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Covariation-based subspace-augmented MUSIC for joint sparse support recovery in impulsive environments

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ABSTRACT

In this paper, we introduce a subspace-augmented MUSIC technique for recovering the joint sparse support of a signal ensemble corrupted by additive impulsive noise. Our approach uses multiple vectors of random compressed measurements and employs fractional lower-order moments stemming from modeling the underlying signal statistics with symmetric alpha-stable distributions. We show through simulations that the recovery performance of the proposed method is particularly robust for a wide range of highly impulsive environments. Our subspace-augmented MUSIC achieves higher recovery rates than a recently introduced sparse Bayesian learning algorithm, which was shown to outperform many state-of-the-art techniques for joint sparse support recovery.

1. Introduction

Joint sparse recovery is of major significance in the framework of compressive sensing (CS) [1,2]. More specifically, given a set of compressed measurements, one aims to recover unknown sparse matrices with non-zero entries restricted to a subset of rows. This extends the widely studied single-measurement-vector (SMV) problem in a multiple-measurement-vector (MMV) framework, where we are interested in recovering the common support of a jointly sparse signal ensemble, that is, of a collection of signals characterized by a small number of non-zero entries, which are placed at the same or similar locations. This problem arises in several distinct applications, such as, in source localization [3] and neuromagnetic imaging [4], where only a subset of the sources transmits data (non-zero rows), whereas all the other sources are inactive (zero rows).

Commonly used approaches to the solution of the MMV problem are based on convex optimization theory employing mixed matrix norms [5] or on a sparse Bayesian learning formulation. In the later case, a recently introduced algorithm for sparse recovery of source vectors using sparse Bayesian learning (T-MSBL) [6] was shown to outperform many state-of-the-art algorithms under various experimental settings. In the favorable case where the submatrix consisting of the non-zero rows of the unknown sparse matrix has full rank, the well known multiple signal classification (MUSIC) algorithm [7] provides a theoretical guarantee for the successful recovery of the joint sparse support. However, rank deficiencies may arise in case of a limited number of measurements or correlations between the signal components. In such cases, the standard MUSIC technique, as well as most of the MMV algorithms, often fails. A subspace-augmented MUSIC (SA-MUSIC) algorithm [8] was introduced recently, which is robust against rank deficiency and ill-conditioning.

In several distinct environments, such as in underwater acoustics, in radar, and communications, the associate

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signals often take large-amplitude values much more frequently than what a Gaussian model implies. In these cases, the use of second-order moment theory (*e.g.,* both in MUSIC and SA-MUSIC) proves to be insufficient for the estimation of the original joint sparse support, with their performance deteriorating rapidly as the degree of impulsiveness increases. To overcome these limitations in cases where we are interested in detecting the active sources in such impulsive environments (*e.g.,* in an underwater network of acoustic sensors), we propose a subspace-augmented MUSIC-like algorithm by employing *fractional lower-order moments* (FLOMs), motivated by the fact that FLOMs best express the underlying heavy-tailed statistics of the signal ensemble and/or the impulsive noise terms. More specifically, members from the family of *symmetric alpha-stable* (SαS) distributions are used as priors for modeling highly impulsive source signals [9–12]. The experimental evaluation reveals a significant increase of the successful recovery rate of the original joint sparse support when compared with T-MSBL, especially in highly impulsive settings.

The rest of the paper is organized as follows: Section 2, introduces the proposed subspace-augmented MUSIC algorithm based on FLOMs, for joint sparse support recovery using multiple measurement vectors. A numerical evaluation is carried out in Section 3, while Section 4 concludes and gives directions for future work.

2. FLOM-based subspace-augmented MUSIC

Consider the scenario of a system consisting of N sources, where S samples are captured from each source at the same time instants. Let $\{\mathbf{x}_i\}_{i=1}^{S} \in \mathbb{R}^N$ be the *sparse signal ensemble* whose vectors share the same or a similar sparse support, that is, their few non-zero entries are placed at the same or similar locations. This ensemble is formed by gathering the i-th sample ($i=1,\dots,S$) of all the N sources into a single vector $\mathbf{x}_i \in \mathbb{R}^N$. A common sparse support in this case means that the same few sources are active when the i-th sample is recorded. The union of the sparse supports has at most K elements. Then, the signal matrix $\mathbf{X}=[\mathbf{x}_1,\dots,\mathbf{x}_S] \in \mathbb{R}^{N\times S}$ is called *row K-sparse* if it has at most K non-zero rows, where the *row support* $\mathcal{I} \subset \{1,\dots,N\}$ of $\mathbf{X}$ consists of the indices of the non-zero rows (active sources). A distinct vector of M random measurements $\mathbf{y}_i \in \mathbb{R}^M$ is generated for each signal $\mathbf{x}_i$, $i=1,\dots,S$ as follows, $\mathbf{y}_i=\mathbf{\Phi}\mathbf{x}_i$, where $\mathbf{\Phi} \in \mathbb{R}^{M\times N}$, with $M<N$, is an appropriate measurement matrix. The theory of compressed sensing states that, with high probability, the signal $\mathbf{x}_i$ can be reconstructed accurately from the measurements $\mathbf{y}_i$ if the matrix $\mathbf{\Phi}$ satisfies certain conditions, such as, the restricted isometry property [1,2]. Typical examples, which are used commonly in practice, include matrices whose entries are independent and identically distributed (i.i.d.) realizations of a random variable, such as Gaussian or Bernoulli. By augmenting the vectors $\{\mathbf{y}_i\}_{i=1}^{S}$ into a single observation matrix $\mathbf{Y}=[\mathbf{y}_1,\dots,\mathbf{y}_S] \in \mathbb{R}^{M\times S}$, the *joint sparse support recovery problem* in the multiple measurement vector (MMV) case consists of estimating the row support of $\mathbf{X}$, given the observations $\mathbf{Y}$ and the measurement matrix $\mathbf{\Phi}$. In the following, we consider the general noisy MMV model given by

$$\mathbf{Y}=\mathbf{\Phi}\mathbf{X}+\mathbf{H}, \tag{1}$$

where $\mathbf{H} \in \mathbb{R}^{M\times S}$ is a noise matrix with i.i.d. components, which is also independent from the signal matrix $\mathbf{X}$. The assumption of i.i.d. noise components is commonly employed in signal processing to simplify a theoretical framework. In addition, this assumption is in fact valid in practice, since we do not expect a set of noisy signals, which are related with the same phenomenon, to be characterized by a significantly different type of noise. In the following, let also $\mathbf{X}^{\mathcal{I}}$ denote the submatrix consisting of the non-zero rows only.

Under a second-order moments approach, MUSIC [7] estimates the dimension of the unknown *signal subspace*, spanned by $\mathbf{\Phi}\mathbf{X}$, and subsequently of the joint sparse support, through the eigenvalue decomposition of the observations covariance matrix, which is given by $\mathbf{\Sigma_Y}=\mathbf{\Phi}\mathbf{\Sigma_X}\mathbf{\Phi}^T+\sigma_\eta^2\mathbf{I}$, where $\mathbf{\Sigma_X}$ is the covariance matrix of the signal ensemble and $\sigma_\eta^2\mathbf{I}$ the noise covariance matrix. In practice, the sample covariance matrix is estimated directly from the observations,

$$\hat{\mathbf{\Sigma}}_{\mathbf{Y}} \triangleq \frac{1}{S}\mathbf{Y}\mathbf{Y}^T, \tag{2}$$

with $\hat{\mathbf{\Sigma}}_{\mathbf{Y}} \to \mathbf{\Sigma_Y}$ as $S\to\infty$. Then, the κ-dimensional signal subspace is generated from the eigenvectors of $\hat{\mathbf{\Sigma}}_{\mathbf{Y}}$ corresponding to the κ largest eigenvalues. A typical way to specify the value of κ is to select those κ largest eigenvalues, which contain a predefined percentage of the total signal energy. The value of this percentage is usually set in an empirical way depending on the specific application. On the other hand, SA-MUSIC (Ref. Algorithm 2 in [8]) was introduced as an efficient and robust alternative of MUSIC in the general noisy case, by estimating the dimension of an *augmented signal subspace*.

However, as it was mentioned in Section 1, there are several practical situations where the signals of interest are acquired in distinct impulsive environments, deviating strongly from a Gaussian assumption. During the 90s, several studies have shown that the family of alpha-stable distributions, and particularly the class of *symmetric alpha-stable* (SαS) distributions, is a powerful statistical tool for modeling highly impulsive source signals. A review of the state-of-the-art on stable processes from a statistical point of view is provided by a collection of papers edited by Cambanis et al. [13], while textbooks in the area have been written by Samorodnitsky and Taqqu [14], and by Nikias and Shao [15].

In the following, we extend the SA-MUSIC framework to tackle the problem of joint sparse support estimation in case of highly impulsive sources. For this purpose, the second-order statistics are substituted by FLOMs, incorporating members of the SαS family of distributions as prior source models.

2.1. Basics of univariate SαS distributions

In this section, we introduce briefly the family of univariate SαS distributions, as well as some of their fundamental properties to be used in the proposed approach. A SαS distribution is best defined by its characteristic function [14],

$$\phi(t) = \exp(i\delta t - \gamma^{\alpha}|t|^{\alpha}), \tag{3}$$

where α ($0<\alpha\leq 2$) is the *characteristic exponent*, a shape parameter which controls the "thickness" of the tails of the density function, $\delta \in \mathbb{R}$ is a *location parameter*, and $\gamma>0$ is the *dispersion*, which determines the spread of the distribution around δ, similar to the variance of the Gaussian. The smaller the α, the heavier the tails of a SαS density function, or, equivalently, the higher the impulsiveness of the associated random variable. In the special case when $\delta=0$ and $\gamma=1$ the SαS distribution is called *standard*. In the following, $X \sim f_{\alpha}(\gamma,\delta)$ will denote a SαS random variable with parameters α, γ, δ. We also note that Gaussian ($\alpha=2$) and Cauchy ($\alpha=1$) are the only SαS members with closed-form expressions for their density functions. Moreover, analogously with a zero-mean assumption, without loss of generality, in the following we focus on SαS distributions centered on zero, that is, with $\delta=0$. If X is not symmetric about zero, it can be easily symmetrized by subtracting the estimated location parameter, $X-\delta$.

An important characteristic of SαS distributions with $\alpha<2$ is the lack of second-order moments. Instead, all moments of order $p<\alpha$ do exist and are called the *fractional lower-order moments* (FLOMs). In particular, the FLOM of order p of a random variable $X \sim f_{\alpha}(\gamma,\delta=0)$ are given by [14]

$$\mathbb{E}\{|X|^p\} = (C(p,\alpha)\cdot\gamma)^p, \quad 0<p<\alpha, \tag{4}$$

where $(C(p,\alpha))^p = \Gamma(1-p/\alpha)/\cos((\pi/2)p)\Gamma(1-p)$. The S$\alpha$S model parameters (α,γ) can be estimated using the consistent Maximum Likelihood (ML) method described by Nolan [16],[1] which gives reliable estimates and provides the tightest possible confidence intervals. Additionally, a statistical norm, the so-called *covariation norm*, is induced for $X \sim f_{\alpha}(\gamma,0)$ with $\alpha\geq 1$ as follows:

$$\|X\|_{\alpha} = \gamma_X \quad \text{for } 1\leq\alpha\leq 2, \tag{5}$$

where γ_X is the dispersion of the random variable X.

Since covariances do not exist for SαS random variables, a quantity called *covariation* has been proposed instead. The covariation of two jointly, that is with the same α, SαS random variables X, Y with $1\leq\alpha\leq 2$, zero location parameters, and dispersions γ_X and γ_Y is defined by

$$[X,Y]_{\alpha} = \frac{\mathbb{E}\{XY^{\langle p-1\rangle}\}}{\mathbb{E}\{|Y|^p\}}\|Y\|_{\alpha}^{\alpha}, \tag{6}$$

where $\mathbf{z}^{\langle a\rangle} = [|z_1|^a\text{sign}(z_1),\ldots,|z_N|^a\text{sign}(z_N)]$ for any $\mathbf{z}\in\mathbb{R}^N$ and $a\geq 0$. The following (pseudo-)linearity property of the covariation in the first argument also holds: If X_1, X_2, Y are jointly SαS, then for any $a,b\in\mathbb{R}$ we have

$$[aX_1+bX_2,Y]_{\alpha} = a[X_1,Y]_{\alpha}+b[X_2,Y]_{\alpha}. \tag{7}$$

Besides, if $X \sim f_{\alpha}(\gamma_X,0)$ and $Y \sim f_{\alpha}(\gamma_Y,0)$ are independent, then, the *stability property* of SαS random variables yields $cX \sim f_{\alpha}(|c|\gamma_X,0)$ ($c\neq 0$) and $X+Y \sim f_{\alpha}((\gamma_X^{\alpha}+\gamma_Y^{\alpha})^{1/\alpha},0)$. Thus, for the i-th noisy random observation vector, $\mathbf{y}_i = \mathbf{\Phi}\mathbf{x}_i+\boldsymbol{\eta}_i$, of the MMV model in (1), if the n-th signal component $\mathbf{x}_{i,n} \sim f_{\alpha}(\gamma_{x,n},0)|_{n=1}^{N}$ and the m-th noise component $\boldsymbol{\eta}_{i,m} \sim f_{\alpha}(\gamma_{\eta,m},0)|_{m=1}^{M}$, then by applying (7) along with the stability property we have that

$$\mathbf{y}_{i,m} \sim f_{\alpha}\left(\left[\sum_{n=1}^{N}(|\phi_{mn}|\gamma_{x,n})^{\alpha}+\gamma_{\eta,m}^{\alpha}\right]^{1/\alpha},0\right), \quad m=1,\ldots,M, \tag{8}$$

where ϕ_{mn} is the element of $\mathbf{\Phi}$ in row m and column n, that is, each compressed measurement follows a SαS distribution with the same α.

Similarly to the correlation coefficient in case of second-order statistics, the *covariation coefficient* is defined for two jointly SαS random variables X and Y as follows:

$$\lambda_{X,Y} = \frac{[X,Y]_{\alpha}}{[Y,Y]_{\alpha}}. \tag{9}$$

Simple estimators for the covariation and covariation coefficient are obtained from (6) and (9), respectively, by substituting the expectations $\mathbb{E}\{\cdot\}$ with sample means.

We emphasize here that the key quantities defined above are based on the assumption of *jointly* SαS random variables. This is a main requirement dictated by the theory of alpha-stable processes [14,15] for the definition of the stability property, as well as the (pseudo-)linearity properties of the covariation and the covariation coefficient. In the noiseless case, this assumption is a reasonable practical simplification for modeling the signal ensemble $\mathbf{X}$, since the source signals are captured in the same environment. In the noisy case, the assumption for jointly SαS signal and noise components serves the theoretical derivation of the proposed framework. Although we do not expect this to be the case in a practical scenario, however, our subsequent experimental evaluation reveals that the proposed algorithm is robust in the disjointly SαS case, too, where the corresponding SαS model parameters are estimated directly from the observations $\mathbf{Y}$. Moreover, the definitions of several quantities, such as, the covariation norm (Ref. Eq. (5)) and covariation (Ref. Eq. (6)), were given for the case $1\leq\alpha\leq 2$. Although extensions of these definitions exist also for $0<\alpha<1$ (Ref. [14]), in the following we focus on the range $\alpha\in[1,2]$, since our previous experience has shown that this is the case which appears usually in distinct practical situations (Ref. [10,12]).

2.2. FLOM-based joint sparse support estimation

In our formulation of the MMV joint sparse support recovery problem in impulsive environments, we assume a SαS model for both the signal and noise components in (1). Accordingly, since the noisy compressed measurements follow a SαS distribution, as stated by (8), we define the *sample covariation matrix* $\hat{\mathbf{C}}$ of the measurements $\mathbf{Y}$ to

[1] Matlab package: http://www.robustanalysis.com.

be the matrix with entries

$$\hat{\mathbf{C}}_{mn} = [\mathbf{Y}^m, \mathbf{Y}^n]_\alpha \overset{(6)}{=} \frac{\frac{1}{S}\sum_{i=1}^{S} \mathbf{Y}_i^m (\mathbf{Y}_i^n)^{\langle p-1 \rangle}}{\frac{1}{S}\sum_{i=1}^{S} |\mathbf{Y}_i^n|^p} \|\mathbf{Y}^n\|_\alpha^\alpha, \tag{10}$$

where $\mathbf{Y}^l \in \mathbb{R}^S$ denotes the l-th row of the measurements matrix $\mathbf{Y}$, and $\mathbf{Y}_i^l$ its i-th component. Moreover, the dispersion $\|\mathbf{Y}^n\|_\alpha \overset{(5)}{=} \gamma_{\mathbf{Y}^n}$ is estimated directly from the components of $\mathbf{Y}^n$ using Nolan's ML method. Similarly, the *sample covariation coefficient matrix* $\hat{\mathbf{\Lambda}}$ is defined to be the matrix whose elements are the estimated covariation coefficients

$$\hat{\mathbf{\Lambda}}_{mn} = \hat{\lambda}_{\mathbf{Y}^m,\mathbf{Y}^n} \overset{(6),(9)}{=} \frac{\sum_{i=1}^{S} \mathbf{Y}_i^m (\mathbf{Y}_i^n)^{\langle p-1 \rangle}}{\sum_{i=1}^{S} |\mathbf{Y}_i^n|^p}. \tag{11}$$

More importantly, we emphasize again that the corresponding SαS model parameters (α,γ) are estimated directly from the observed measurements $\mathbf{Y}$.

Notice that most of the key quantities introduced above depend on the parameter p, whose optimal value depends on α. In [17], we generated a lookup table (Ref. Table 1) via an extensive set of Monte-Carlo runs for $\alpha \in [0.9,2]$, which gives the optimal p as the one that *minimizes the standard deviation* of the covariation estimator given by (6), where the expectations are replaced by the sample means. In our proposed algorithm the optimal p for $\alpha \in [1,2]$ is obtained by interpolating the entries of this table.

From the above we deduce that the estimation of the underlying signal subspace, and subsequently the joint sparse support of the source signal ensemble, can be accomplished by means of a typical subspace technique applied to the sample covariation or the sample covariation coefficient matrices, which are now adapted to the underlying heavy-tailed statistics of the noisy signal ensemble via the FLOMs. For numerical reasons the sample covariation coefficient matrix $\hat{\mathbf{\Lambda}}$ is employed in our proposed algorithm. This is justified by the fact that the elements of $\hat{\mathbf{\Lambda}}$ are estimated using the compressed measurements $\mathbf{Y}$ directly (Ref. Eq. (11)), while the use of the sample covariation matrix $\hat{\mathbf{C}}$ would require the estimation of dispersions (Ref. Eq. (10)), which entails the danger of inaccurate estimate when a limited number of measurement vectors, S, is available.

Due to the non-symmetry of $\hat{\mathbf{\Lambda}}$, its eigenvectors do not form an orthogonal system, while its eigenvalues may be complex in general. We overcome this limitation by employing the symmetrized sample covariation coefficient matrix, which is given by $\hat{\mathbf{\Lambda}}_{sym} = \frac{1}{2}(\hat{\mathbf{\Lambda}} + \hat{\mathbf{\Lambda}}^T)$. Similarly to SA-MUSIC, our proposed algorithm, hereafter referred to as FLOM-SA-MUSIC, proceeds in two steps for the estimation of the row-sparse support of $\mathbf{X}$.

Let $|\mathcal{I}| = K$ be the size of the true sparse support (or equivalently the number of active sources). At the first step, FLOM-SA-MUSIC performs a symmetric rank-revealing eigendecomposition [18], $\hat{\mathbf{\Lambda}}_{sym} = \mathbf{V}_r \mathbf{D}_r \mathbf{V}_r^T$, resulting in a rough estimate of the rank r of $\hat{\mathbf{\Lambda}}_{sym}$ along with the corresponding matrix of the r most significant eigenvectors $\mathbf{V}_r$. A $(K-r)$-dimensional partial support $\overline{\mathcal{I}}$ of the true support $\mathcal{I}$ is estimated by applying the standard Orthogonal Matching Pursuit (OMP)[2] taking as inputs the measurement matrix Φ and the matrix $\mathbf{V}_r$. At the second step, a QR factorization is applied on the augmented matrix $[\mathbf{V}_r | \mathbf{\Phi}_{\overline{\mathcal{I}}}]$, where $\mathbf{\Phi}_{\overline{\mathcal{I}}}$ consists of the columns of $\mathbf{\Phi}$ corresponding to the indices of $\overline{\mathcal{I}}$, resulting in an orthogonal matrix $\tilde{\mathbf{Q}}$ and an upper triangular matrix $\tilde{\mathbf{R}}$. Finally, the estimate of the true sparse support is completed by finding the K-element index set $\hat{\mathcal{I}}^* \subset \{1,\ldots,N\}$ maximizing the following objective:

$$\hat{\mathcal{I}}^* = \underset{\substack{|\hat{\mathcal{I}}| = K \\ \forall i_k \in \hat{\mathcal{I}}}}{\arg\max} \|\tilde{\mathbf{Q}}^T \boldsymbol{\phi}_{i_k}\|_2^2, \tag{12}$$

where $\boldsymbol{\phi}_{i_k}$ is the i_k-th column of $\mathbf{\Phi}$. Algorithm 1 summarizes our proposed FLOM-SA-MUSIC method.

Algorithm 1. FLOM-SA-MUSIC.

Inputs: Compressed measurements $\mathbf{Y}$, measurement matrix $\mathbf{\Phi}$, support size K, signal dimension N
Output: Sparse support $\hat{\mathcal{I}}^*$
// Main algorithm
Compute: estimated covariation coefficient matrix $\hat{\mathbf{\Lambda}}_{sym}$ (from Eq. (11))
// Perform symmetric rank-revealing eigendecomposition (SRREIG) [18]
$(\mathbf{V}_r, \mathbf{D}_r, r) = \text{SRREIG}(\hat{\mathbf{\Lambda}}_{sym})$
// Initial estimate of sparse support using OMP
$\overline{\mathcal{I}} \leftarrow \text{OMP}(\mathbf{\Phi}, \mathbf{V}_r, K-r)$
// QR factorization of augmented matrix
$(\tilde{\mathbf{Q}}, \tilde{\mathbf{R}}) = \text{QR}([\mathbf{V}_r | \mathbf{\Phi}_{\overline{\mathcal{I}}}])$
// Final sparse support estimate
for $n = 1,\ldots,N$ **do**
 $w_n = \|\tilde{\mathbf{Q}}^T \boldsymbol{\phi}_n\|_2^2$
endfor
Estimated support $\hat{\mathcal{I}}^* =$ {indices of K largest w_n}

3. Experimental evaluation

In this section, we evaluate the performance of FLOM-SA-MUSIC in terms of the successful recovery of the true joint sparse support, and compare with the performance of T-MSBL[3] [6], which was shown to outperform many state-of-the-art algorithms under various experimental settings. Although the derivation of the several SαS-based quantities in Section 2 assumes jointly SαS signal ($\mathbf{X}$) and noise ($\mathbf{H}$) components, that is, with the same characteristic exponent, in the subsequent evaluation we also examine the more general disjointly SαS case.

The non-zero entries of the row-sparse signal matrix $\mathbf{X}$ are i.i.d. realizations of a standard Sα_1S distribution, while the elements of the noise matrix $\mathbf{H}$ are drawn from a standard Sα_2S distribution, with $\alpha_1, \alpha_2 \in [1,\ldots,2]$. The original signal dimension (number of sources) is equal to $N=256$ and the row-sparsity level $\zeta = K/N$ varies between 5% and 20% of the signal dimension N. The sampling ratio $\mu = M/N \in [0.15,\ldots,0.5]$, while the elements of the measurement matrix $\mathbf{\Phi}$ are drawn from a Bernoulli distribution with equiprobable 0 and 1 values. Finally, all the subsequent results correspond to an

[2] Matlab code: http://www.mathworks.com/matlabcentral/fileexchange/32402.

[3] Matlab code: http://dsp.ucsd.edu/~zhilin/TMSBL.html

averaging over 1000 Monte-Carlo runs, where in each run the same measurement matrix is used for all the (α_1,α_2) pairs. Moreover, if $\mathcal{I}$ and $\hat{\mathcal{I}}$ denote the true and the estimated sparse support, respectively, then the successful recovery rate $\mathcal{P}_{SR}$ is defined by

$$\mathcal{P}_{SR} = \frac{||\mathcal{I}|-|\mathcal{I}\backslash\hat{\mathcal{I}}||}{|\mathcal{I}|}, \tag{13}$$

where $|\mathcal{I}|$ denotes the cardinality of set $\mathcal{I}$ and $\mathcal{I}\backslash\hat{\mathcal{I}}$ is the set difference.

We start with an evaluation of the support estimation accuracy of the proposed FLOM-SA-MUSIC algorithm in the jointly SαS case, and for sparse signals having exactly the same support. Fig. 1(a) shows the average percentages of successful support recovery, as a function of the sampling rate μ, for a fixed sparsity level $\zeta = 0.10$ and number of measurement vectors $S=256$, and for four distinct characteristic exponents $\alpha_1 = \alpha_2 = \alpha \in \{1.2,1.5,1.7,2\}$. First, we observe that the support recovery success rate increases for both FLOM-SA-MUSIC and T-MSBL as the statistics tend to a Gaussian ($\alpha \rightarrow 2$). In addition, as it was expected, the estimation accuracy also increases by increasing the sampling rate μ. Interestingly, this is not the case with T-MSBL, for which an increase in the sampling rate when dealing with highly impulsive signal and noise terms (that is, small values of α), does not affect its decreased estimation accuracy. This also reveals the inefficiency of techniques based on second-order statistics, such as the T-MSBL, to describe heavy-tailed phenomena, which are best expressed via the FLOMs.

Fig. 1(b) shows similar results to those described before, but for a smaller sparsity level, or equivalently, a larger $\zeta = 0.20$. As it can be seen, for a lower sparsity level the performance of FLOM-SA-MUSIC decreases for very small sampling rates. In fact, T-MSBL is better than FLOM-SA-MUSIC for sampling rates up to $20\%-25\%$, above which our proposed FLOM-SA-MUSIC results again in a significantly increased estimation accuracy outperforming T-MSBL, especially for small values of α. This is not surprising, since, as the number of non-zero rows of $\mathbf{X}$ increases, more measurements are necessary to estimate accurately the underlying heavy-tailed statistics. On the other hand, the estimated statistics of a very small number of measurements tend to a Gaussian, thus favoring T-MSBL, even if they are drawn from a heavy-tailed distribution. From these two figures we observe that a good compromise between the required sampling rate and a relatively high average success rate of FLOM-SA-MUSIC is achieved for values of μ larger than $25\%-30\%$, which is used in the subsequent results wherever we need to keep the sampling rate fixed.

A comparison between the above two figures indicates that the degree of sparsity is a parameter that affects the performance of every recovery algorithm. In Fig. 2(a), we examine the effect of a varying sparsity level to the

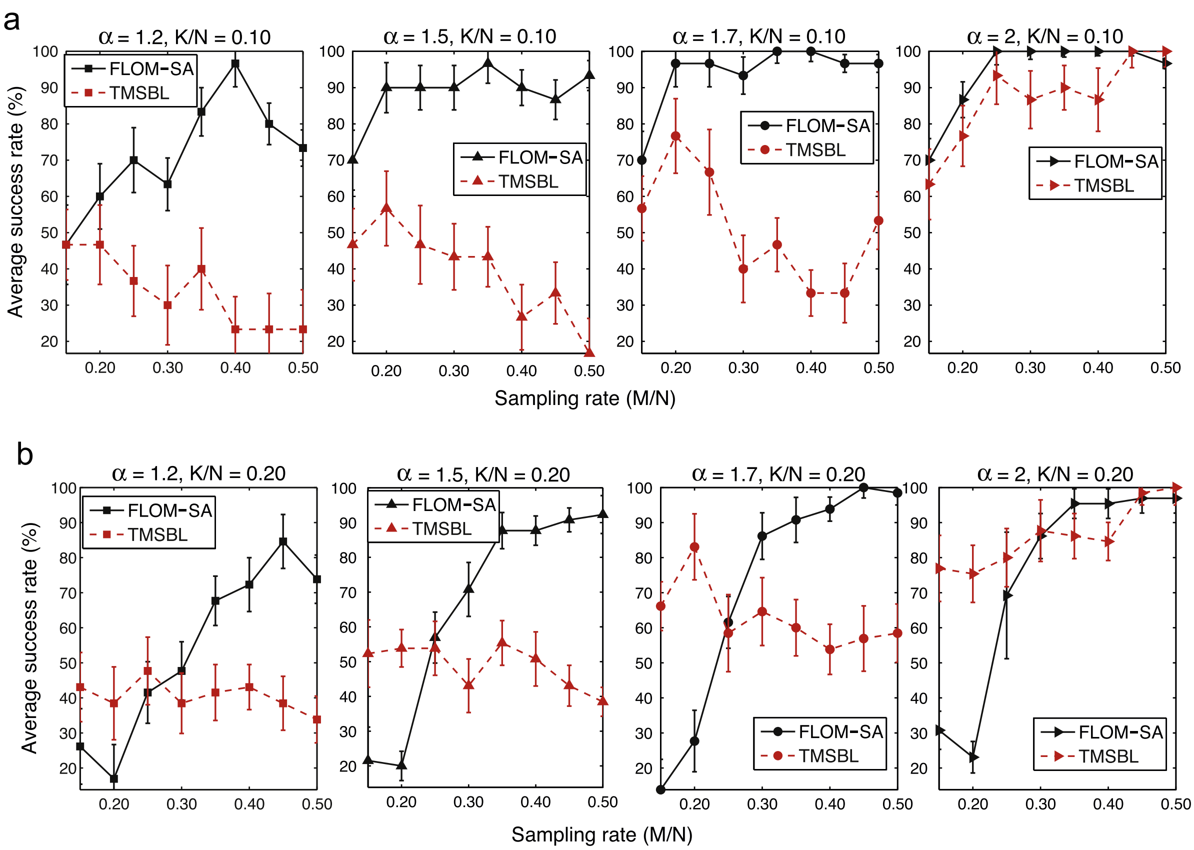

Fig. 1. Average success rates as a function of the sampling rate ($\mu \in [0.15,\ldots,0.5]$, $S=256$). (a) Average success rate vs. sampling rate for $K/N = 0.10$. (b) Average success rate vs. sampling rate for $K/N = 0.20$.

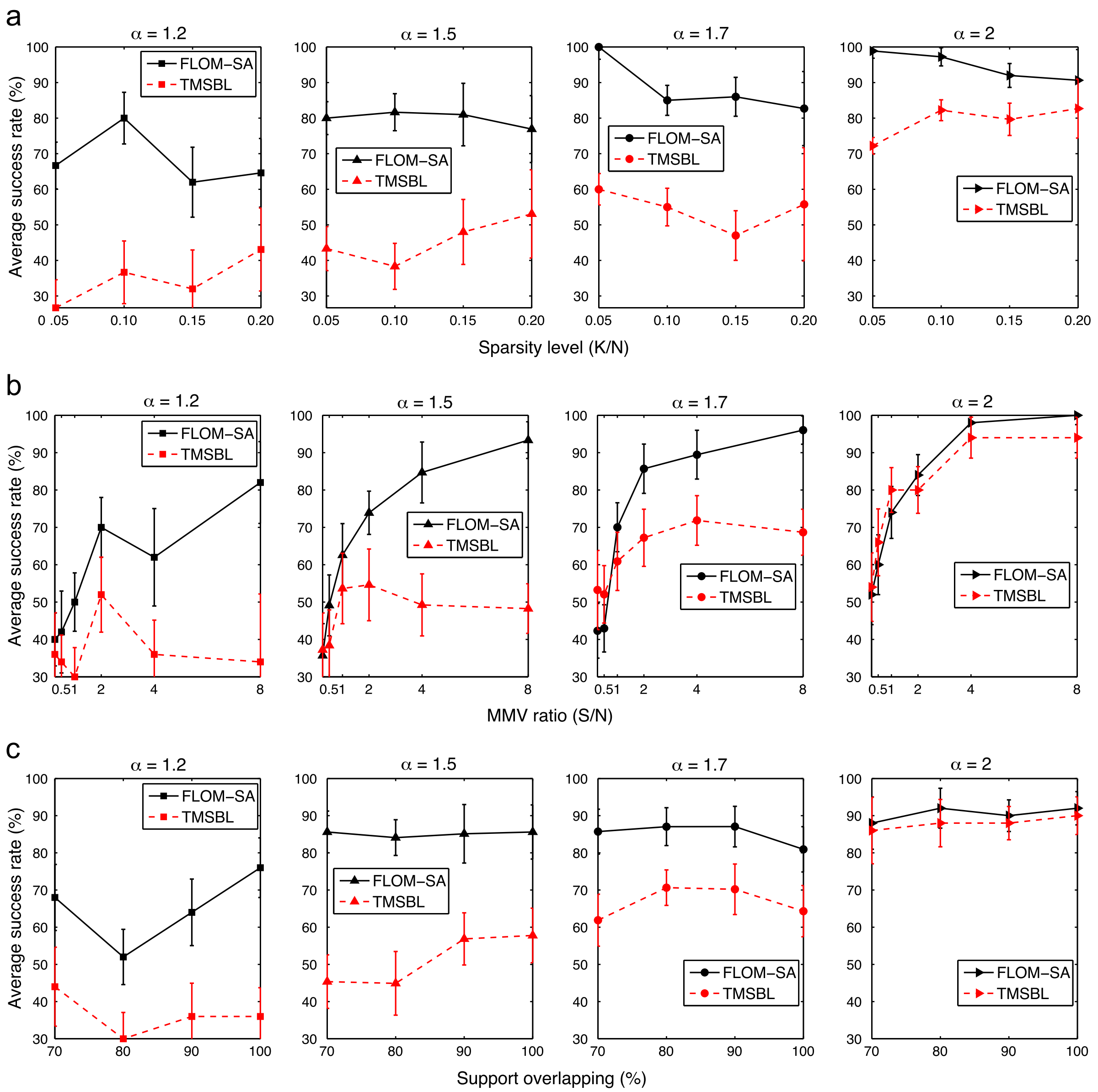

Fig. 2. Average success rates as a function of: (a) sparsity level, (b) number of measurements ratio, and (c) support overlapping, in the jointly SαS case. (a) Average success rate vs. sparsity level K/N ($M/N = 0.30$). (b) Average success rate vs. number of measurements ratio S/N ($M/N = 0.30$, $K/N = 0.15$). (c) Average success rate vs. support overlapping ($M/N = 0.30$, $K/N = 0.15$, $S/N = 2$).

estimation accuracy of the joint sparse support, for both FLOM-SA-MUSIC and T-MSBL. More specifically, the sparsity level $\zeta \in [0.05,\ldots,0.20]$, while the sampling rate μ and the number of measurement vectors S are kept fixed at 0.30 and 256, respectively. As we expected, the average success rate increases as the sparsity degree increases (smaller ζ). In addition, for the same ζ the support estimation accuracy improves as the statistics tend to a Gaussian ($\alpha = 2$) by keeping fixed the number of measurement vectors. Moreover, in all cases FLOM-SA-MUSIC outperforms T-MSBL, with their difference in performance to decrease as α increases.

Both, our proposed FLOM-SA-MUSIC algorithm and T-MSBL, are designed in a statistical framework with their performance depending largely on the accurate estimation of the underlying signal statistics. As in every statistical method, an accurate estimate of the several model parameters is affected by the number of available samples. Fig. 2(b) depicts the average success rate as a function of the number of measurement vectors (columns of $\mathbf{Y}$), which is expressed in terms of the ratio $\omega = S/N$. The sampling rate is fixed at $\mu = 0.30$, the sparsity level at $\zeta = 0.15$, while the characteristic exponent varies in $\alpha \in \{1.2,1.5,1.7,2\}$. As it can be seen, for a very small number of measurement vectors (small ω) T-MSBL outperforms FLOM-SA-MUSIC almost always. However, as ω increases the performance of the proposed FLOM-SA-MUSIC increases rapidly and outreaches the one of

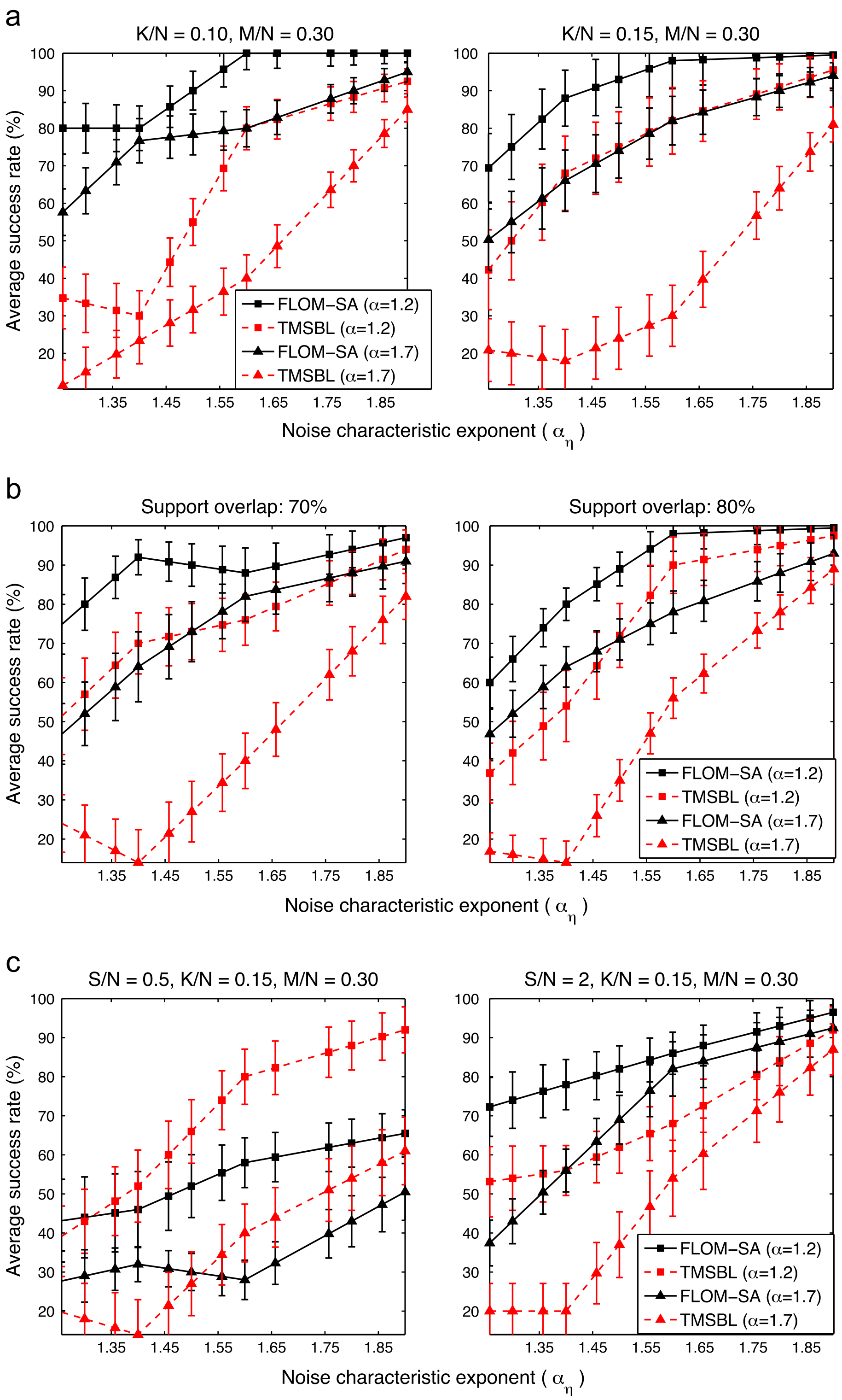

Fig. 3. Average success rates as a function of the noise characteristic exponent for varying: (a) sparsity level, (b) support overlapping, and (c) number of measurements ratio, in the disjointly SαS case. (a) Average success rate vs. noise characteristic exponent for varying K/N ($M/N = 0.30$). (b) Average success rate vs. noise characteristic exponent for varying support overlap ($M/N = 0.30$, $K/N = 0.15$, $S/N = 2$). (c) Average success rate vs. noise characteristic exponent for varying S/N ($M/N = 0.30$, $K/N = 0.15$).

T-MSBL, with the difference in their average success rates to increase for smaller values of α. This was an expected behavior, since the accuracy in estimating the SαS model parameters increases by increasing ω, whereas for very small values of ω the model tends to a Gaussian, thus favoring T-MSBL, which has been designed specifically under this assumption.

Finally, all the previous results refer to the case of a sparse signal ensemble with exactly the same support. As a last experiment in the jointly SαS case, we study the scenario where the sparse supports of the signal ensemble are not exactly the same, but they overlap. That is, the support of every signal is different but has a common subset which occupies the corresponding sparse support. Fig. 2(c) shows the average percentages of successful support recovery for a support overlapping varying between 70% and 100%, for $\mu = 0.30$, $\zeta = 0.15$, and $S/N = 2$. Clearly, the proposed FLOM-SA-MUSIC algorithm is quite robust in case of a reduced support overlapping and for a varying impulsiveness (values of α), in contrast to T-MSBL whose support recovery accuracy drops quickly as the percentage of support overlapping reduces.

As we mentioned before, the proposed FLOM-SA-MUSIC method was designed based on the assumption of jointly SαS signal and noise components. However, this should not be expected to be the case in a practical situation. For this purpose, we test the efficiency of FLOM-SA-MUSIC in case of disjointly SαS signal and noise terms. Fig. 3(a) shows the average success rates of FLOM-SA-MUSIC and T-MSBL for a varying noise characteristic exponent $\alpha_2 \in [1.25, \ldots, 1.95]$, which is different than the signal characteristic exponent $\alpha_1 \in \{1.2, 1.7\}$. The sampling rate is fixed at $\mu = 0.30$ and the number of measurements ratio at $\omega = 2$, while two distinct cases for the sparsity level are shown, namely, for $\zeta = 0.10$ (left subplot) and $\zeta = 0.15$ (right subplot). Clearly, FLOM-SA-MUSIC is very robust in the disjointly SαS case, in contrast to T-MSBL.

Similar robustness in the successful support recovery is also observed for a decreased support overlapping. Fig. 3(b) shows the average success rates as a function of the noise characteristic exponent, for a support overlapping of 70% (left subplot) and 80% (right subplot), where the other parameters are set as in Fig. 3(a). Finally, the effect of the number of measurements in the support recovery accuracy is tested under a disjointly SαS assumption. Fig. 3(c) depicts the average success rates as a function of the noise characteristic exponent, for two different ratios $S/N = 0.5$ and $S/N = 2$, for $\zeta = 0.15$ and $\mu = 0.30$. As it was expected, similarly to the jointly SαS case, the performance of FLOM-SA-MUSIC degrades for a small number of measurements (left subplot), due to inaccuracies in estimating the SαS model parameters. However, as the number of measurements increases the accuracy of FLOM-SA-MUSIC in recovering the exact sparse support increases significantly outperforming T-MSBL for all the (α_1, α_2) pairs.

4. Conclusion

In this study, we proposed a FLOM-based subspace-augmented MUSIC algorithm for joint sparse support recovery, when the signal ensemble and/or the additive noise component are characterized by (high) impulsiveness. We did this by exploiting the notion of fractional lower-order moments, which best approximate the underlying heavy-tailed statistics of the environment. The experimental evaluation revealed a clear improvement of the average successful recovery rate, especially when both the signal and noise components are highly impulsive, when compared with the recently introduced T-MSBL technique, which was shown to outperform many state-of-the-art algorithms.

In the current implementation, a single estimate of the SαS parameters (α, γ) is carried out using the measurement matrix **Y**. However, if this estimate is not accurate enough (*e.g.,* in case of a small number of measurement vectors) the subsequent support estimate could be inaccurate. An extension of the proposed FLOM-SA-MUSIC scheme will incorporate a recursive derivation being able to overcome an inaccurate initial estimate by employing an appropriate updating rule.

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