Wideband Array Signal Processing with Alpha-Stable Distributions*

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Abstract

The importance of extending the statistical array signal processing methodology to what we call the alpha-stable framework is apparent. First, scientists and engineers have started to appreciate infinite moments and the elegant scaling and self-similarity properties of stable distributions. Additionally, real life applications exist in which impulsive channels tend to produce large-amplitude, short-duration interferences more frequently than Gaussian channels do. The stable law has been shown to successfully model noise over certain impulsive channels. In this paper, we propose new robust techniques for wideband source detection and localization in the presence of noise modeled as a complex isotropic stable process.

1 Introduction

Statistical array processing based on the linear theory of random processes with finite second-order moments has been the focus of considerable academic research. Critical problems such as high-resolution direction finding, null- and beam-steering, and detection of the number of sources illuminating an array of sensors have been studied under the assumption of a Gaussian or second-order model. Looking toward real life applications, we are interested in developing array processing methods for a larger class of random processes which include the Gaussian processes as special elements. The availability of such methods would make it possible to operate in environments which, while sharing many characteristics, also differ from Gaussian environments in significant ways.

In [1] we addressed the solution of the signal parameter estimation problem through the use of sensor array data retrieved in the presence of impulsive interference. We introduced optimal, maximum likelihood-based (ML) approaches to the localization problem of **narrow-band** sources in the presence of noise modeled as a complex isotropic Cauchy process. In this paper, we develop techniques for direction finding of **wide-band** sources in impulsive noise environments. Focused wide-band methods are used so that broadband sources can be represented by rank-one models at the receiver. According to our approach, the covariation matrix of sensor outputs is formed after steering delays are inserted to form a conventional delay-and-sum beamformer. The resulting *steered covariation matrix (SCM)* focuses wide-bands arrivals from its steering direction. By using a different SCM for every direction of interest, all broadband sources can be handled by a rank-one model. With this approach the full signal bandwidth is exploited and thus, the observation time to achieve high-resolution performance is reduced.

In Section 2, we present some necessary preliminaries on α-stable processes. In Section 3, we present subspace source localization methods based on geometrical properties of the data model. Considerable research has been done in this area under the framework of Gaussian distributed signals and/or noise [2]. The better known of the so-called eigenvector-based methods are the MUSIC, Minimum Norm, and the ESPRIT method. These methods estimate the bearings of the source signals by performing an eigendecomposition on the spatial covariance matrix of the array sensor outputs. Since $S\alpha S$ processes do not possess finite pth order moments for $p \geq \alpha$, traditional subspace techniques employing second- and higher-order moments cannot be applied in impulsive noise environments modeled under the stable law. Instead, properties of fractional lower-order moments (FLOM's) and covariations should be used. In Section 4, we introduce the steered covariation matrix (SCM) and present wide-band source localization methods in the presence of

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α-stable distributed signals and noise. Simulation experiments demonstrating the performance of the proposed methods are presented in Section 5. Finally, conclusions are drawn in Section 6.

2 Mathematical Preliminaries

In this section we introduce the statistical model that will be used to describe the additive noise. The model is based on the class of *isotropic* $S\alpha S$ distributions, and is well-suited for describing impulsive noise processes [3].

Stable processes satisfy the stability property which states that linear combinations of jointly stable variables are indeed stable. They arise as limiting processes of sums of independent, identically-distributed random variables via the generalized central limit theorem. They are described by their characteristic exponent α, taking values $0 < \alpha \leq 2$. Gaussian processes are stable processes with $\alpha = 2$. Stable distributions have heavier tails than the normal distribution, possess finite pth order moments only for $p < \alpha$, and are appropriate for modeling noise with outliers.

A complex random variable (r.v.) $X = X_1 + \jmath X_2$ is isotropic $S\alpha S$ if X_1 and X_2 are jointly $S\alpha S$ and have a symmetric distribution. The characteristic function of X is given by

$$\varphi(\omega) = \mathcal{E}\{\exp(\jmath\Re[\omega X^*])\} = \exp(-\gamma|\omega|^\alpha), \quad (1)$$

where $\omega = \omega_1 + \jmath\omega_2$. The *characteristic exponent* α is restricted to the values $0 < \alpha \leq 2$ and it determines the shape of the distribution. The smaller the characteristic exponent α, the heavier the tails of the density. The *dispersion* γ ($\gamma > 0$) plays a role analogous to the role that the variance plays for second-order processes. Namely, it determines the spread of the probability density function around the origin.

Several complex r.v.'s are jointly $S\alpha S$ if their real and imaginary parts are jointly $S\alpha S$. When X and Y are jointly $S\alpha S$ with $1 < \alpha \leq 2$, the *covariation* of X and Y is defined by

$$[X, Y]_\alpha = \frac{\mathcal{E}\{XY^{<p-1>}\}}{\mathcal{E}\{|Y|^p\}}\gamma_Y, \quad 1 \leq p < \alpha, \quad (2)$$

where $\gamma_Y = [Y, Y]_\alpha$ is the dispersion of the r.v. Y, and we use throughout the convention $Y^{<p>} = |Y|^{p-1}Y^*$. Also, the *covariation coefficient* of X and Y is defined by

$$\lambda_{X,Y} = \frac{[X, Y]_\alpha}{[Y, Y]_\alpha}, \quad (3)$$

and by using (2), it can be expressed as

$$\lambda_{X,Y} = \frac{E\{XY^{<p-1>}\}}{E\{|Y|^p\}}, \quad \text{for } 1 \leq p < \alpha. \quad (4)$$

The covariation of complex jointly $S\alpha S$ r.v.'s is not generally symmetric and has the following properties:

P1 If X_1, X_2 and Y are jointly $S\alpha S$, then for any complex constants a and b,

$$[aX_1 + bX_2, Y]_\alpha = a[X_1, Y]_\alpha + b[X_2, Y]_\alpha;$$

P2 If Y_1 and Y_2 are independent and X_1, X_2 and Y are jointly $S\alpha S$, then for any complex constants a, b and c,

$$[aX_1, bY_1 + cY_2]_\alpha = ab^{<\alpha-1>}[X_1, Y_1]_\alpha + ac^{<\alpha-1>}[X_1, Y_2]_\alpha;$$

P3 If X and Y are independent $S\alpha S$, then $[X, Y]_\alpha = 0$.

3 Narrow-Band Eigenstructure Methods

Consider an array of r sensors that receive signals generated by q narrow-band sources with center frequency ω and locations $\theta_1, \theta_2, \ldots, \theta_q$. Since the signals are narrow-band, the propagation delay across the array is much smaller than the reciprocal of the signal bandwidth, and it follows that, by using a complex envelop representation, the array output can be expressed as

$$\mathbf{x}(t) = \mathbf{A}(\Theta)\mathbf{s}(t) + \mathbf{n}(t), \quad 1 \leq t \leq M, \quad (5)$$

where $\mathbf{s}(t)$ is the vector of signals emitted by the sources as received at the reference sensor of the array, $\mathbf{n}(t)$ is the noise vector and $\mathbf{A}(\Theta)$ is the $r \times q$ matrix of the array steering vectors

$$\mathbf{A}(\Theta) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_q)], \quad (6)$$

with $\mathbf{a}(\theta_k) = [1, e^{-\jmath\omega\tau_2(\theta_k)}, \ldots, e^{-\jmath\omega\tau_r(\theta_k)}]^T$.

To proceed, assume that the q signal waveforms are noncoherent, complex isotropic $S\alpha S$ random processes with diagonal covariation matrix $\mathbf{\Gamma}_S = \text{diag}(\gamma_{s_1}, \ldots, \gamma_{s_q})$. Also, the noise vector $\mathbf{n}(t)$ is a complex $S\alpha S$ random process with the same characteristic exponent α as the signals. The noise is independent of the signals and has covariation matrix $\mathbf{\Gamma}_N = \gamma_n\mathbf{I}$.

We can write (5) as follows:

$$\mathbf{x}(t) = \mathbf{w}(t) + \mathbf{n}(t), \quad (7)$$

where $\mathbf{w}(t) = \mathbf{A}(\Theta)\mathbf{s}(t)$. By the stability property, it follows that $\mathbf{w}(t)$ is also a complex $S\alpha S$ random vector, independent of $\mathbf{n}(t)$, with components:

$$w_i(t) = \mathbf{A}_i(\Theta)\mathbf{s}(t) = \sum_{k=1}^{q} a_i(\theta_k)s_k(t). \quad (8)$$

Now, we define the *covariation matrix* of the observation vector $\mathbf{x}(t)$ as the matrix whose elements are the covariations $[x_i(t), x_j(t)]_\alpha$ of the components of $\mathbf{x}(t)$. It can be shown that the covariation matrix of the observation vector can be written as:

$$[\mathbf{x}(t), \mathbf{x}(t)]_\alpha = \mathbf{A}(\Theta)\mathbf{\Gamma}_S \mathbf{A}^{<\alpha-1>}(\Theta) + \gamma_n \mathbf{I}, \qquad (9)$$

where the (i,j)th element of matrix $\mathbf{A}^{<\alpha-1>}(\Theta)$ results from the (j,i)th element of $\mathbf{A}(\Theta)$ according to the operation

$$[\mathbf{A}^{<\alpha-1>}(\Theta)]_{i,j} = |\,[\mathbf{A}(\Theta)]_{j,i}\,|^{\alpha-2}\,[\mathbf{A}(\Theta)]^*_{j,i}\,.$$

Clearly, when $\alpha = 2$, i.e., for Gaussian distributed signals and noise, the expression for the covariation matrix is reduced to the well-known form of the covariance matrix.

When the amplitude response of the sensors equals unity, i.e., for steering vectors of the form

$$\mathbf{a}(\theta_k) = [1, e^{-\jmath\omega\tau_2(\theta_k)}, \ldots, e^{-\jmath\omega\tau_r(\theta_k)}]^T,$$

it follows that

$$[\mathbf{x}(t), \mathbf{x}(t)]_\alpha = \mathbf{A}(\Theta)\mathbf{\Gamma}_S \mathbf{A}^H(\Theta) + \gamma_n \mathbf{I}, \qquad (10)$$

Hence, in this case, subspace techniques such as the MUSIC algorithm can be applied to the covariation matrix of the observation vector to extract the bearing information. We will refer to the new algorithm resulting from the eigendecomposition of the array covariation coefficient matrix as the **R̲obust C̲ovariation-Based MUSIC** or **ROC-MUSIC** [4].

In practice, we have to estimate the covariation matrix from a finite number of array sensor measurements. A proposed estimator for the covariation coefficient $\lambda_{X,Y}$ is called the *modified fractional lower order (FLOM) estimator* and is given by [4]

$$\hat{\lambda}_{X,Y} = \frac{\sum_{i=1}^n X_i Y_i^{<p-1>}}{\sum_{i=1}^n |Y_i|^p} \qquad (11)$$

for some $0 \leq p < \alpha$ and independent observations $(X_1, Y_1), \ldots, (X_n, Y_n)$.

4 The Wide-Band Problem: The Steered Covariation Matrix

Consider, as before, an array of r sensors that receive signals generated by q *wide-band* sources with identical bandwidth and locations $\theta_1, \theta_2, \ldots, \theta_q$. The signals are assumed to be noncoherent, ergodic and stationary complex isotropic $S\alpha S$ random processes. Also, the noise vector $\mathbf{n}(t)$ is a complex $S\alpha S$ random process with the same characteristic exponent α as the signals. Then, the signal received at the ith sensor can be expressed as

$$x_i(t) = \sum_{k=1}^q a_i(\theta_k) s_k(t - \tau_i(\theta_k)) + n_i(t). \qquad (12)$$

Unlike the narrow band case where the problem was formulated in terms of the sampled data, in the wide-band case it is convenient to formulate the problem in the frequency domain. Assuming the availability of observations in the interval $(-T/2, T/2)$, let $\mathbf{X}(m)$ denote the frequency-domain vector with elements $X_i(m)$ corresponding to the Fourier series coefficients of $x_i(m)$ at frequency $\omega_m = 2\pi m/T$. Note that, by the stability property of the alpha-stable processes, the frequency-domain elements $X_i(m)$ are also $S\alpha S$ processes with the same characteristic exponent as the time domain samples $x_i(t)$.

Under the assumption that the sensor outputs are approximately band-limited to $\omega_l \leq \omega \leq \omega_h$, and that the observation time $T >> 2\pi/(\omega_h - \omega_l) + \max(\tau_i(\theta))$, the *steered sensor output vector* $\mathbf{y}(t, \theta)$ can be expressed as [5]

$$\mathbf{y}(t, \theta) = \sum_{m=l}^h \mathbf{T}_m(\theta)\mathbf{Y}(m)e^{\jmath\omega_m t}, \qquad (13)$$

where

$$\mathbf{T}_m(\theta) = \mathrm{diag}\{e^{\jmath\omega_m\tau_0(\theta)}, \ldots, e^{\jmath\omega_m\tau_{r-1}(\theta)}\}. \qquad (14)$$

Assuming that the observation time T is large, the Fourier coefficients $\mathbf{Y}(m)$ and $\mathbf{Y}(n)$ are independent for $m \neq n$. Then, we the Steered Covariation Matrix (SCM) $\mathbf{C}(\theta)$ can be expressed as

$$\mathbf{C}(\theta) = \sum_{m=l}^h \mathbf{T}_m(\theta)\mathbf{K}(\omega_m)\mathbf{T}_m^H(\theta), \qquad (15)$$

where $\mathbf{K}(\omega_m) = [\mathbf{Y}(m), \mathbf{Y}(m)]_\alpha$ is the covariation matrix of the Fourier coefficient vector at frequency ω_m. In the SCM-methods the SCM matrix $\mathbf{C}(\theta)$ is computed for each steering direction, θ, of interest. Hence, SCM-based methods are more computationally intensive than coherent subspace methods. Their advantage lies in the fact that they take advantage of the full time-bandwidth product of the observations, thus resulting in more stable statistical estimates.

In practice, in order to estimate $\mathbf{K}(\omega_m)$, we divide the T second observation into N nonoverlapping segments of ΔT seconds each and apply the discrete Fourier transform to obtain uncorrelated frequency-domain vectors $\mathbf{Y}_n(m)$ for each segment. Then, the covariation matrix $\mathbf{K}(\omega_m)$ is estimated as

$$\hat{\mathbf{K}}(\omega_m) = \frac{1}{N}\sum_{n=1}^N \mathbf{Y}_n(m)\mathbf{Y}_n^{<p-1>}(m), \qquad (16)$$

for some $0 < p < \alpha$.

5 Simulation Results

In this experiment we compare the performance of the proposed ROC-MUSIC algorithm to MUSIC. The array is linear with five sensors spaced a half-wavelength apart. Two QAM communication signals independent of each other and of the same power impinge to the array. The number of signals is assumed to be known. The noise is assumed to follow the complex isotropic $S\alpha S$ distribution with dispersion γ. In every experiment we perform 200 Monte-Carlo runs and compute the resolution event probability, and the mean-square error (MSE) of the direction-of-arrival estimates averaged for the two sources. The MSE of the DOA estimates was calculated by taking into consideration only the Monte-Carlo runs for which the two algorithms resolved the two sources.

The resolution analysis of the two algorithms was studied by using a popular resolution criterion defined by the following threshold equation [6],[7]:

$$\Lambda(\theta_1,\theta_2) \triangleq P(\theta_m) - \frac{1}{2}\{P(\theta_1)+P(\theta_2)\} > 0, \quad (17)$$

where θ_1 and θ_2 are the angles of arrival of the two signals, $\theta_m = (\theta_1 + \theta_2)/2$ is the mid-range between them, and the *null spectrum* $P(\theta) \triangleq 1/S(\theta)$ is defined as the reciprocal of the spatial spectrum $S(\theta)$. The two signals are said to be resolvable if inequality (17) holds. The inequality implies that the null spectrum magnitude at the mid-angle should lie above the line segment linking the two signal valleys, in order for the two sources to be resolvable.

Since the alpha-stable family with $\alpha < 2$ determines processes with infinite variance, we use two alternative signal-to-noise ratios (SNR's) defined in [8], namely the *Generalized SNR (GSNR)* which is the ratio of the signal power over the noise dispersion γ:

$$GSNR = 10\log(\frac{1}{\gamma M}\sum_{t=1}^{M}|s(t)|^2), \quad (18)$$

and for finite sample realizations, the *Pseudo-SNR (PSNR)*:

$$PSNR = 10\log(\frac{\sum_{t=1}^{M}|s(t)|^2}{\sum_{t=1}^{M}|n(t)|^2}). \quad (19)$$

In the following simulation experiments, we study the resolution capability and estimation accuracy of ROC-MUSIC versus MUSIC as a function of the noise characteristic exponent α, and the angular separation of the two sources.

Characteristic exponent α. Figure 1 illustrates the performance of the two algorithms in a wide range of noise environments, from the more impulsive (α in the neighborhood of 1) to the Gaussian ones ($\alpha = 2$).

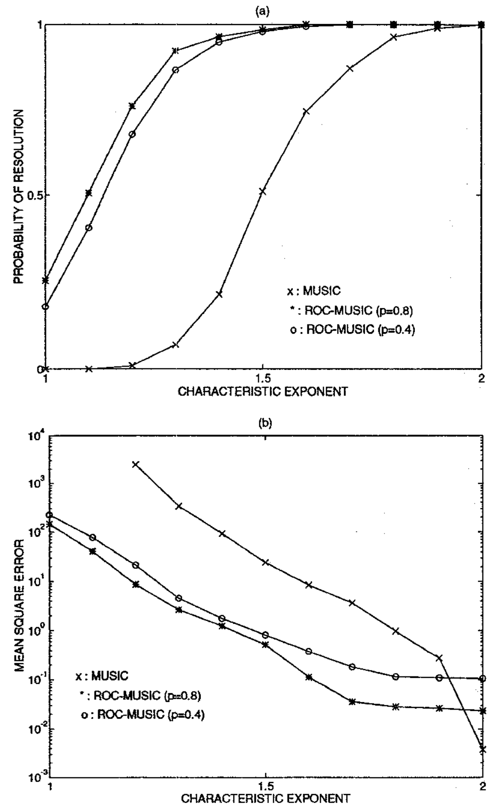

Figure 1: Probability of resolution (a) and mean square error (b) as a function of the characteristic exponent α.

The angles of arrival for the two signals are $\theta_1 = -5^o$ and $\theta_2 = 5^o$. The number of snapshots available to the algorithms is $M = 1000$. The GSNR is 22.3 dB ($\gamma = 1$) and the average PSNR varies from -18 dB ($\alpha = 1.01$) to 0 dB ($\alpha = 2.0$). The characteristic exponent α of the additive noise is unknown to the ROC-MUSIC algorithm. We use two values of the parameter p in the estimation of the covariation matrix (c.f. (11)): $p = 0.8$ and $p = 0.4$. Clearly, MUSIC can be thought as a special case of ROC-MUSIC with $p = 2$.

Figure 1 depicts the improved performance of ROC-MUSIC over that of MUSIC both in terms of resolution probability and MSE, for values of α in the range $(1, 2)$. Note that for $\alpha < 1.2$ MUSIC does not resolve the two sources in any of the 200 Monte Carlo runs. The results suggest that in impulsive noise environments modeled under the stable law, it is more beneficial to use the covariation matrix (lower-order moments) instead of the covariance matrix (second-order moments). Of course, for Gaussian additive noise ($\alpha = 2$) the use of second-order moments ($p = 2$) gives better results.

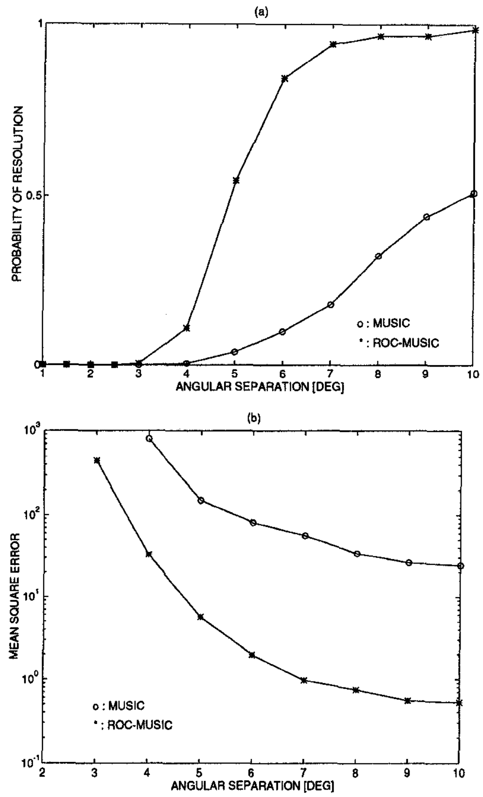

Figure 2: Probability of resolution (a) and mean square error (b) as a function of the source angular separation ($\alpha = 1.5$).

Angular separation. Figure 2 illustrates the variation of the algorithmic performance with respect to the angle separation of the two incoming signals, for $M = 1000$, GSNR= 22.3 dB, PSNR= -1.56 dB ($\alpha = 1.5$). As expected, the resolution capability of both algorithms improves with increased angle separation between the two sources. But for a given probability of resolution, the ROC-MUSIC algorithm requires a lower angle separation threshold than MUSIC.

6 Concluding Remarks

Until recently, statistical signal processing with alpha-stable distributions has not been popular due to the fact that the linear space of a stable process is not a Hilbert space, as in the case of Gaussian processes, but either a Banach ($1 < \alpha < 2$) or a metric space ($0 < \alpha \leq 1$) both of which are more unyielding in their structure. In this paper, we demonstrated that techniques designed under the stable law perform more robustly than their counterparts designed under the Gaussian law. In particular, we presented novel approaches to the wide-band DOA estimation problem in the presence of impulsive interference. We defined the steered covariation matrix of an array of sensors, and applied subspace-based bearing estimation techniques resulting to improved bearing estimates in the presence of impulsive additive noise.

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