The Robust Covariation-Based MUSIC (ROC-MUSIC) Algorithm for Bearing Estimation in Impulsive Noise Environments

Panagiotis Tsakalides, *Member, IEEE*, and Chrysostomos L. Nikias, *Fellow, IEEE*

***Abstract*— This paper presents a new subspace-based method for bearing estimation in the presence of impulsive noise which can be modeled as a complex symmetric alpha-stable ($S\alpha S$) process. We define the covariation matrix of the array sensor outputs and show that eigendecomposition-based methods, such as the MUSIC algorithm, can be applied to the sample covariation matrix to extract the bearing information from the measurements. A consistent estimator for the marginals of the covariation matrix is presented and its asymptotic performance is studied. The improved performance of the proposed source localization method in the presence of a wide range of impulsive noise environments is demonstrated via Monte Carlo experiments.**

I. Introduction

STATISTICAL array processing based on the linear theory of random processes with finite second-order moments has been the focus of considerable academic research. Critical problems such as high-resolution direction finding, null- and beam-steering, and detection of the number of sources illuminating an array of sensors have been studied under the assumption of a Gaussian or second-order model. Many different classes of methods, compromising optimality for the sake of computational efficiency, have been proposed under the aforementioned statistical framework [1].

Looking toward real world applications, we are interested in developing array processing methods for a larger class of random processes which include the Gaussian processes as special elements. The availability of such methods would make it possible to operate in environments which differ from Gaussian environments in significant ways.

The class of stable distributions has some important characteristics which make it very attractive for modeling impulsive noise environments. Stable processes satisfy the stability property which states that linear combinations of jointly stable variables are indeed stable. They arise as limiting processes of sums of independent, identically distributed random variables via the generalized central limit theorem. They are described by their characteristic exponent α, taking values $0 < \alpha \leq 2$. Gaussian processes are stable processes with $\alpha = 2$. Stable distributions have heavier tails than the normal distribution, possess finite pth-order moments only for $p < \alpha$, and are appropriate for modeling noise with outliers. The main reason for the difficulty in developing signal processing methods based on stable processes is due to the fact that the linear space of a stable process is not a Hilbert space, as in the case of Gaussian processes, but either a Banach $(1 < \alpha < 2)$ or a metric space $(0 < \alpha \leq 1)$ both of which are more unyielding in their structure.

Recently, it has been shown that the improved performance gained by designing signal processing algorithms in an alpha-stable framework justifies the mathematical and computational complexity involved. In [2] we dealt with optimal approaches (optimal in the maximum likelihood (ML) sense) to the direction of arrival (DOA) problem in the presence of impulsive noise. The analysis was based on the assumption that the additive noise could be modeled as a complex *symmetric α-stable* ($S\alpha S$) process. The optimal ML techniques employed in [2] are often regarded as exceedingly complex due to the high computational load of the multivariate nonlinear optimization problem involved with these techniques. Hence, sub-optimal methods need to be developed for the solution of the DOA problem in the presence of impulsive noise, when reduced computational cost is a crucial design requirement.

In this paper, we present subspace techniques for the source localization problem, techniques which are based on the geometrical properties of the data model. Considerable research has been done in this area under the framework of Gaussian distributed signals and/or noise. The better known of the so-called eigendecomposition-based methods are the MUSIC [3], Minimum Norm [4], [5], and the ESPRIT method [6]. These methods estimate the bearings of the source signals by performing an eigendecomposition on the spatial covariance matrix of the array sensor outputs. Studies concerning the statistical efficiency of the most popular eigendecomposition-based method, namely the MUSIC algorithm, have been done in [7], [8]. The relationship between the MUSIC and ML methods has also been studied in [7]. Since $S\alpha S$ processes do not possess finite pth-order moments for $p \geq \alpha$, traditional subspace techniques employing both second-order and higher order moments [9] cannot be applied in impulsive noise environments modeled under the stable law. Instead, properties of fractional lower order moments (FLOM's) and covariations should be used [10].

This paper extends the subspace-based techniques for bearing estimation to processes with finite moments of order p

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The authors are with the Signal & Image Processing Institute, Department of Electrical Engineering-Systems, University of Southern California, Los Angeles, CA 90089-2564 USA (e-mail: tsakalid@sipi.usc.edu).

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($p < 2$) and to complex isotropic $S\alpha S$ processes. The paper is organized as follows: In Section II, we present some necessary introduction on α-stable processes and their properties. In Section III, we discuss the development of subspace techniques in the presence of α-stable distributed signals and noise. Our analysis is based on the formulation of the covariation matrix of the array sensor outputs. Finally, simulation experiments are presented in Section IV, and conclusions are drawn in Section V.

II. Mathematical Preliminaries: Complex $S\alpha S$ Random Variables

A complex random variable (RV) $X = X_1 + \jmath X_2$ is symmetric α-stable ($S\alpha S$) if X_1 and X_2 are jointly $S\alpha S$ and then its characteristic function is written as

$$\begin{aligned}\varphi(\omega) &= E\{\exp[\jmath\Re(\omega X^*)]\}E\{\exp[\jmath(\omega_1 X_1 + \omega_2 X_2)]\} \\ &= \exp\left[-\int_{S_2} |\omega_1 x_1 + \omega_2 x_2|^\alpha d\Gamma_{X_1,X_2}(x_1,x_2)\right] \quad (1)\end{aligned}$$

where $\omega = \omega_1 + \jmath\omega_2$, $\Re[\cdot]$ is the real part operator, and Γ_{X_1,X_2} is a symmetric measure on the unit sphere S_2, called the *spectral measure* of the random variable X. The *characteristic exponent* α is restricted to the values $0 < \alpha \leq 2$ and it determines the shape of the distribution. The smaller the characteristic exponent α, the heavier the tails of the density.

A complex random variable $X = X_1 + \jmath X_2$ is *isotropic* if and only if (X_1, X_2) has a uniform spectral measure. In this case, the characteristic function of X can be written as

$$\varphi(\omega) = E\{\exp(\jmath\Re[\omega X^*])\} = \exp(-\gamma|\omega|^\alpha) \quad (2)$$

where γ ($\gamma > 0$) is the *dispersion* of the distribution. The dispersion plays a role analogous to the role that the variance plays for second-order processes. Namely, it determines the spread of the probability density function around the origin. A method for generating complex isotropic $S\alpha S$ random variables is presented in Appendix A.

Several complex RV's are jointly $S\alpha S$ if their real and imaginary parts are jointly $S\alpha S$. When $X = X_1 + \jmath X_2$ and $Y = Y_1 + \jmath Y_2$ are jointly $S\alpha S$ with $1 < \alpha \leq 2$, the *covariation* of X and Y is defined by

$$\begin{aligned}[X,Y]_\alpha = \int_{S_4} &(x_1 + \jmath x_2)(y_1 + \jmath y_2)^{\langle\alpha-1\rangle} \\ &\times d\Gamma_{X_1,X_2,Y_1,Y_2}(x_1,x_2,y_1,y_2) \quad (3)\end{aligned}$$

where we use throughout the convention

$$Y^{\langle\beta\rangle} = |Y|^{\beta-1}Y^*. \quad (4)$$

It can be shown that for every $1 \leq p < \alpha$, the covariation can be expressed as a function of moments [11]

$$[X,Y]_\alpha = \frac{E\{XY^{\langle p-1\rangle}\}}{E\{|Y|^p\}}\gamma_Y \quad (5)$$

where γ_Y is the dispersion of the RV Y given by

$$\gamma_Y^{p/\alpha} = \frac{E\{|Y|^p\}}{C(p,\alpha)} \quad \text{for } 0 < p < \alpha \quad (6)$$

with

$$C(p,\alpha) = \frac{2^{p+1}\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(-\frac{p}{\alpha}\right)}{\alpha\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{p}{2}\right)} \quad (7)$$

where $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt. \quad (8)$$

Obviously, from (5) it holds that

$$[X,X]_\alpha = \gamma_X. \quad (9)$$

Also, the *covariation coefficient* of X and Y is defined by

$$\lambda_{X,Y} = \frac{[X,Y]_\alpha}{[Y,Y]_\alpha}, \quad (10)$$

and by using (5) it can be expressed as

$$\lambda_{X,Y} = \frac{E\{XY^{\langle p-1\rangle}\}}{E\{|Y|^p\}} \quad \text{for } 1 \leq p < \alpha. \quad (11)$$

The covariation of complex jointly $S\alpha S$ RV's is not generally symmetric and has the following properties [12]:

P1 If X_1, X_2 and Y are jointly $S\alpha S$, then

$$[aX_1 + bX_2, Y]_\alpha = a[X_1,Y]_\alpha + b[X_2,Y]_\alpha \quad (12)$$

for any complex constants a and b.

P2 If Y_1 and Y_2 are independent and Y_1, Y_2, and X are jointly $S\alpha S$, then

$$\begin{aligned}&[aX, bY_1 + cY_2]_\alpha \\ &\quad = ab^{\langle\alpha-1\rangle}[X,Y_1]_\alpha + ac^{\langle\alpha-1\rangle}[X,Y_2]_\alpha \quad (13)\end{aligned}$$

for any complex constants a, b, and c.

P3 If X and Y are independent $S\alpha S$, then $[X,Y]_\alpha = 0$.

III. Subspace Techniques in the α-Stable Framework

A. Problem Formulation

Consider an array of r sensors with arbitrary locations and arbitrary directional characteristics, that receive signals generated by q narrow-band sources with known center frequency ω and locations $\theta_1, \theta_2, \cdots, \theta_q$. Since the signals are narrow-band, the propagation delay across the array is much smaller than the reciprocal of the signal bandwidth, and it follows that, by using a complex envelop representation, the array output can be expressed as

$$\mathbf{x}(t) = \sum_{k=1}^{q} \mathbf{a}(\theta_k)s_k(t) + \mathbf{n}(t) \quad (14)$$

where

- $\mathbf{x}(t) = [x_1(t), \cdots, x_r(t)]^T$ is the vector of the signals received by the array sensors,
- $s_k(t)$ is the signal emitted by the kth source as received at the reference sensor 1 of the array,

- $\mathbf{a}(\theta_k) = [1, e^{-j\omega\tau_2(\theta_k)}, \cdots, e^{-j\omega\tau_r(\theta_k)}]^T$ is the steering vector of the array toward direction θ_k,
- $\tau_i(\theta_k)$ is the propagation delay between the first and the ith sensor for a waveform coming from direction θ_k,
- $\mathbf{n}(t) = [n_1(t), \cdots, n_r(t)]^T$ is the noise vector.

Equation (14) can be expressed in a compact form as

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t) \tag{15}$$

where $\mathbf{A}(\boldsymbol{\theta})$ is the $r \times q$ matrix of the array steering vectors

$$\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_q)] \tag{16}$$

and $\mathbf{s}(t)$ is the $q \times 1$ vector of the signals

$$\mathbf{s}(t) = [s_1(t), \cdots, s_q(t)]^T. \tag{17}$$

Assuming that M snapshots are taken at time instants $t_1, \cdots, t_M$, the data can be expressed as

$$\mathbf{X} = \mathbf{A}(\boldsymbol{\theta})\mathbf{S} + \mathbf{N} \tag{18}$$

where $\mathbf{X}$ and $\mathbf{N}$ are the $r \times M$ matrices

$$\mathbf{X} = [\mathbf{x}(t_1), \cdots, \mathbf{x}(t_M)] \tag{19}$$

$$\mathbf{N} = [\mathbf{n}(t_1), \cdots, \mathbf{n}(t_M)] \tag{20}$$

and $\mathbf{S}$ is the $q \times M$ matrix

$$\mathbf{S} = [\mathbf{s}(t_1), \cdots, \mathbf{s}(t_M)]. \tag{21}$$

Our objective is to estimate the directions of arrival $\theta_1, \cdots, \theta_q$ of the sources from the M snapshots of the array $\mathbf{x}(t_1), \cdots, \mathbf{x}(t_M)$.

B. The Array Covariation Matrix

In this section, we assume that the q signal waveforms are noncoherent, statistically independent, complex isotropic $S\alpha S$ $(1 < \alpha \leq 2)$ random processes with zero location parameter and covariation matrix $\mathbf{\Gamma}_S = \text{diag}(\gamma_{s_1}, \cdots, \gamma_{s_q})$. Also, the noise vector $\mathbf{n}(t)$ is a complex isotropic $S\alpha S$ random process with the same characteristic exponent α as the signals. The noise is assumed to be independent of the signals with covariation matrix $\mathbf{\Gamma}_N = \gamma_n \mathbf{I}$.

Equation (15) can be written as

$$\mathbf{x}(t) = \mathbf{w}(t) + \mathbf{n}(t) \tag{22}$$

where $\mathbf{w}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t)$. By the stability property, it follows that $\mathbf{w}(t)$ is also a complex isotropic $S\alpha S$ random vector with components

$$w_i(t) = \mathbf{A}_i(\boldsymbol{\theta})\mathbf{s}(t) = a_i(\theta_1)s_1(t) + \cdots + a_i(\theta_q)s_q(t) \quad i = 1, \cdots, r. \tag{23}$$

In addition, it holds that $\mathbf{w}(t)$ is independent of $\mathbf{n}(t)$.

Now, we define the *covariation matrix*, $\mathbf{\Gamma}_X$, of the observation vector process $\mathbf{x}(t)$ as the matrix whose elements are the covariations $[x_i(t), x_j(t)]_\alpha$ of the components of $\mathbf{x}(t)$. We have that

$$\begin{aligned}[x_i(t), x_j(t)]_\alpha &= [w_i(t) + n_i(t), w_j(t) + n_j(t)]_\alpha \\ &= [w_i(t), w_j(t)]_\alpha + [w_i(t), n_j(t)]_\alpha \\ &\quad + [n_i(t), w_j(t)]_\alpha + [n_i(t), n_j(t)]_\alpha.\end{aligned} \tag{24}$$

By the independence assumption of $\mathbf{w}(t)$ and $\mathbf{n}(t)$ and by property $P3$ we have that

$$[w_i(t), n_j(t)]_\alpha = 0 \tag{25}$$

and

$$[n_i(t), w_j(t)]_\alpha = 0. \tag{26}$$

Also, by using (23) and properties $P1$ and $P2$ it follows that

$$\begin{aligned}[w_i(t), w_j(t)]_\alpha &= \left[\sum_{k=1}^{q} a_i(\theta_k)s_k(t), w_j(t)\right]_\alpha \\ &= \sum_{k=1}^{q} a_i(\theta_k)[s_k(t), w_j(t)]_\alpha \\ &= \sum_{k=1}^{q} a_i(\theta_k)\left[s_k(t), \sum_{l=1}^{q} a_j(\theta_l)s_l(t)\right]_\alpha \\ &= \sum_{k=1}^{q} a_i(\theta_k)a_j^{\langle\alpha-1\rangle}(\theta_k)\gamma_{s_k}\end{aligned} \tag{27}$$

where $\gamma_{s_k} = [s_k, s_k]_\alpha$. Finally, due to the noise assumption made earlier, it holds that

$$[n_i(t), n_j(t)]_\alpha = \gamma_n \delta_{i,j} \tag{28}$$

where $\delta_{i,j}$ is the Kronecker delta function. Combining (24)–(28) we obtain the following expression for the covariations of the sensor measurements

$$[x_i(t), x_j(t)]_\alpha = \sum_{k=1}^{q} a_i(\theta_k)a_j^{\langle\alpha-1\rangle}(\theta_k)\gamma_{s_k} + \gamma_n\delta_{i,j} \quad i, j = 1, \cdots, r. \tag{29}$$

In addition, the dispersion and covariation coefficients of the array sensor measurements are given, respectively, by

$$\gamma_{x_j(t)} = \sum_{k=1}^{q} |a_j(\theta_k)|^\alpha \gamma_{s_k} + \gamma_n \quad j = 1, \cdots, r \tag{30}$$

and

$$\lambda_{x_i(t), x_j(t)} = \frac{\sum_{k=1}^{q} a_i(\theta_k)a_j^{\langle\alpha-1\rangle}(\theta_k)\gamma_{s_k} + \gamma_n\delta_{i,j}}{\sum_{k=1}^{q} |a_j(\theta_k)|^\alpha\gamma_{s_k} + \gamma_n} \quad i, j = 1, \cdots, r. \tag{31}$$

In matrix form, (29) gives the following expression for the covariation matrix of the observation vector

$$\mathbf{\Gamma}_X \triangleq [\mathbf{x}(t), \mathbf{x}(t)]_\alpha = \mathbf{A}(\theta)\mathbf{\Gamma}_S\mathbf{A}^{\langle\alpha-1\rangle}(\theta) + \gamma_n\mathbf{I} \tag{32}$$

where the (i, j)th element of matrix $\mathbf{A}^{\langle\alpha-1\rangle}(\boldsymbol{\theta})$ results from the (j, i)th element of $\mathbf{A}(\boldsymbol{\theta})$ according to the operation

$$\begin{aligned}[\mathbf{A}^{\langle\alpha-1\rangle}(\boldsymbol{\theta})]_{i,j} &= [\mathbf{A}(\boldsymbol{\theta})]_{j,i}^{\langle\alpha-1\rangle} \\ &= |[\mathbf{A}(\boldsymbol{\theta})]_{j,i}|^{\alpha-2}[\mathbf{A}(\boldsymbol{\theta})]_{j,i}^*.\end{aligned} \tag{33}$$

Clearly, when $\alpha = 2$, i.e., for Gaussian distributed signals and noise, the expression for the covariation matrix is identical to the well-known expression for the covariance matrix

$$\mathbf{R}_X = \mathbf{A}(\boldsymbol{\theta})\mathbf{\Sigma}\mathbf{A}^H(\boldsymbol{\theta}) + \sigma^2\mathbf{I}, \tag{34}$$

where $\mathbf{\Sigma}$ is the signal covariance matrix.

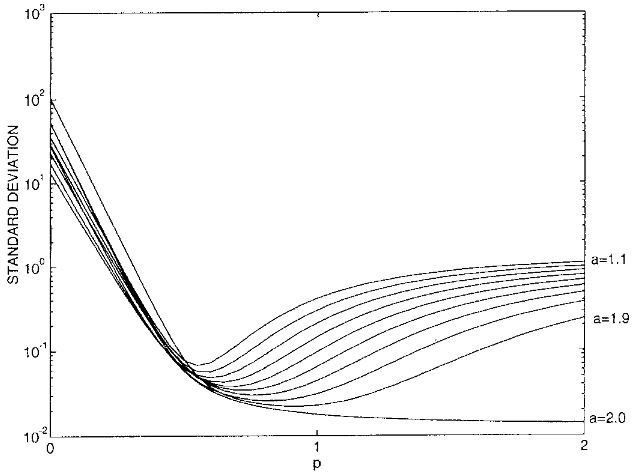

Fig. 1. Standard deviation of the MFLOM estimates of the modified covariation coefficient as a function of the parameter p.

When the amplitude response of the sensors equals unity, i.e., for steering vectors of the form $\mathbf{a}(\theta_k) = [1, e^{-\jmath\omega\tau_2(\theta_k)}, \cdots, e^{-\jmath\omega\tau_r(\theta_k)}]^T$, it follows that

$$[\mathbf{A}^{\langle\alpha-1\rangle}(\boldsymbol{\theta})]_{i,j} = |e^{-\jmath\omega\tau_j(\theta_i)}|^{\alpha-2}e^{\jmath\omega\tau_j(\theta_i)} = [\mathbf{A}(\boldsymbol{\theta})]^*_{j,i} \quad (35)$$

and thus, the covariation matrix can be written as

$$\mathbf{\Gamma}_X = \mathbf{A}(\boldsymbol{\theta})\mathbf{\Gamma}_S\mathbf{A}^H(\boldsymbol{\theta}) + \gamma_n\mathbf{I}. \quad (36)$$

In addition, from (30) and (31) the dispersion and covariation coefficients of the array sensor measurements can be written as

$$\gamma_{x_j(t)} = \sum_{k=1}^{q}\gamma_{s_k} + \gamma_n \qquad j = 1, \cdots, r \quad (37)$$

and

$$\lambda_{x_i(t),x_j(t)} = \frac{\sum_{k=1}^{q} a_i(\theta_k)a_j^*(\theta_k)\gamma_{s_k} + \gamma_n\delta_{i,j}}{\sum_{k=1}^{q}\gamma_{s_k} + \gamma_n} \qquad i, j = 1, \cdots, r. \quad (38)$$

Observing (36), we conclude that standard subspace techniques can be applied to the covariation or the covariation coefficient matrices of the observation vector to extract the bearing information. We will refer to the new algorithm resulting from the eigendecomposition of the array covariation coefficient matrix as the **Robust Covariation-Based MUSIC** or **ROC-MUSIC**. More specifically, it follows that the rank of matrix $\mathbf{A}(\boldsymbol{\theta})\mathbf{\Gamma}_S\mathbf{A}^H(\boldsymbol{\theta})$ is q, with the smallest $(r-q)$ of its eigenvalues equal to zero. In other words, if we let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_r$ denote the eigenvalues of matrix $\mathbf{\Gamma}_X$, then

$$\rho_{q+1} = \cdots = \rho_r = \gamma_n. \quad (39)$$

By denoting the corresponding eigenvectors of $\mathbf{\Gamma}_X$ by $\{\mathbf{v}_i\}_{i=1}^r$, the ROC-MUSIC spectrum can be expressed as

$$S_{\text{ROC-MUSIC}}(\theta) = \frac{1}{\sum_{i=q+1}^{r}|\mathbf{a}^H(\theta)\mathbf{v}_i|^2}. \quad (40)$$

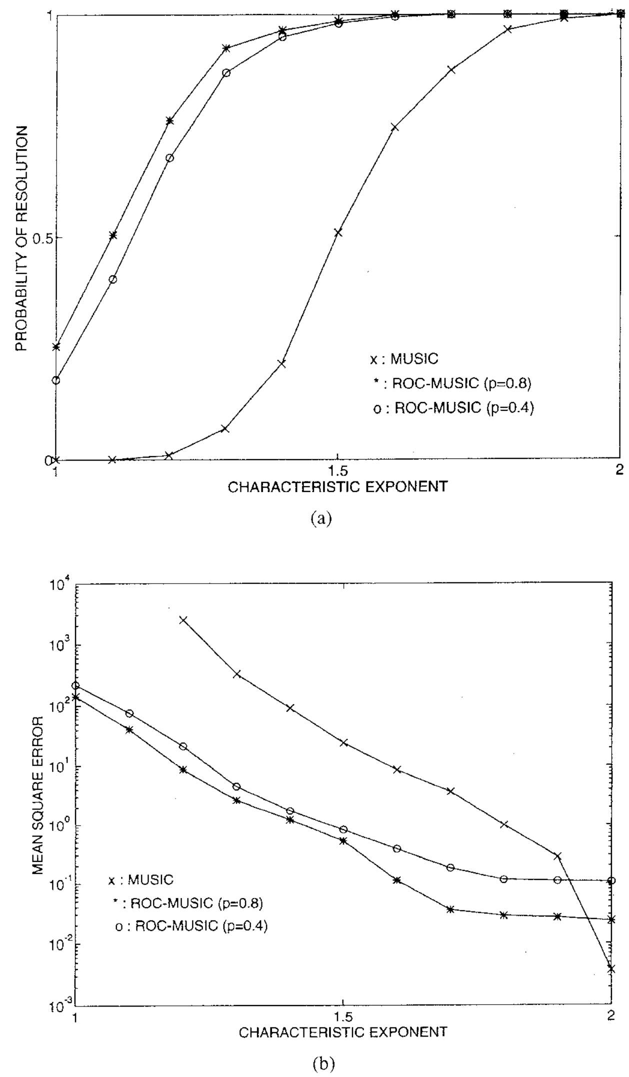

Fig. 2. Probability of resolution (a) and mean square error (b) as a function of the characteristic exponent α.

TABLE I
GSNR AND AVERAGE PSNR FOR DIFFERENT VALUES OF α

	Noise Characteristic Exponent α					
	$\alpha = 1.01$	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2.0$
GSNR [dB]	22.3 ($\gamma = 1$)					
PSNR [dB]	-17.9690	-7.8245	-2.8001	-0.8380	-0.2164	-0.0441

The locations of the source signals are determined by the values of θ for which the spectrum given by (40) peaks.

In practice, we have to estimate the covariation matrix from a finite number of array sensor measurements. One such estimator for the covariation coefficient $\lambda_{X,Y}$, is called the *fractional lower order (FLOM) estimator*. The FLOM estimator was proposed in [13] and is based on fractional lower order moments of the stable process. Expanding the p-support range of the FLOM estimator we define the *modified FLOM (MFLOM)* estimator

$$\hat{\lambda}_{X,Y}(p) = \frac{\sum_{i=1}^{n} X_i Y_i^{\langle p-1\rangle}}{\sum_{i=1}^{n}|Y_i|^p} \quad (41)$$

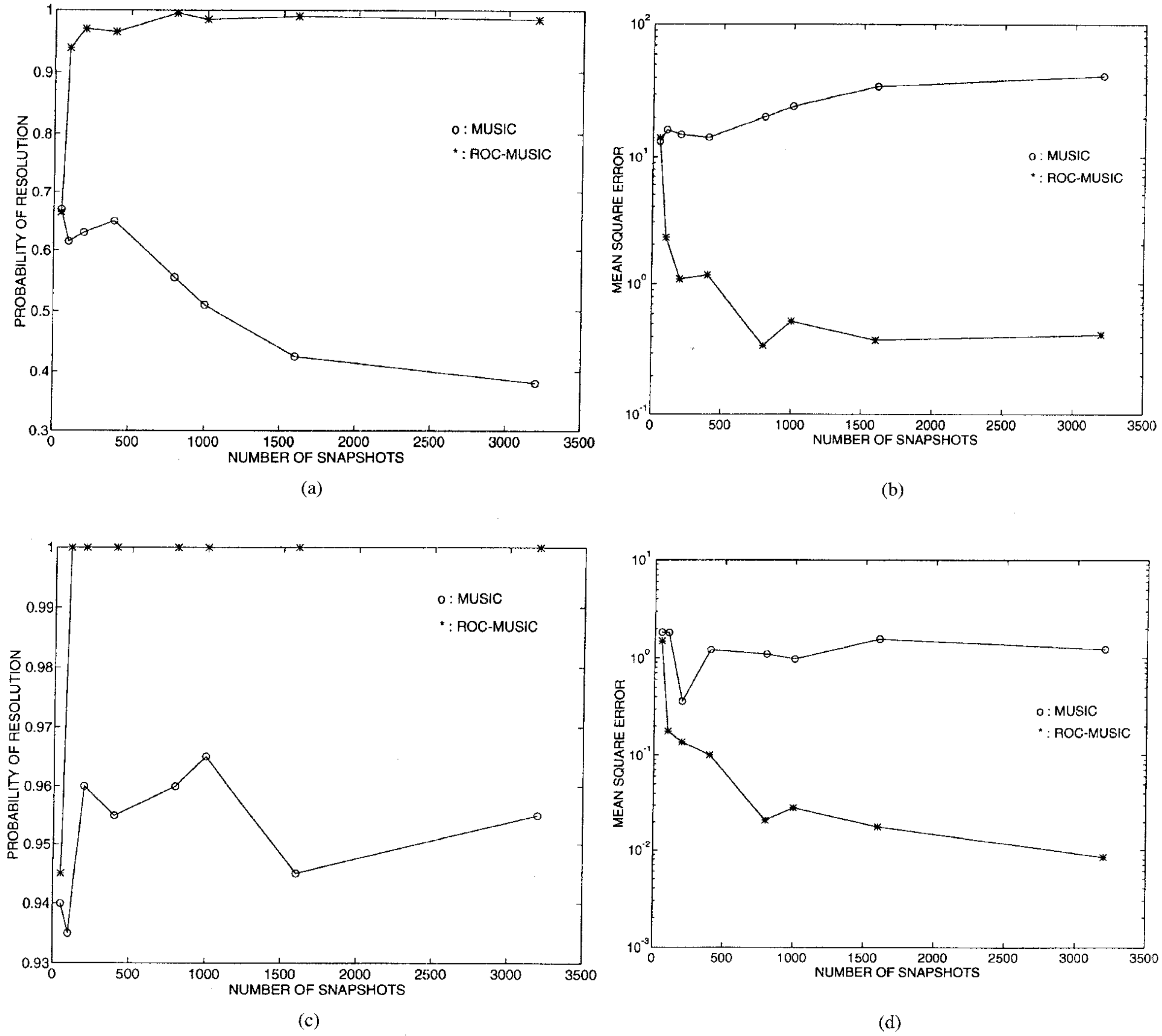

Fig. 3. Probability of resolution and mean square error as a function of the number of snapshots, (a)–(b): $\alpha = 1.5$, (c)–(d): $\alpha = 1.8$.

TABLE II
GSNR AND AVERAGE PSNR FOR DIFFERENT VALUES OF M

	Number of snapshots M							
	$M = 50$	$M = 100$	$M = 200$	$M = 400$	$M = 800$	$M = 1000$	$M = 1600$	$M = 3200$
GSNR [dB]	22.4003	22.2257	22.2015	22.3551	22.2832	22.3099	22.2907	22.3166
PSNR [dB] ($\alpha = 1.5$)	-0.6420	-0.6034	-1.0301	-1.4491	-1.2882	-1.5614	-2.1307	-1.9418
PSNR [dB] ($\alpha = 1.8$)	-0.1551	0.1494	-0.1808	-0.5530	-0.1396	-0.2164	-0.4561	-0.3315

for independent observations $(X_1, Y_1), \cdots, (X_n, Y_n)$. In the above expression, $1/2 < p < \alpha$ if X and Y are real $S\alpha S$ random variables, and $0 < p < \alpha$ if X and Y are complex isotropic $S\alpha S$ random variables. The estimator given in (41) is a moments-based estimator of the *modified covariation coefficient function* defined as

$$\lambda_{X,Y}(p) = \frac{E\{XY^{\langle p-1\rangle}\}}{E\{|Y|^p\}}, \qquad \begin{array}{ll} 1/2 < p < \alpha & \text{if } X \text{ and } Y \text{ are real} \\ 0 < p < \alpha & \text{if } X \text{ and } Y \text{ are complex} \end{array} \tag{42}$$

The modified covariation coefficient function is well defined (finite) for the aforementioned values of the parameter p as shown in [14]. The theoretical performance of the MFLOM estimator is studied also in [14]. Clearly, when $1 \leq p < \alpha$ the function $\lambda_{X,Y}(p)$ equals the covariation coefficient $\lambda_{X,Y}$ as defined in (10) and (11).

Fig. 1 illustrates the influence of the parameter p to the performance of the MFLOM estimator of the covariation coefficient. As we can see, for the case of non-Gaussian stable signals ($1 < \alpha < 2$), the values of p in the range

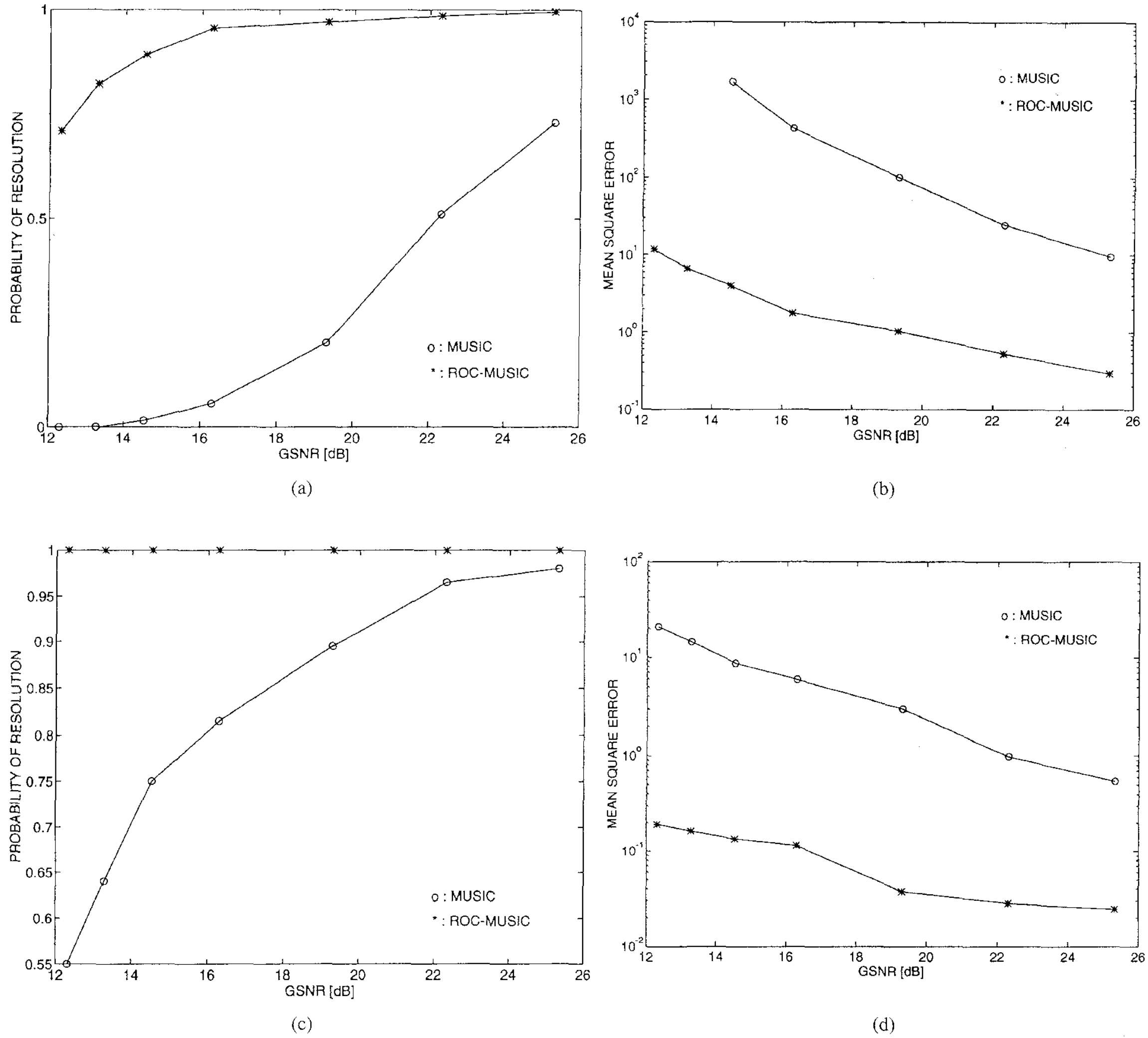

Fig. 4. Probability of resolution and mean square error as a function of the generalized signal-to-noise ratio (GSNR), (a)–(b) $\alpha = 1.5$. (c)–(d) $\alpha = 1.8$.

$(1/2, \alpha/2)$ result into the smallest standard deviations. For Gaussian signals the optimal value of p is 2 and the resulting MFLOM estimator is simply the least-squares estimator, as expected.

IV. Simulations

We performed two simulation experiments to assess the relative performance of the MUSIC and ROC-MUSIC algorithms. The two experiments study the performance of the MUSIC and ROC-MUSIC algorithms in the presence of simulated $S\alpha S$ noise and real radar clutter, respectively. The improved performance of the ROC-MUSIC method in terms of resolution capability, bias, and mean square error is apparent in the simulation results.

A. Performance Comparison of ROC-MUSIC versus MUSIC for Simulated Data

In this experiment we compare the performance of the proposed ROC-MUSIC algorithm to MUSIC for simulated

TABLE III
GSNR and Average PSNR for Different Values of γ (Real Clutter Data)

	Noise Dispersion γ						
	$\gamma = 0.5$	$\gamma = 1$	$\gamma = 2$	$\gamma = 4$	$\gamma = 6$	$\gamma = 8$	$\gamma = 10$
GSNR [dB]	25.3202	22.3099	19.2996	16.2893	14.5284	13.2790	12.3099
PSNR [dB] ($\alpha = 1.5$)	-0.7486	-1.5614	-2.9538	-5.0779	-6.6677	-7.9334	-8.9824
PSNR [dB] ($\alpha = 1.8$)	-0.0730	-0.2164	-0.5031	-1.0462	-1.5426	-1.9969	-2.4147

data. The sample covariation coefficient matrix (SCCM), as estimated by (41), is not symmetric and hence it has complex eigenvalues in general. The more snapshots are available at the array sensors, the more nearly symmetric SCCM becomes. We come around this problem by performing the eigenvalue decomposition to the sum of the sample covariation coefficient matrix and its Hermitian transpose.

The array is linear with five sensors spaced a half-wavelength apart. Two QAM communication signals independent of each other and of the same power impinge to the array.

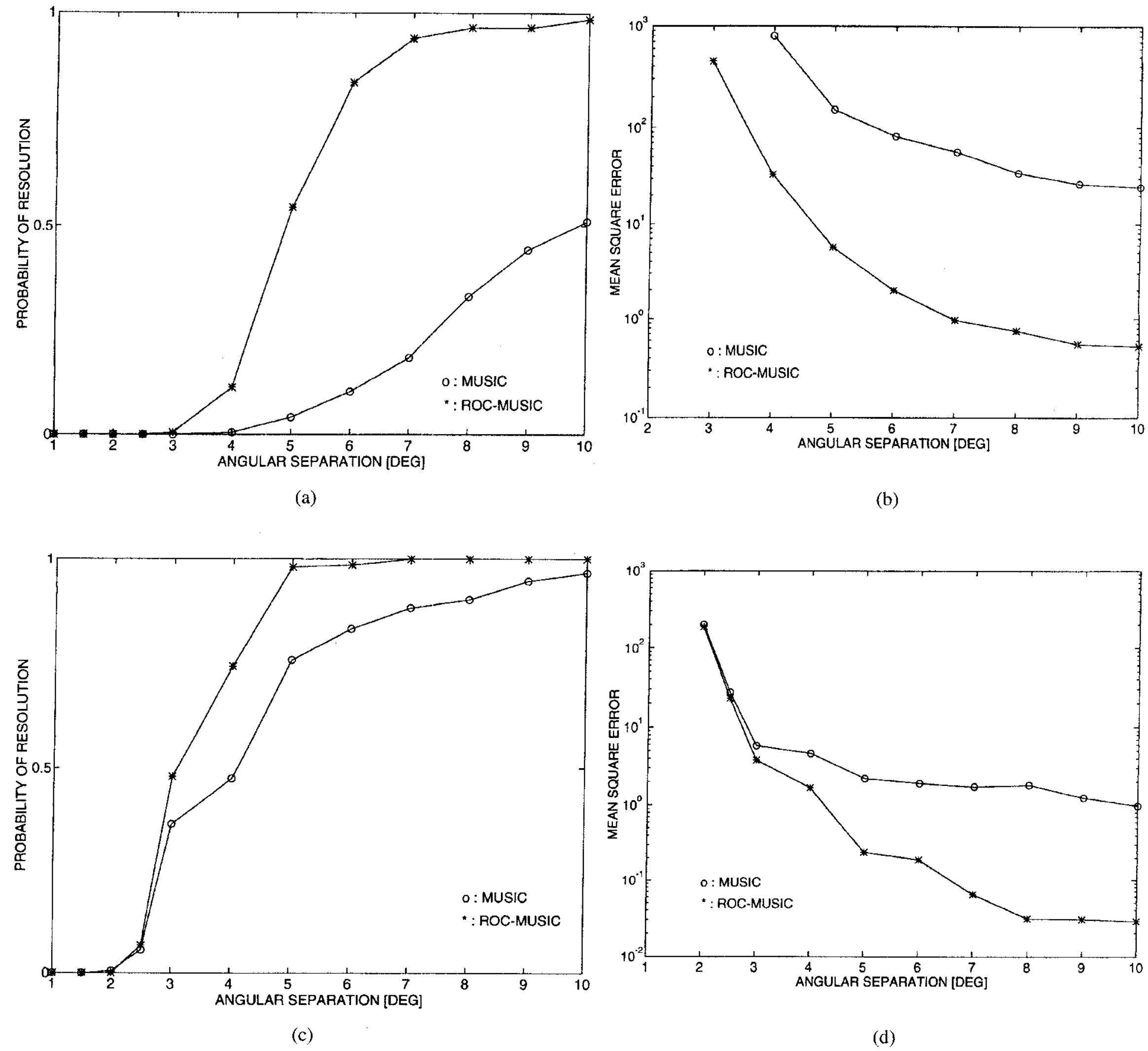

Fig. 5. Probability of resolution and mean square error as a function of the source angular separation. (a)–(b) $\alpha = 1.5$. (c)–(d) $\alpha = 1.8$.

The number of signals is assumed to be known. The noise is assumed to follow the complex isotropic $S\alpha S$ distribution with dispersion γ. In every experiment we perform 200 Monte Carlo runs and compute the resolution event probability, and the mean-square error (MSE) of the direction-of-arrival estimates averaged for the two sources. The MSE of the DOA estimates was calculated by taking into consideration only the Monte Carlo runs for which the two algorithms resolved the two sources.

The resolution analysis of the two algorithms was studied by using a popular resolution criterion defined by the following threshold equation [8], [15]

$$\Lambda(\theta_1, \theta_2) \triangleq P(\theta_m) - \frac{1}{2}\{P(\theta_1) + P(\theta_2)\} > 0 \qquad (43)$$

where θ_1 and θ_2 are the angles of arrival of the two signals, $\theta_m = (\theta_1 + \theta_2)/2$ is the mid-range between them, and the *null spectrum* $P(\theta) \triangleq 1/S(\theta)$ is defined as the reciprocal of the spatial spectrum $S(\theta)$ given in (40). The two signals are said to be resolvable if inequality (43) holds. The inequality implies that the null spectrum magnitude at the mid-angle should lie above the line segment linking the two signal valleys, in order for the two sources to be resolvable. In our experiments we estimated the null spectrum from a finite number of array sensor measurements, and the probability of resolution was determined by averaging the resolution events over the 200 independent Monte Carlo runs.

Since the alpha-stable family with $\alpha < 2$ determines processes with infinite variance, we use two alternative signal-to-noise ratios (SNR's) defined in [2], namely the *Generalized SNR (GSNR)* which is the ratio of the signal power over the noise dispersion γ

$$\text{GSNR} = 10\log\left(\frac{1}{\gamma M}\sum_{t=1}^{M}|s(t)|^2\right) \qquad (44)$$

and for finite sample realizations, the *Pseudo-SNR (PSNR)*

$$\text{PSNR} = 10\log\left(\frac{\sum_{t=1}^{M}|s(t)|^2}{\sum_{t=1}^{M}|n(t)|^2}\right). \qquad (45)$$

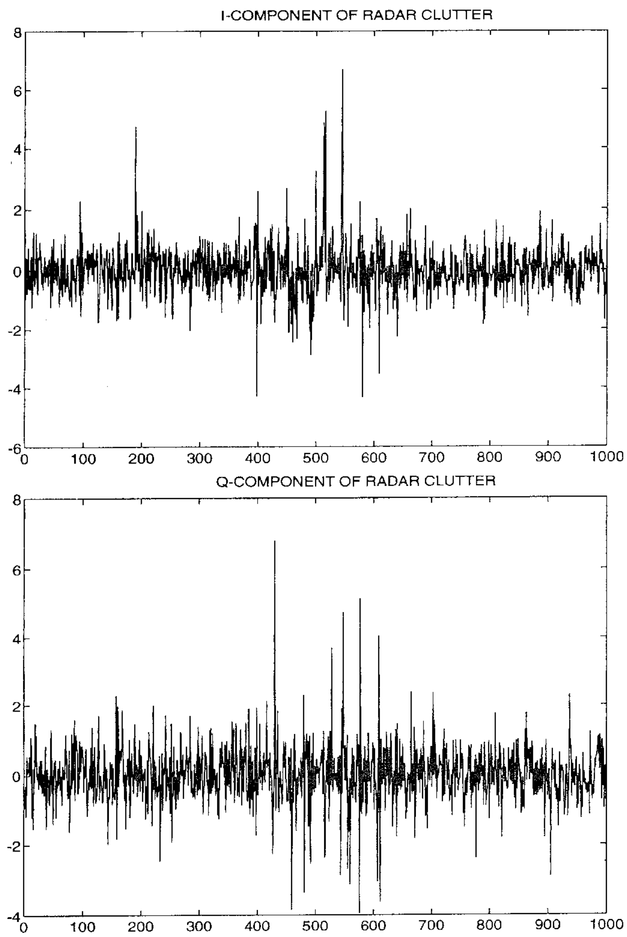

Fig. 6. In-phase (I) and quadrature (Q) components of radar clutter.

In the following simulation experiments, we study the resolution capability and estimation accuracy of ROC-MUSIC versus MUSIC as a function of four parameters, namely the noise characteristic exponent α, the number of snapshots M, the noise dispersion γ, and the angular separation of the two sources.

Characteristic Exponent α: Fig. 2 illustrates the performance of the two algorithms in a wide range of noise environments, from the more impulsive (α in the neighborhood of 1) to the Gaussian ones ($\alpha = 2$). The angles of arrival for the two signals are $\theta_1 = -5°$ and $\theta_2 = 5°$. The number of snapshots available to the algorithms is $M = 1000$. The GSNR is 22.3 dB ($\gamma = 1$) and is shown together with the average PSNR on Table I. The characteristic exponent α of the additive noise is unknown to the ROC-MUSIC algorithm. We use two values of the parameter p in the estimation of the covariation matrix (c.f. (41)): $p = 0.8$ and $p = 0.4$. Clearly, MUSIC can be thought as a special case of ROC-MUSIC with $p = 2$.

Fig. 2 depicts the improved performance of ROC-MUSIC over that of MUSIC both in terms of resolution probability and MSE, for values of α in the range $(1, 2)$. Note that for $\alpha < 1.2$ MUSIC does not resolve the two sources in any of the 200 Monte Carlo runs. The results suggest that in impulsive noise environments modeled under the stable law, it is more beneficial to use the covariation matrix (lower order moments) instead of the covariance matrix (second-order moments). Of course, for Gaussian additive noise ($\alpha = 2$) the use of second-order moments ($p = 2$) gives better results. Fig. 2(b) shows that the choice of $p = 0.8$ in ROC-MUSIC gives better results than the ones obtained when using $p = 0.4$, especially when $\alpha \geq 1.6$ in which case $p < \alpha/2$. It is not clear however from this result how the parameter p affects the performance of ROC-MUSIC. An analytical study on the way the choice of p affects the localization and resolution capabilities of the algorithm is currently under way.

Number of Snapshots M: The influence of the number of snapshots M to the performance of the algorithms is shown on Fig. 3. For this and the following experiments we chose two values for the characteristic exponent of the noise, namely $\alpha = 1.5$ and $\alpha = 1.8$, corresponding to a fairly impulsive noise environment ($\alpha = 1.5$) and an almost Gaussian one ($\alpha = 1.8$). Also, in the implementation of ROC-MUSIC, we used $p = 0.8$ for both cases of the stable noise. For this experiment, the GSNR is kept almost constant at 22.3 dB as shown on Table II. As we can also see from Table II, the PSNR is greater for the case of the less impulsive noise ($\alpha = 1.8$). As illustrated in Fig. 3, the performance of ROC-MUSIC is superior to that of MUSIC especially as the number of snapshots M increases and more and more impulsive noise samples are incorporated into the data.

Noise Dispersion γ: In this experiment we study the influence of the noise dispersion γ, i.e., the influence of the GSNR to the performance of the methods. The number of snapshots available to the algorithms is $M = 1000$. The GSNR and average PSNR for this experiment are shown on Table III. The results are depicted in Fig. 4. Again, for $\alpha = 1.5$ and GSNR < 14 dB, MUSIC fails to resolve the two sources in all 200 Monte Carlo runs.

Angular Separation: Fig. 5 illustrates the variation of the algorithmic performance with respect to the angle separation of the two incoming signals, for $M = 1000$, GSNR $= 22.3$ dB, PSNR$(\alpha = 1.5) = -1.56$ dB, and PSNR$(\alpha = 1.8) = -0.22$ dB. As expected, the resolution capability of both algorithms improves with increased angle separation between the two sources. However, for a given probability of resolution, the ROC-MUSIC algorithm requires a lower angle separation threshold than MUSIC.

B. Performance Comparison of ROC-MUSIC versus MUSIC for Real Clutter Data

The proposed algorithm has been tested with real radar sea clutter data provided by the Naval Surface Warfare Center, Carderock Division, Bethesda, MD. Clutter is a group of unwanted radar returns due to scattering centers such as precipitation, birds, and ocean waves. The received clutter signal can be represented in terms of its *in-phase* (I) and *quadrature* (Q) components. A typical sample set of the sea clutter data is shown in Fig. 6. The spiky nature of these radar returns is obvious, and it has been shown, using the algorithms developed in [16], that they can be modeled as $S\alpha S$ processes with $\alpha = 1.85$ and $\gamma = 0.19$.

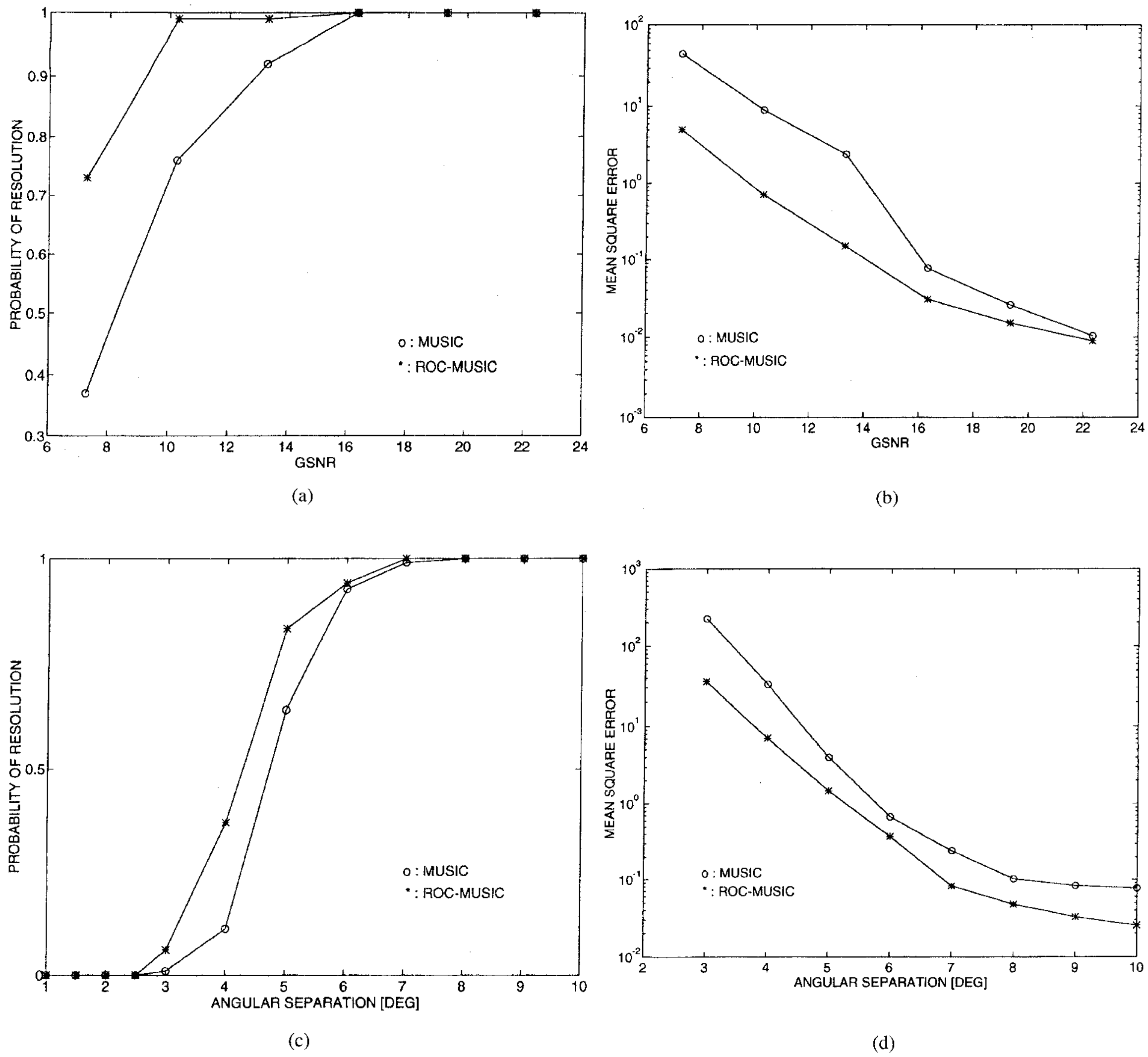

Fig. 7. Probability of resolution and mean square error as a function of the generalized signal-to-noise ratio (GSNR) (a)–(b) and source angular separation (c)–(d). Real clutter data experiment.

TABLE IV
GSNR AND AVERAGE PSNR FOR DIFFERENT VALUES OF γ (REAL CLUTTER DATA)

	Noise Dispersion γ					
	$\gamma = 1$	$\gamma = 2$	$\gamma = 4$	$\gamma = 8$	$\gamma = 16$	$\gamma = 32$
GSNR [dB]	22.3290	19.3187	16.3084	13.2981	10.2878	7.2775
PSNR [dB]	-0.4998	-0.8081	-1.4173	-2.5602	-4.5344	-7.5873

We studied the performance of ROC-MUSIC versus MUSIC in the presence of radar clutter and the results are shown in Fig. 7. Two uncorrelated radar waveforms were used and their number was assumed to be known. For the ROC-MUSIC we used $p = 0.85$ in the expression for the FLOM estimator of the covariation. Fig. 7(a)–(b) shows the probability of resolution and the MSE for various values of GSNR (cf. Table IV). The number of snapshots was $M = 600$ and the two sources had angles of arrival taking values $\theta_1 = -5°$ and $\theta_2 = 5°$. As we can see, for values of GSNR less than 13 dB (PSNR < -2 dB), ROC-MUSIC exhibits a 10 dB gain in MSE performance. Hence, for low GSNR values in which most radars operate, ROC-MUSIC shows a significant performance improvement over MUSIC. Fig. 7(c)–(d) depicts the algorithmic behavior as a function of the angle separation when GSNR $= 19.3$ dB (PSNR $= -0.8$ dB).

V. CONCLUDING REMARKS

We have formulated the covariation matrix of the array outputs for the case of complex isotropic $S\alpha S$ signals and noise. We showed that, for the special case of uncorrelated signals and of an array of sensors with unit amplitude response, the covariation matrix has similar form to the covariance matrix of Gaussian distributed signals. Therefore, subspace-based bearing estimation techniques can be applied to the covariation matrix resulting to improved direction of arrival estimates in the presence of impulsive noise environments. We presented a consistent estimator for the marginals of the co-variation matrix, whose asymptotic performance is analyzed in [14]. The improved performance of the proposed ROC-MUSIC algorithm in the presence of a wide range of impulsive noise environments was demonstrated via Monte Carlo experiments. Currently, we study the asymptotics of the joint distribution of the elements of the covariation matrix. From such an analysis we will be able to obtain an in-depth understanding of the statistical behavior of the ROC-MUSIC algorithm with respect to the localization accuracy and resolution capability.

This paper assumed knowledge of the number of the sources generating the signals that illuminated the antenna array. In many practical situations, this knowledge may not be available a priori. Hence, future research includes the development of methods for the detection of the number of signals in the presence of impulsive noise based on application of information-theoretic criteria. Finally, we will address the problem of detecting and localizing multiple wide-band sources in impulsive noise [17].

Appendix A
Generation of Complex Isotropic $S\alpha S$ Random Variables

The generation of complex isotropic $S\alpha S$ deviates of characteristic exponent α is based on the following proposition [18].

Proposition 1: A complex $S\alpha S$ ($\alpha < 2$) random variable $X = X_1 + jX_2$ is isotropic if and only if there exist two i.i.d. zero-mean Gaussian variables G_1 and G_2 and a real stable random variable A of characteristic exponent $\alpha/2$, dispersion $\cos^2(\pi\alpha/4)$ and skewness $\beta = 1$ (we write $A \sim S_{\alpha/2}(\cos^2(\pi\alpha/4), 1)$), independent of (G_1, G_2) such that $(X_1, X_2) \stackrel{d}{=} (A^{1/2}G_1, A^{1/2}G_2)$.

We say that the vector (X_1, X_2) is *sub-Gaussian* with underlying vector (G_1, G_2). It can be shown that the real and imaginary parts of X are always dependent, unless G_1 and G_2 are degenerate. Hence, every complex isotropic $S\alpha S$ random variable with $\alpha < 2$ can be expressed as

$$X = A^{1/2}(G_1 + jG_2) \tag{A.1}$$

and its generation involves the generation of a real, totally skewed stable random variable. The problem of generating a real stable deviate is studied in [19] and [20]. Here, we present the result for easy reference.

To generate a real standard stable random variable $A \sim S_\alpha(1, \beta)$ of characteristic exponent α, skewness β and unit dispersion $\gamma = 1$, the following representations can be deduced

$$S(\alpha, \beta, 1) = D_{\alpha,\beta}\frac{\sin\alpha(U - U_0)}{(\cos U)^{1/\alpha}} \cdot \left(\frac{\cos(U - \alpha(U - U_0))}{W}\right)^{\frac{1-\alpha}{\alpha}}, \quad \text{for } \alpha \neq 1 \tag{A.2}$$

and

$$S(1, \beta, 1) = \frac{2}{\pi}\left[\left(\frac{\pi}{2} + \beta U\right)\tan U - \beta\ln\left(\frac{\frac{\pi}{2}W\cos U}{\frac{\pi}{2} + \beta U}\right)\right] \tag{A.3}$$

where W is standard exponential with $\Pr\{W > w\} = e^{-w}, w > 0$, and U is uniform on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Also, $D_{\alpha,\beta} = [\cos(\arctan(\beta\tan(\pi\alpha/2)))]^{-1/\alpha}$, and $U_0 = -\frac{\pi}{2}\beta[k(\alpha)/\alpha]$ with $k(\alpha) = 1 - |1 - \alpha|$. Then, a stable variate, A_1, of dispersion γ can be obtained from A by $A_1 = \gamma^{1/\alpha}A$.

To conclude, the following proposition gives the relationship between the dispersion γ of the complex r.v. $X = X_1 + jX_2$ and the variance σ of the complex Gaussian RV $G = G_1 + jG_2$.

Proposition 2: The dispersion γ of the complex RV $X = X_1 + jX_2$ generated according to Proposition 1 is given by $\gamma = (\sigma/2)^\alpha$, where σ is the variance of the underlying complex Gaussian random variable.

Proof: The Laplace transform of the RV $A \sim S_{\alpha/2}(\cos^2(\pi\alpha/4), 1)$ is given by [18]

$$E\{\exp(-sA)\} = \exp(-s^{\alpha/2}), \qquad s > 0. \tag{A.4}$$

Also, since $G = G_1 + jG_2 \sim N_C(\sigma, 0)$, its characteristic function is given by

$$\varphi_G(\omega) = \exp(-\sigma^2|\omega|^2/4) \tag{A.5}$$

where $\omega = \omega_1 + j\omega_2$. Then, the characteristic function of X can be expressed as

$$\begin{aligned}\varphi_X(\omega) &= E\{\exp(j\Re[\omega X^*])\}\\ &= E\{\exp(j(\omega_1 A^{1/2}G_1 + \omega_2 A^{1/2}G_2))\}\\ &= E\{E\{\exp(j(\omega_1 A^{1/2}G_1 + \omega_2 A^{1/2}G_2)) \mid A\}\}\\ &= E\{\exp(-\sigma^2(\omega_1^2 + \omega_2^2)A/4)\} \text{ [by use of (A.5)]}\\ &= \exp((\sigma/2)^\alpha|\omega|^\alpha) \text{ [by use of (A.4)].}\end{aligned} \tag{A.6}$$

Comparing (A.6) with (2) we conclude that $\gamma = (\sigma/2)^\alpha$. □

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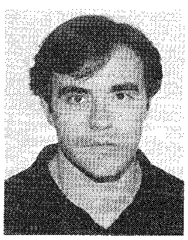
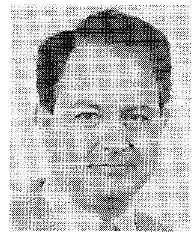


Panagiotis Tsakalides (M'95) was born in Serres, Greece, on December 15, 1967. He received the Diploma in electrical engineering from the Aristotle University of Thessaloniki, Greece, in 1990 and the M.S. degree in computer engineering and the Ph.D. degree in electrical engineering from the University of Southern California (USC), Los Angeles, in 1991 and 1995, respectively.

He is currently a Research Assistant Professor with the Signal and Image Processing Institute at USC. His general research interests lie in the area of statistical signal processing with emphasis in estimation and detection theory and applications in array processing, sonar, and radar problems.

In 1990, Dr. Tsakalides received scholarships from the Fulbright and Bodossakis Foundations for graduate studies.



Chrysostomos L. Nikias (F'91) received the Diploma in mechanical and electrical engineering from the National Technical University of Athens, Greece, in 1977 and the M.S. and Ph.D. degrees in electrical engineering from the State University of New York at Buffalo in 1980 and 1982, respectively.

He is Professor of Electrical Engineering—Systems at the University of Southern California (USC), Los Angeles and the Director of USC's Integrated Media Systems Center. Prior to that, he held academic appointments at Northeastern University, Boston, MA, and at the University of Connecticut, Storrs. He is the author of over 90 published journal papers, holds four patents, and is the co-author of the following textbooks: *Higher-Order Spectra Analysis: A Nonlinear Signal Processing Framework* (Englewood Cliffs, NJ: Oppenheim Series of Signal Processing, Prentice-Hall, 1993) (with A. Petropulu); *Advanced Signal Processing* (New York: Macmillan, 1992) (with J. Proakis, C. Rader, and F. Li); and *Signal Processing with Alpha-Stable Distributions and Applications* (New York: Wiley, 1994) (with M. Shao). His research interests span the fields of statistical signal processing with applications to communications, sonar, radar, and speech processing problems. He has organized and taught short courses in signal processing devoted to continuing engineering education. He is also the designer and instructor of several videotape tutorial seminars on modern spectral analysis.

Dr. Nikias served as the chairman of the IEEE Signal Processing (SP) Committee on Statistical Signal and Array Processing; an elected ADCOM member of the IEEE SP Society; as Associate Editor of the IEEE TRANSACTIONS ON ACOUSTICS, SPEECH, AND SIGNAL PROCESSING; and as the organizer and co-chairman of the following workshops: *Third ASSP Workshop on Spectrum Estimation and Modeling* (Boston, MA, 1986); *Higher Order Spectral Analysis* (Vail, CO, 1989); and *Higher Order Statistics* (France, 1991). He received both the 1993 outstanding teacher award of the National Technological University (TV university in USA) and the "Fred W. Ellersick Award of Outstanding Unclassified Paper at Military Communications (MILCOM)'92." He is a member of the California Council on Science and Technology (CCST).