

Broadband Beamforming with Alpha-Stable Distributions

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Abstract

In this paper, we study the problem of localizing wide-band sources in the presence of noise modeled as a complex isotropic stable process. We consider the frequency-domain representation of the sensor outputs and show that the spectral density of complex stable processes plays a role, in array processing problems, analogous to that played by the power spectral density of second-order processes.

1 Introduction

In [1], we addressed the solution of the signal parameter estimation problem through the use of sensor array data retrieved in the presence of impulsive interference. We introduced optimal, maximum likelihood-based approaches to the localization problem of *narrow-band* sources in the presence of noise modeled as a complex isotropic Cauchy process. In addition, in [2], we developed subspace methods based on fractional lower-order statistics and introduced the ROC-MUSIC algorithm for array processing applications where reduced computational cost is a crucial design parameter. In this paper, we develop techniques for direction finding of *wide-band* sources in impulsive noise environments. Focused wide-band methods are used so that broad-band sources can be represented by rank-one models at the receiver. Following an approach similar to the one in [3], the covariation matrix of sensor outputs is formed after steering delays are inserted to form a conventional delay-and-sum beamformer. The resulting *steered spectral covariation matrix (SSCM)* focuses wide-band arrivals from its steer-

ing direction. By using a different SSCM for every direction of interest, all broad-band sources can be handled by a rank-one model. With this approach, the full signal bandwidth is exploited and thus, the observation time to achieve high-resolution performance is reduced.

In Section 2, we present some necessary preliminaries on the spectral representation of complex stationary α -stable processes. In Section 3, we introduce the steered spectral covariation matrix (SSCM) and present wide-band source localization methods in the presence of α -stable distributed signals and noise. Simulation experiments demonstrating the performance of the proposed methods are presented in Section 4. Finally, conclusions are drawn in Section 5.

2 Spectral Representation of Alpha-Stable Processes

Statistical signal processing has traditionally been based on the theory of second-order stationary Gaussian processes, $\{X(t); -\infty < t < \infty\}$. In this paper, we consider an important class of stationary processes which have the spectral representation:

$$X(t) = \int_{-\infty}^{\infty} e^{jt\omega} d\xi(\omega); \quad -\infty < t < \infty, \quad (1)$$

and whose second-order moments are infinite [4]. Namely, we consider strictly stationary complex symmetric alpha-stable ($S\alpha S$) processes ($0 < \alpha < 2$) having the spectral representation (1) where $\xi(\omega)$ is a $S\alpha S$ process with independent increments satisfying

$$\{E|d\xi(\omega)|^p\}^{\alpha/p} = C(p, \alpha)\phi(\omega) d\omega \quad \text{for all } 0 < p < \alpha, \quad (2)$$

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where $C(p, \alpha)$ is a constant depending on p and α and $\phi(\omega)$ is a nonnegative function called the *spectral density* of $X(t)$. The spectral density $\phi(\omega)$ describes fully the distribution of the process $X(t)$. A process $\{X(t); -\infty < t < \infty\}$ is strictly stationary if and only if ξ has isotropic or rotationally invariant increments, i.e., the distribution of the process of increments $e^{j\vartheta} d\xi(\omega)$, $-\infty < \omega < \infty$, does not depend on the rotation ϑ [5].

The *covariation function* plays for stable processes a role analogous to the role that the covariance plays for Gaussian processes and is given by:

$$[X(t), X(s)]_\alpha \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{j(t-s)\omega} \phi(\omega) d\omega. \quad (3)$$

As we see in (3), the covariation function of an alpha-stable process has an identical representation in terms of the spectral density ϕ as the covariance function of a second-order process. The covariation of complex jointly $S\alpha S$ r.v.'s is not generally symmetric and has the following properties [6]:

P1 If $X(r)$, $X(s)$ and $X(t)$ are jointly $S\alpha S$, then

$$[aX(r) + bX(s), X(t)]_\alpha = a[X(r), X(t)]_\alpha + b[X(s), X(t)]_\alpha$$

for any complex constants a and b .

P2 If $X(s)$ and $X(t)$ are independent and $X(r)$, $X(s)$ and $X(t)$ are jointly $S\alpha S$, then

$$[aX(r), bX(s) + cX(t)]_\alpha = ab^{<\alpha-1>} [X(r), X(s)]_\alpha + ac^{<\alpha-1>} [X(r), X(t)]_\alpha$$

for any complex constants a , b and c ¹.

P3 If $X(s)$ and $X(t)$ are independent $S\alpha S$, then $[X(s), X(t)]_\alpha = 0$.

3 The Wide-Band Source Localization Problem

Consider an array of r sensors that receive signals generated by q *wide-band* sources s_1, s_2, \dots, s_q with identical bandwidth and locations $\theta_1, \theta_2, \dots, \theta_q$. The signals are assumed to be noncoherent, ergodic, and stationary complex isotropic $S\alpha S$ random processes. In addition, the noise vector $\mathbf{n}(t) = [n_1(t), n_2(t), \dots, n_r(t)]^T$ is a complex $S\alpha S$ random process with the same characteristic exponent α as

¹We use throughout the convention $Y^{<p>} = |Y|^{p-1}Y^*$.

the signals. The signal received at the i th sensor can be expressed as

$$x_i(t) = \sum_{k=1}^q s_k(t - \tau_i(\theta_k)) + n_i(t), \quad (4)$$

where $\tau_i(\theta_k)$ is the propagation delay between the first and the i th sensor for a waveform coming from direction θ_k . Unlike the narrow band case where the problem is usually formulated in terms of the sampled data, in the wide-band case it is convenient to formulate the problem in the frequency domain. Assuming the availability of M observations in the interval $(-T/2, T/2)$, let $\mathbf{X}(m)$ denote the frequency-domain vector with elements $X_i(m)$ corresponding to the Fourier series coefficients of $x_i(t)$ at frequency $\omega_m = 2\pi m/T$. Under the assumption that the sensor outputs are approximately band-limited to $\omega_l \leq \omega \leq \omega_h$, the frequency-domain sensor output vector $\mathbf{X}(m)$ can be expressed in matrix form:

$$\mathbf{X}(m) = \mathbf{A}(\omega_m, \boldsymbol{\theta})\mathbf{S}(m) + \mathbf{N}(m); \quad \omega_l \leq \omega_m \leq \omega_h, \quad (5)$$

where

- $\mathbf{A}(\omega_m, \boldsymbol{\theta}) = [\mathbf{a}(\omega_m, \theta_1), \dots, \mathbf{a}(\omega_m, \theta_q)]$ is the $r \times q$ matrix of the array steering vectors at frequency ω_m , with $\mathbf{a}(\omega_m, \theta_k) = [1, e^{-j\omega_m \tau_2(\theta_k)}, \dots, e^{-j\omega_m \tau_r(\theta_k)}]^T$ being the frequency steering vector of the array toward direction θ_k and,
- $\mathbf{S}(m) = [S_1(m), \dots, S_q(m)]^T$ and $\mathbf{N}(m) = [N_1(m), \dots, N_r(m)]^T$ are the frequency-domain signal and noise vectors, respectively.

Note that, by the stability property of the alpha-stable processes, the frequency-domain elements $X_i(m)$ are also $S\alpha S$ processes with the same characteristic exponent as the time domain samples $x_i(t)$. Also, assuming that the observation time T is large, the Fourier coefficients $\mathbf{X}(m)$ and $\mathbf{X}(n)$ are independent for $m \neq n$ [5].

3.1 The Spectral Covariation Matrix

Denote by $\mathbf{K}_s(\omega_m)$ and $\mathbf{K}_n(\omega_m)$ the source and noise spectral covariation matrices, respectively. Under the assumptions made above about the statistical independence of the signals and the noise, both matrices are diagonal with $\mathbf{K}_s(\omega_m) = \text{diag}(\gamma_{s_1}(\omega_m), \dots, \gamma_{s_q}(\omega_m))$ and $\mathbf{K}_n(\omega_m) = \gamma_n(\omega_m)\mathbf{I}$, where $\gamma_{s_k}(\omega_m) = [S_k(m), S_k(m)]_\alpha$, $k = 1, \dots, q$, and $\gamma_n(\omega_m) = [N_i(m), N_i(m)]_\alpha$, $i = 1, \dots, r$.

Equation (5) can be written as

$$\mathbf{X}(m) = \mathbf{W}(m) + \mathbf{N}(m); \quad \omega_l \leq \omega_m \leq \omega_h, \quad (6)$$

where $\mathbf{W}(m) = \mathbf{A}(\omega_m, \boldsymbol{\theta})\mathbf{S}(m)$. By the stability property, it follows that $\mathbf{W}(m)$ is also a complex isotropic $S\alpha S$ random vector. In addition, it holds that $\mathbf{W}(m)$ is independent of $\mathbf{N}(m)$.

Now, we define the *spectral covariation matrix*, $\mathbf{K}(\omega_m)$, of the frequency-domain vector process $\mathbf{X}(m)$ as the matrix whose elements are the covariations $[X_i(m), X_j(m)]_\alpha$ of the components of $\mathbf{X}(m)$. We have that

$$\begin{aligned} [\mathbf{K}(\omega_m)]_{ij} &= [X_i(m), X_j(m)]_\alpha \\ &= [W_i(m), W_j(m)]_\alpha + [W_i(m), N_j(m)]_\alpha + \\ &\quad [N_i(m), W_j(m)]_\alpha + [N_i(m), N_j(m)]_\alpha. \end{aligned} \quad (7)$$

By the independence assumption of $\mathbf{W}(m)$ and $\mathbf{N}(m)$ and by property P3 we have that

$$[W_i(m), N_j(m)]_\alpha = 0, \quad (8)$$

and

$$[N_i(m), W_j(m)]_\alpha = 0. \quad (9)$$

In addition, by using properties P1 and P2 it follows that

$$\begin{aligned} [W_i(m), W_j(m)]_\alpha &= \\ &\sum_{k=1}^q a_i(\omega_m, \theta_k) a_j^{\langle \alpha-1 \rangle}(\omega_m, \theta_k) \gamma_{s_k}(\omega_m), \end{aligned} \quad (10)$$

where $\gamma_{s_k}(\omega_m) = [S_k(m), S_k(m)]_\alpha$. Finally, due to the noise assumption made earlier, it holds that

$$[N_i(m), N_j(m)]_\alpha = \gamma_n(\omega_m) \delta_{i,j}, \quad (11)$$

where $\delta_{i,j}$ is the Kronecker delta function. Combining (7)-(11) we obtain the following expression for the spectral covariation matrix of the observation vector:

$$\begin{aligned} \mathbf{K}_X(\omega_m) &\triangleq [\mathbf{X}(m), \mathbf{X}(m)]_\alpha = \\ &\mathbf{A}(\omega_m, \boldsymbol{\theta}) \mathbf{K}_S(\omega_m) \mathbf{A}^{\langle \alpha-1 \rangle}(\omega_m, \boldsymbol{\theta}) + \gamma_n(\omega_m) \mathbf{I}, \end{aligned} \quad (12)$$

where the (i, j) th element of matrix $\mathbf{A}^{\langle \alpha-1 \rangle}(\omega_m, \boldsymbol{\theta})$ results from the (j, i) th element of $\mathbf{A}(\omega_m, \boldsymbol{\theta})$ according to the operation

$$[\mathbf{A}^{\langle \alpha-1 \rangle}(\omega_m, \boldsymbol{\theta})]_{i,j} = [\mathbf{A}(\omega_m, \boldsymbol{\theta})]_{j,i}^{\langle \alpha-1 \rangle} \quad (13)$$

Since the frequency steering vectors are of the form $\mathbf{a}(\omega_m, \theta_k) = [1, e^{-j\omega_m \tau_2(\theta_k)}, \dots, e^{-j\omega_m \tau_r(\theta_k)}]^T$, it follows that

$$[\mathbf{A}^{\langle \alpha-1 \rangle}(\omega_m, \boldsymbol{\theta})]_{i,j} = [\mathbf{A}(\omega_m, \boldsymbol{\theta})]_{j,i}^*, \quad (14)$$

and thus the spectral covariation matrix can be written as

$$\mathbf{K}_X(\omega_m) = \mathbf{A}(\omega_m, \boldsymbol{\theta}) \mathbf{K}_S(\omega_m) \mathbf{A}^H(\omega_m, \boldsymbol{\theta}) + \gamma_n(\omega_m) \mathbf{I}. \quad (15)$$

Clearly, when $\alpha = 2$, i.e., for Gaussian distributed signals and noise, the expression for the spectral covariation matrix is identical to the well-known expression for the power spectral density matrix:

$$\mathbf{P}_X(\omega_m) = \mathbf{A}(\omega_m, \boldsymbol{\theta}) \mathbf{P}_S(\omega_m) \mathbf{A}^H(\omega_m, \boldsymbol{\theta}) + \sigma^2(\omega_m) \mathbf{I}, \quad (16)$$

where $\mathbf{P}_S(\omega_m)$ is the source power spectral density matrix.

In practice, in order to estimate $\mathbf{K}_X(\omega_m)$, we divide the T second observation into N nonoverlapping segments of ΔT seconds each and apply the discrete Fourier transform to obtain uncorrelated frequency-domain vectors $\mathbf{X}_n(m)$ for each segment. Then, the spectral covariation matrix $\mathbf{K}_X(\omega_m)$ is estimated as

$$\hat{\mathbf{K}}_X(\omega_m) = \frac{1}{N} \sum_{n=1}^N \mathbf{X}_n(m) \mathbf{X}_n^{\langle p-1 \rangle}(m), \quad (17)$$

for some $0 < p < \alpha/2$ [2].

Extending the narrow-band source localization techniques of [2] to the wide-band case, the series of spectral covariation matrices over the receiver band can be used to obtain narrow-band spatial estimation results in each frequency. Then, the bearings of the wide-band sources can be estimated by combining the narrow-band location estimates. Hence, we define $P_{IRM}(\theta)$ as the incoherent wide-band ROC-MUSIC estimator obtained by summing the narrow-band ROC-MUSIC results over the band of interest:

$$P_{IRM}(\theta) = \sum_{\omega_m=\omega_l}^{\omega_h} \frac{1}{\sum_{i=q+1}^r |\mathbf{a}^H(\omega_m, \boldsymbol{\theta}) \mathbf{v}_i(\omega_m)|^2}, \quad (18)$$

where $\{\mathbf{v}_i(\omega_m); i = q+1, \dots, r\}$ are the eigenvectors corresponding to the $r - q$ smallest eigenvalues of the sample spectral covariation matrix $\hat{\mathbf{K}}_X(\omega_m)$ estimated in (17).

The performance of the incoherent ROC-MUSIC in localizing wide-band sources is limited because it requires long observation times to obtain statistically stable estimates of the frequency-dependent spectral covariation matrices [3]. In many practical situations, the observation time available at the receiver may be limited due to nonstationary propagation environments and fast moving signal sources. Such a scenario causes a severe degradation to the performance of the

incoherent wide-band methods. To reduce the observation time required to achieve high-resolution broadband source localization, we present in the following section a focused wide-band array processing method which extends results obtained by Krolik [3] to alpha-stable signals and noise.

3.2 The Steered Spectral Covariation Matrix

According to the steered spectral covariation methods, steering delays are inserted into the sensor output measurements to ensure that sources from a particular direction have the same rank-one representation at all temporal frequencies. Under the assumption that the sensor outputs are approximately band-limited to $\omega_l \leq \omega \leq \omega_h$, and that the observation time $T \gg 2\pi/(\omega_h - \omega_l) + \max(\tau_i(\theta))$, the *steered sensor output vector* $\mathbf{y}(t, \theta)$ at the direction of interest θ is defined as [3]

$$\mathbf{y}(t, \theta) = \sum_{m=l}^h \mathbf{T}_m(\theta) \mathbf{X}(m) e^{j\omega_m t}, \quad (19)$$

where

$$\mathbf{T}_m(\theta) = \text{diag}\{e^{j\omega_m \tau_0(\theta)}, \dots, e^{j\omega_m \tau_{r-1}(\theta)}\}. \quad (20)$$

By substituting $\mathbf{T}_m(\theta)$ into (19), we can express the i th component $y_i(t, \theta)$ of the steered time-domain sensor output vector $\mathbf{y}(t, \theta)$ as follows

$$y_i(t, \theta) = \sum_{m=l}^h e^{j\omega_m \tau_i(\theta)} X_i(m) e^{j\omega_m t} = x_i(t + \tau_i(\theta)). \quad (21)$$

Hence, the steered output vector can be expressed as

$$\mathbf{y}(t, \theta) = [x_0(t + \tau_0(\theta)), \dots, x_{r-1}(t + \tau_{r-1}(\theta))]^T \quad (22)$$

Now, we define the *Steered Spectral Covariation Matrix (SSCM)*, $\mathbf{C}(\theta)$, of the steered sensor output process $\mathbf{y}(t, \theta)$ as the matrix whose elements are the covariations $[y_i(t, \theta), y_j(t, \theta)]_\alpha$ of the components of $\mathbf{y}(t, \theta)$. Assuming that the observation time T is large, the Fourier coefficients $\mathbf{X}(m)$ and $\mathbf{X}(n)$ are independent for $m \neq n$ [5]. Then, by following similar steps as in the derivation of the spectral covariation matrix in Section 3.1, $\mathbf{C}(\theta)$ can be expressed as

$$\mathbf{C}(\theta) = \sum_{m=l}^h \mathbf{T}_m(\theta) \mathbf{K}(\omega_m) \mathbf{T}_m^H(\theta), \quad (23)$$

where $\mathbf{K}(\omega_m) = [\mathbf{X}(m), \mathbf{X}(m)]_\alpha$ is the covariation matrix of the Fourier coefficient vector at frequency ω_m .

The structure of $\mathbf{C}(\theta)$ is apparent when we consider the case of a line array with sensors equally spaced at distance d apart. In this case, $\tau_i(\theta) = i\tau(\theta)$, where $\tau(\theta) = (d/c)\sin(\theta)$ and c is the propagation speed. Then, according to the model of (4) the ij th element of $\mathbf{C}(\theta)$ for steering direction $\theta = \theta_l$ is given by

$$[\mathbf{C}(\theta_l)]_{ij} = \sum_{k=1, k \neq l}^q \zeta_k(h(\tau(\theta_l) - \tau(\theta_k))) + \gamma_{s_l} + \gamma_n \delta_{i,j}, \quad (24)$$

where $h = i - j$, $\zeta_k(h(\tau(\theta) - \tau(\theta_k))) = [s_k(t + i(\tau(\theta) - \tau(\theta_k))), s_k(t + j(\tau(\theta) - \tau(\theta_k)))]_\alpha$ is the covariation function of the k th source signal, $\gamma_{s_l} = [s_l(t), s_l(t)]_\alpha$ is the covariation of the l th source and $\gamma_n = [n_i(t), n_i(t)]_\alpha$. Hence, the spectral covariation matrix $\mathbf{C}(\theta)$ steered at the direction of a source contains a constant component equal to the source covariation regardless of the source's spectral signature. Also, the off-steering direction summation terms in (24) decrease with increasing h and increasing separation $|\tau(\theta_l) - \tau(\theta_k)|$ from the steering direction. It follows that the spatial spectral signature of a source in the steering direction $\theta = \theta_l$ can be estimated by measuring the constant component γ_{s_l} . The resulting spatial spectral estimate is called the *Steered Spectral Covariation (SSC)* method given by

$$P_{SSC}(\theta) = [\mathbf{1}^H \mathbf{C}(\theta)^{-1} \mathbf{1}]^{-1}, \quad (25)$$

where $\mathbf{1}$ is an $r \times 1$ vector of ones, and $\mathbf{C}(\theta)$ is given by (23). In practice, the SSCM $\mathbf{C}(\theta)$ is computed for each steering direction, θ , of interest. Hence, SSCM-based methods are more computationally intensive than incoherent subspace methods. Their advantage lies in the fact that they take advantage of the full time-bandwidth product of the observations, thus resulting in more stable statistical estimates.

4 Simulation Results

In this section, the performance of the proposed steered spectral covariation (SSC) method for localizing wide-band sources is assessed through simulation experiments. We use a uniformly spaced linear array consisting of $r = 10$ omnidirectional sensors which are spaced a half-wavelength apart at the normalized frequency of π radians. The two source signals are modeled as mutually uncorrelated communication signals with bandpass spectral spectra with $\omega_l = 0.125\pi$ and $\omega_h = \pi$. The source directions are -20° and 30° . The additive noise process is modeled as a stable process with flat spectral density over the same bandpass

range as the signal. A total of 8 narrow-band spectral covariation matrices $\mathbf{K}(\omega_m)$, equally spaced from ω_l to ω_h , were used when applying the IRM and SSC methods (cf. (18) and (25), respectively). We also implement the incoherent MUSIC (IM) and steered minimum variance (STMV) [3] methods which are based on a second-order statistics formulation.

We study the estimation accuracy of the four methods as a function of the observation time (number of snapshots at each frequency, N) and the characteristic exponent α of the signals and noise. The so-called generalized signal-to-noise ratio $GSNR = 10 \log(\frac{1}{\gamma M} \sum_{t=1}^M |s(t)|^2)$ is approximately equal to 0dB. Figure 1(a) shows the resulting MSE of the estimated DOA as a function of the number of snapshots, N , at each frequency ω_m for the different methods. Figure 1(b) shows the resulting MSE curves as functions of the characteristic exponent α . The number of snapshots at each frequency available to the receiver is $N = 625$. Clearly, for non-Gaussian additive noise, SSC exhibits less mean-square estimation error than the other three methods.

5 Concluding Remarks

Until recently, statistical signal processing with alpha-stable distributions has not been popular due to the fact that the linear space of a stable process is not a Hilbert space, as in the case of Gaussian processes, but either a Banach ($1 < \alpha < 2$) or a metric space ($0 < \alpha \leq 1$) both of which are more unyielding in their structure. In this chapter, we presented new approaches to the wide-band DOA estimation problem in the presence of impulsive interference. We defined the steered spectral covariation matrix of an array of sensors, and applied steered subspace-based bearing estimation techniques resulting to improved bearing estimates in the presence of impulsive additive noise.

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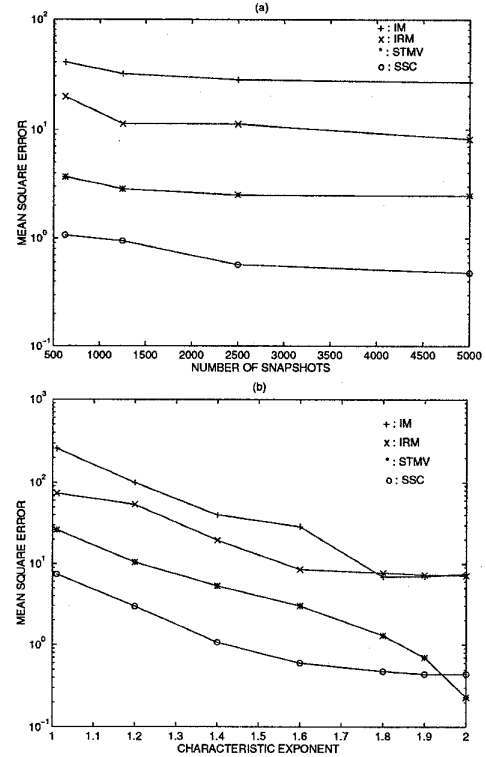


Figure 1: MSE curves of the estimated DOA as functions of the number of snapshots N ($\alpha = 1.5$) (a), and the characteristic exponent α (b).

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