SPECTRAL METHODS FOR STATIONARY HARMONIZABLE ALPHA-STABLE PROCESSES

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ABSTRACT

We address the problem of estimation of the fractional-power spectrum of certain classes of symmetric, alpha-stable (SαS) processes. We start with a summary of the key definitions and results from the theory of *stationary, harmonizable* SαS processes and proceed to discuss the performance of fractional-power periodograms. Next, we present a high resolution fractional-power spectrum estimation algorithm that we term "the minimum dispersion distortionless response." The algorithm is a generalization of the classical Maximum Likelihood Method of Capon. Preliminary tests of the algorithms are run on simulated data.

Key words - Impulsive Process, Symmetric Alpha-Stable Process, Power-Spectrum Estimation

1 Introduction

The design of modern signal processing algorithms needs to account for the possibility of operation of the algorithms in a highly non-Gaussian environment in which signals and/or noises may be characterized by stochastic processes with distributions with tails that are significantly heavier than the tails of the Gaussian distribution [7, 2, 6]. Such an environment is termed "impulsive" and is quite common in radio links, underwater sonar and submarine communications, radar, telephone lines, and mobile communications [7, 2, 6]. In an impulsive environment, traditional Gaussian algorithms will perform poorly. Thus, a need arises to design signal processing algorithms that maintain high performance when operating in an impulsive environment and are robust to fluctuations in the characteristics of this environment. This task can be achieved only if good statistical models are available to describe and quantify the interference.

In a number of applications in the above areas, the signal processing tasks include the determination of the frequency content (spectrum) of measured signals/noises. In radar and sonar, for example, random amplitude modulation results from Doppler spreading from fast moving targets. Time-selective fading in communication channels or dispersive propagation in underwater acoustic channels also results in random amplitude modulation. Finally, co-channel interference from multiple users in communication systems can also be described as a superposition of randomly amplitude-modulated sinusoids. For these applications, signal processing tasks include the estimation of the distribution of signal/noise power over a frequency range. Collectively, this problem is referred to as *power spectrum estimation.*

Traditional power spectrum estimation techniques rely on a Gaussianity assumption for the measured data or on the assumption of existence of finite second- and fourth-order statistics. The relevant literature is vast and includes *nonparametric*, *parametric*, and *subspace* techniques. Recently, related array signal processing techniques were extended to account for lack of existence of finite second- or higher-order statistics in the signal/noise models [4, 5]. It was shown, that maximum likelihood techniques [4] based on the Cauchy, as opposed to the Gaussian, distribution and subspace techniques [5] based on the concept of covariation, as opposed to the concept of covariance, outperformed existing techniques when the data distribution deviated from the Gaussian and remain robust in the entire class of alpha-stable distributions. The purpose of the present paper is three-fold: (i) we review the concept of *fractional-power* spectrum of a *stationary, harmonizable* SαS process, (ii) we present fractional-power periodograms and their performance as nonparametric fractional-power spectrum estimators, and (iii) we present a parametric high resolution algorithm that extends the classical Maximum Likelihood Method of Capon. The paper is accompanied by preliminary tests of the proposed algorithms on simulated data.

2 Harmonizable SαS Processes

A. Power-Spectrum of Stationary Fourth-Order Processes

The representation of fourth-order, zero-mean, mean-

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square continuous, stationary processes Z_t, $t = 0, \pm 1, \pm 2, \ldots$, is a well-studied problem in the statistical and signal processing literature [3]. The assumption is that the process Z_t is the Fourier transform of a process with orthogonal increments. More specifically, such processes admit a representation of the form

$$Z_t = \int_{-\pi}^{\pi} e^{it\omega}\, d\xi(\omega), \qquad (2\text{-}1)$$

in terms of a zero-mean process $\xi(\omega)$, $-\pi < \omega < \pi$, with orthogonal increments and

$$\mathcal{E}\{|d\xi(\omega)|^2\} = \phi(\omega)\, d\omega, \qquad (2\text{-}2)$$

where $\phi(\omega)$ is a non-negative function.

The *spectral density*, or simply the power spectrum, of the process Z_t is defined to be the function $\phi(\omega)$ and relates to the *covariance function* of Z_t through [3]

$$\begin{aligned} \text{Covariance}(Z_t, Z_s) &\equiv \mathcal{E}\{Z_t Z_s^*\} \\ &= \int_{-\pi}^{\pi} e^{i(t-s)\omega} \phi(\omega)\, d\omega. \end{aligned} \qquad (2\text{-}3)$$

Therefore, the power spectrum of the process Z_t is the Fourier transform of the covariance function of the process.

The spectral density function of a stationary *Gaussian* process provides a complete characterization of the process in the sense that its finite distributions of arbitrary order can be expressed in terms of the spectral density function.

B. Fractional-Power Spectrum of Stationary Harmonizable SαS Processes

Harmonizable SαS processes are Fourier Transforms of SαS processes with independent increments [1]. More specifically, a harmonizable SαS process Z_t, $t = 0, \pm 1, \pm 2, \ldots$, is defined as

$$Z_t = \int_{-\pi}^{\pi} e^{it\omega}\, d\xi(\omega), \qquad (2\text{-}4)$$

where the process $\xi(\omega)$ is SαS with independent increments and

$$\begin{aligned} \mathcal{E}\{|d\xi(\omega)|^p\}^{\alpha/p} &= C(p,\alpha)\phi(\omega)\, d\omega, \\ &\quad \text{for } 0 < p < \alpha. \end{aligned} \qquad (2\text{-}5)$$

In Eq.(2-5), $C(p, \alpha)$ is a constant, independent of the process $\xi(\omega)$ and $\phi(\omega)$ is a non-negative function that we will call the fractional-power spectral density or, simply, the fractional-power spectrum of Z_t.

We note that the integral in Eq.(2-4) is defined by means of convergence in probability or, equivalently, in the p-th mean, $0 < p < \alpha$, and the finite-dimensional characteristic functions of arbitrary order N of the process Z_t are given by

$$\begin{aligned} &\mathcal{E}\{\exp[i\Re(\sum_{n=1}^{N} z_n^* Z_{t_n})]\} = \\ &\exp[-c_\alpha \int_{-\pi}^{\pi} |\sum_{n=1}^{N} z_n^* e^{it_n\omega}|^\alpha \phi(\omega)\, d\omega], \end{aligned} \qquad (2\text{-}6)$$

where $c_\alpha = \frac{1}{\alpha\pi}\int_0^\pi |\cos\theta|^\alpha\, d\theta$. It can also be shown that, for a characteristic exponent $1 < \alpha < 2$, the *covariation function* of Z_t can be expressed as

$$\text{Covariation}(Z_t, Z_s) = \int_{-\pi}^{\pi} e^{i(t-s)\omega}\phi(\omega)\, d\omega. \qquad (2\text{-}7)$$

Therefore, the fractional-power spectral density of a stationary, harmonizable symmetric, alpha-stable process plays a role completely analogous to that played by the (usual) power spectral density of a stationary Gaussian process.

C. Fractional-Power Spectrum Estimation Problem Formulation

Consider the measurements (data) $\{Z_t, t = 0 \pm 1, \pm 2, \ldots, \pm T\}$ from a model as in Eqs.(2-4) and (2-5). We want to deduce (estimate) the fractional-power spectrum $\phi(\omega)$, $-\pi < \omega < \pi$ from these measurements.

3 Fractional-Power Periodograms

In this section, we look into the design of *non-parametric* estimators for the fractional-power spectrum of a harmonizable SαS process. This design is based on a fractional-power version of the usual periodogram as in the following theorem (summary of several theorems in [1]):

Proposition

Given the data $\{Z_t, t = 0 \pm 1, \pm 2, \ldots, \pm T\}$, define the p-th power periodogram

$$I_T(\omega) = C_{p,\alpha}|d_T(\omega)|^p, \qquad (3\text{-}1)$$

where

$$d_T(\omega) = \Re\{\sum_{t=-T}^{T} e^{-it\omega} h_T(t) Z_t\}. \qquad (3\text{-}2)$$

Let $h_T(t)$ be a bounded even function (window), vanishing for $|t| > T$ and having real, non-negative Fourier transform

$$H_T(\omega) = \sum_{t=-T}^{t=T} h_T(t) e^{-it\omega},$$

with $\int_{-\pi}^{\pi} |H_T(\omega)|^\alpha\, d\omega = 1$, for all T. If $C_{p,\alpha}$ is a proper normalization constant, p is chosen as $0 < p < \alpha$, and $h_T(t)$ satisfies certain mild conditions as $T \to \infty$ [1], then $I_T(\omega)$ is an asymptotically unbiased estimator of $\phi(\omega)^{\frac{p}{\alpha}}$. Additionally, assume that p is chosen as $0 <$

$p < \alpha/2$ and appropriate smoothing spectral windows $W_T(\omega)$ are applied to $I_T(\omega)$ to obtain

$$f_T(\omega) = \int_{-\pi}^{\pi} W_T(\omega - u) I_T(u)\, du. \qquad (3\text{-}3)$$

Then $\phi_T(\omega) \equiv [f_T(\omega)]^{\frac{\alpha}{p}}$ is an asymptotically unbiased and consistent estimator for the fractional-power spectral density $\phi(\omega)$.

4 High Resolution in Fractional-Power Spectrum Estimation

Let us now consider high resolution methods for fractional-power spectrum estimation. More specifically, we look into extensions of Capon's maximum likelihood method. Other high resolution methods will be considered elsewhere.

Consider a FIR filter with impulse response $h_{\omega_0}(n)$, $n = 0, 1, 2, \ldots, q-1$. Let

$$H_{\omega_0}(\omega) \equiv \sum_{n=0}^{q} h_{\omega_0}(n) e^{-i\omega n} \qquad (4\text{-}1)$$

be the filter frequency response. We will design the filter so that

$$H_{\omega_0}(\omega_0) = 1 \qquad (4\text{-}2)$$

and its passband around the frequency ω_0 be as narrow as possible. This second requirement will be satisfied if we design the filter so that the dispersion of its output is as small as possible and at the same time the constraint in Eq.(4-2) is satisfied.

Let Z_t be the input to the FIR filter. Then, the filter output will be

$$\begin{aligned} V_t &= \sum_{k=0}^{q} h_{\omega_0}(k) Z_{t-k} \\ &= \sum_{k=0}^{q} h_{\omega_0}(k) \int_{-\pi}^{\pi} e^{i(t-k)\omega} \phi(\omega)\, d\xi(\omega) \\ &= \int_{-\pi}^{\pi} e^{it\omega} [\phi(\omega)]^{1/\alpha} [\sum_{k=0}^{q} h_{\omega_0}(k) e^{-ik\omega}]\, d\xi(\omega) \\ &= \int_{-\pi}^{\pi} e^{it\omega} [\phi(\omega)]^{1/\alpha} H_{\omega_0}(\omega)\, d\xi(\omega). \qquad (4\text{-}3) \end{aligned}$$

Therefore,

$$\text{Covariation}(V_t, V_s) = \int_{-\pi}^{\pi} e^{i(t-s)\omega} \phi(\omega) |H_{\omega_0}(\omega)|^{\alpha}\, d\omega. \qquad (4\text{-}4)$$

Setting $R_Z(t-s) = \text{Covariation}(Z_t, Z_s)$ and taking Eq.(2-7) into account, we get

$$\text{Covariation}(V_t, V_s) = \int_{-\pi}^{\pi} e^{i(t-s)\omega} [\sum_k e^{-i\omega k} R_Z(k)] | \sum_{n=0}^{q} e^{-i\omega n} h_{\omega_0}(n)|^{\alpha}\, d\omega. \qquad (4-5)$$

We, thus, need to minimize Covariation(V_t, V_s) w.r.t the sequence $h_{\omega_0}(n), n = 0, 1, 2, \ldots, q$, s.t.c. $H_{\omega_0}(\omega_0) = \sum_{n=0}^{q} h_{\omega_0}(n) e^{-i\omega_0 n} = 1$.

5 Illustration on Computer-Synthesized Data

The performance of the fractional-power periodogram is illustrated next on computer simulated data. Fig. 1 shows typical records with a spectrum $\phi(\omega)$, given by

$$\phi(\omega) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} R_t e^{-i\omega t}, \qquad (5\text{-}1)$$

where

$$\begin{aligned} R_t &= \exp(-0.02|t|)[\cos(0.35\pi t) \\ &\quad + \frac{1}{15\pi} \sin(0.35\pi t)] \\ &+ \exp(-0.04|t|)[\cos(0.4\pi t) \\ &\quad + \frac{1}{15\pi} \sin(0.4\pi t)]. \end{aligned} \qquad (5\text{-}2)$$

for the Gaussian and a non-Gaussian alpha-stable (with $\alpha = 1.5$) case. In Fig. 2, the estimate is shown as obtained from 10 non-overlapping records, each of length 128. Clearly, the use of a fractional power of order $p = 2$ degrades in the non-Gaussian case. On the other hand, the use of a fractional power of order $p = 0.6$ allows the estimation of the true spectrum in both the Gaussian and the non-Gaussian cases.

6 Summary, Conclusions, and Future Work

In this paper, we reviewed *harmonizable* alpha-stable random processes as Fourier transforms of alpha-stable random processes with independent increments. We discussed non-parametric, as well as high-resolution, algorithms for fractional-power spectrum estimation and illustrated the algorithms with synthetic data. Future work in this area will address the problem of high resolution in fractional-power spectrum estimation and extensively test the algorithms proposed in here, as well as other approaches. Additionally, future work will also concentrate on similar problems for classes of alpha-stable processes other than the harmonizable.

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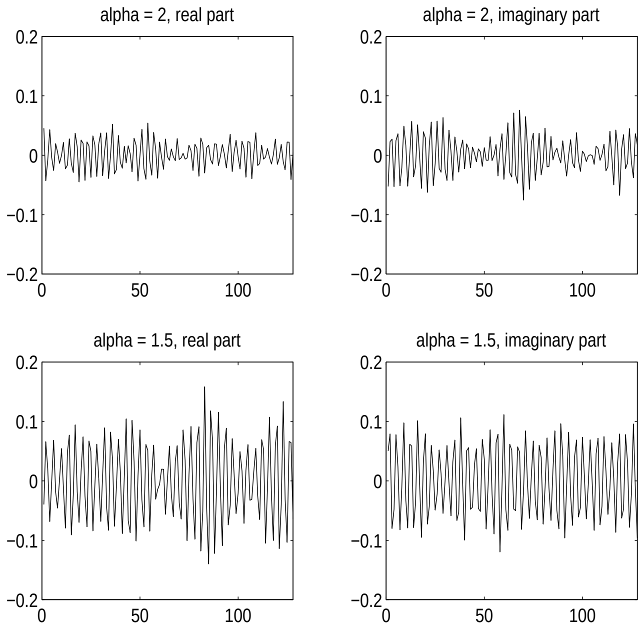

Figure 1: Typical data sets

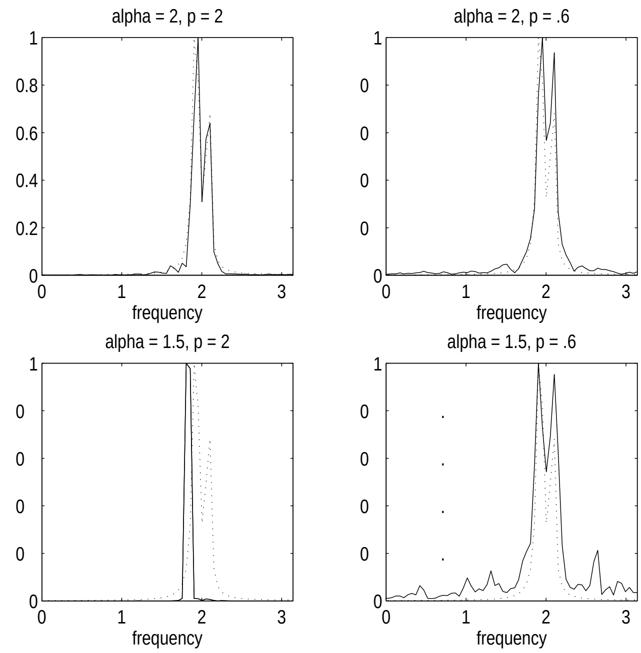

Figure 2: True spectrum (dotted line) and its estimates (solid line)