representation and Reasoning, AAAI Press, 2012.

[39] G. Antoniou, Nonmonotonic Reasoning, MIT Press, 1997.

[40] H. Enderton, A Mathematical Introduction to Logic, Academic Press Inc., New York, 1972.

[41] G. Gratzer, Lattice Theory: First Concepts And Distributive Lattices, W. H. Freeman and co., San Francisco, 1971.

[42] F. Manola, E. Miller, B. McBride, RDF primer, available at `http://www.w3.org/TR/rdf-primer` (2004).

[43] D. Brickley, R. Guha, RDF vocabulary description language 1.0: RDF Schema, available at `http://www.w3.org/TR/2004/REC-rdf-schema-20040210` (2004).

[44] M. Dean, G. Schreiber, S. Bechhofer, F. van Harmelen, J. Hendler, I. Horrocks, D. L. McGuinness, P. F. Patel-Schneider, L. A. Stein, OWL web onotology language reference, W3C Recomendation, available at `http://www.w3.org/TR/owl-ref/` (2004).

[45] I. Horrocks, P. F. Patel-Schneider, F. van Harmelen, From SHIQ and RDF to OWL: The making of a Web Ontology Language, Journal of Web Semantics 1 (1) (2003) 7–26.

[46] W3C OWL Working Group, OWL 2 Web Ontology Language: Document Overview, W3C Recommendation, 27 October 2009, available at `http://www.w3.org/TR/owl2-overview/`.

[47] B. Motik, B. C. Grau, I. Horrocks, Z. Wu, A. Fokoue, C. Lutz, OWL 2 web ontology language: Profiles, World Wide Web Consortium, Working Draft, available at `http://www.w3.org/TR/2008/WD-owl2-profiles-20081202` (December 2008).

[48] G. Mints, A Short Introduction to Intuitionistic Logic, Kluwer Academic Publishers, Norwell, MA, USA, 2000.

[49] R. Epstein, Classical Mathematical Logic: The Semantic Foundations of Logic, Princeton University Press, 2006.

[50] G. Flouris, D. Plexousakis, G. Antoniou, On generalizing the AGM postulates, in: Proceedings of the 3^{rd} European Starting AI Researcher Symposium (STAIRS-06), 2006.

[51] G. Qi, J. Du, Model-based revision operators for terminologies in Description Logics, in: Proceedings of the 21^{st} International Joint Conference on Artificial Intelligence (IJCAI-09), 2009.

[52] M. M. Ribeiro, R. Wassermann, AGM revision in Description Logics, in: Proceedings of the Workshop on Automated Reasoning about Context and Ontology Evolution (ARCOE-09), 2009.